SOLUTIONS

OF THE

CAMBRIDGE PROBLEMS,

FROM 1800 TO 1820.

BY I. M. F. WRIGHT, B.A.,

LATE SCHOLAR OF TRINITY COLLEGE, CAMBRIDGE.

THE FIRST EDITION REVISED.

VOL. I.

1

BLACK AND ARMSTRONG,

(FOREIGN BOOKSELLERS TO THE KING,)

TAVISTOCK-STREET, COVENT-GARDEN.

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PREFACE

TO THE REVISED EDITION.

THE volume containing the questions themselves, entitled 'Cambridge Problems, or a Collection of the Questions, &c.,' which comprised the enunciations of the problems solved in this work, having long been out of print and being very scarce, it was considered essential to the further utility of these pages that they should go again forth to the public accompanied by a reprint of those enunciations. Accordingly the questions that are actually here solved, omitting all such as are termed 'book-work,' and not problems, are made to precede the solutions, with a reference at each to the number of its solution in either volume. For the convenience also of those who may already possess the solutions without the questions, this reprint of the enunciations is published separately.

The work has been carefully revised throughout its entire extent, and many of the solutions, where any doubt existed in the author's mind as to their accuracy, re-wrought by two of his pupils; but although some errors have been found to have escaped him at first, the result has been such as to convince him that the work was originally composed with more than usual care and assiduity. Indeed, besides the 'Corrigenda and Addenda' appended to the last edition, it has been found necessary to make but two alterations of any importance, which are to be found in articles 502 and 611 of Vol. I., in the former of which the arbitrary constants had been omitted, and in the latter the question itself, being imperfectly stated, had led to errors in the reasoning. New solutions are given in both instances. But whatever defects may still be found in a work of such extraordinary magnitude and variety of research, the reader will easily discover that no labour or expense has been spared to render it as immaculate and as useful, which is a far more important consideration, as the nature and extent of the subjects which it treats, and the fallibility of all things human, will admit.

It may not be quite irrelevant to a work of this nature, considering that it is honoured by the perusal of many persons not educated there, to contain a slight sketch of the system pursued at the University of Cambridge; and the less so from the attacks that have recently been perpetrated against that institution, not only by persons who are totally ignorant of the nature of its workings, but by those who have had every opportunity of noticing and appreciating them—but whose merited disappointments have rendered them malignant.

An Undergraduate commences his course as a 'Freshman,' or 'First-Year-Man,' with Euclid, which occupies the lectures during the First Term. Algebra is given to the Second Term; and the Third Term is

employed in Trigonometry. These subjects occupy, in the daily lectures, the whole of the 'Freshman's Year,' at the termination of which a general examination takes place at each of the seventeen colleges, lasting several hours for three or four days. The examination consists of questions being proposed, *viva voce*, to each student, which are taken from the Elementary Treatises in which he has been lectured; and then follows a printed paper of some fifteen or five-and-twenty problems relating to the same subjects, specimens of which are given at the end of Vol. II. of this work. Each examinée is furnished with one of these papers; and no books or references being permitted in the Hall of the College when the examination is carried on, he must rely entirely upon his own acquirements for solution. Each question has been privately considered by the examiners to be worth a certain number of 'marks;' and it is the sum total of such marks, as estimated at the close of the examination, that determines the place of each student in the classes. Thus, one question of but little difficulty being deemed worth ten marks, another will be counted twenty marks, and so on, according to the time or talent it may require for solution. Those that attain a place in the first class are rewarded by prizes of books bearing the college arms, &c.; and it is this equitable method of adjusting claims by the number of marks that inspires a degree of competition in the students that would be incredible to all who have not *felt* it.

During the First Term of the second year the subject is usually Mechanics; the Second is devoted to Hydrostatics, Hydrodynamics, and Pneumatics; and the Third to Plain Astronomy, and the first three sections of Newton's 'Principia;' but the 'Junior Sophs' are not lectured in the same subjects at all the colleges. The Terms of the third year are spent in like preparation for examination in the other and higher departments of Mathematical Science, scil. Optics, Physical Astronomy, &c.; and a similar examination, classification, and distribution of honorary and substantial rewards takes place at the end of this as also of the second year.

The Tenth Term is applied to a recapitulation of all those subjects, when the student, who is now called a 'Questionist,' from having to answer questions proposed to him by opponents in the schools, undergoes an examination of six days for the degree of Bachelor of Arts. The *mark* method here again is adopted, and the aspirant becomes a 'Wrangler,' a 'Senior Optime,' a 'Junior Optime,' one of the οἱ πολλοι, or is 'Plucked,' according as his 'marks' number him one on the 'Tripos' of the three first classes, make him one of the *many*, or deprive him of his degree. Every one knows what is meant by 'Senior Wrangler.'

It is the problems which have been proposed at this most important of all the Cambridge *Mathematical Examinations* that have been attempted in solution in the following pages.

TO THE

TUTORS OF THE SEVERAL COLLEGES,

THIS WORK,

WHICH IS INTENDED TO PROMOTE THE STUDIES OVER WHICH THEY SO ABLY PRESIDE,

IS RESPECTFULLY INSCRIBED,

BY THEIR

DEVOTED SERVANT,

I. M. F. WRIGHT.

London, Feb. 11, 1825.

PREFACE.

An elaborate Preface to a work upon the "Exact Sciences" is better suited to display the Author's English than his Mathematics. A work of this nature being, from the most cursory perusal, readily appreciable by those for whose use or entertainment it is designed, and totally unintelligible to others,—the Author of "Solutions of the Cambridge Problems" merely announces to his readers, that he has laboured long and hard for their benefit, and awaits with some degree of confidence that most pleasing, and in its consequences most substantial, reward—their approbation.

London, Feb. 11, 1825.

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REFERRING

TO THE PROBLEMS.

N. B. The column marked N gives the number of the solution. Those which are headed by *p* and *n* give the page and number in the page of the volume containing the enunciations. Consequently the three columns give the enunciation of any problem corresponding to its solution.

GEOMETRY.

ARITHMETIC.

ALGEBRA.

TRIGONOMETRY.

DIFFERENTIALS.

INTEGRALS.

FINITE DIFFERENCES.

FINITE INTEGRALS.

SERIES.

PROBABILITIES.

MISCELLANIES.

SOLUTIONS.

GEOMETRY.

1. Fig. 1. WITH a radius = given radius describe a $\odot$, in which taking any point A, draw from it a line touching the $\odot$, make the $\angle$ BAC = the given $\angle$ a (suppose given $\angle$s a, b, c,) from C, draw CD making $\angle$ C = $\angle$ b, and join DA. ADC is the $\triangle$ required.

$$\text{For} \because \angle D = \angle CAB = \angle a$$
$$\angle C = \angle b$$

and $\angle$ CAD = supplem. of C + D = supp. of a + b = c
$\therefore$ $\triangle$ CAD has been described having its $\angle$s = given $\angle$s, and radius of its circumscribing $\odot$ = given radius.

2. Since AEB subtends a R. $\angle$ D, ADB = $\frac{1}{4}$ $\odot$ whose rad. is AD.

Now circles are to one another as the squares of their diameters (*Euc.* Book 12.)

$$\therefore \odot ADBC : 4 \times AEBD :: AB^2 : 4 AD^2$$
$$\text{Or } \frown ACB : 2 \times AEBD :: 2 AD^2 : 4 AD^2$$
$$:: 1 : 2$$
$$\therefore 2 \frown ACB = 2 AEBD$$
$$\text{and } \frown ACB = AEBD.$$

Take away the common part AEB of these equals, and the lune ACBEA = $\triangle$ ADB.

3. Fig. 2. Let a b c be a $\triangle$ having its $\angle$s $=$ given $\angle$s respectively. With the given radius, and O as a centre, describe a $\odot$, and at O make the $\angle$ m O n $=$ supp. of $\angle$ c, and $\angle$ m O r $=$ supp. of $\angle$ a. Draw lines touching the $\odot$ in points m, r, n, and meeting in A, B, C; (for they must meet, or O n, O m, would be in same straight line, and $\angle$ c $=$ O) A B C is the $\triangle$ required.

For $\because$ $\angle$s at m and n are R $\angle$s, and $\angle$s of quadrilateral figure together $=$ 4 R. $\angle$s, the $\angle$ C $=$ supp. of $\angle$ m O n $=$ $\therefore$ $\angle$ c. Similarly it may be proved that $\angle$ A $=$ $\angle$ a and $\angle$ B $=$ $\angle$ b.

$\therefore$ the $\triangle$ A B C has been described having its $\angle$s $=$ given $\angle$s and rad. of its inscribed $\odot$ $=$ a given straight line.

4. Fig. 3. Let the circles A B P, C P D touch in the point P. Draw the diameter C D $\parallel$ diameter A B, and join A P, P B, C P, D P. Also join O P, Q P, O, Q being the centres of the respective circles.

Now since P Q and P O are both $\perp$ to the straight line touching the circles in point P, they are in the same straight line.

$\therefore$ $\angle$ D O P $=$ alternate $\angle$ P Q A,

or, 2 $\angle$ C $=$ 2 $\angle$ B

$\therefore$ $\angle$ C $=$ $\angle$ B

or $\angle$ B P Q $=$ $\angle$ O P C, and they are vertical $\angle$s,

$\therefore$ B P, P C are in same straight line.

Similarly A P, P D may be proved to be in same straight line.

5. Since the $\odot$ is given, its radius is, and the $\angle$s being equal, are each $\frac{1}{3}$ of two R. $\angle$s, and are $\therefore$ known. Hence it appears that this problem is only a particular case of Problem I.

The problem may, however, be solved by bisecting the radius of the given $\odot$, and through the point of bisection, drawing a $\perp$ to the radius. The $\perp$ terminated both ways by the $\odot$ will be the side of the equilateral $\triangle$.

6. Fig. 4. Let A B, B C, the two given lines, meet in B, and lie in the same straight line A C. With B as centre, and radius $=$ B C describe a $\odot$ cutting A C in D, and from A draw A E touching the $\odot$ in E. A E is the line required.

$$\text{For } AE^2 = AD \cdot AC$$
$$= (AB - BD) \times (AB + BC)$$
$$= (AB - BC) \times (AB + BC)$$

$\therefore$ AE is a mean proportional between AB + BC and AB—BC.

7. Fig. 5. Let ABCD be a ▱ whose diagonals AC, BD cut each other at R. ∠ in P.

$\because$ AB is ∥ DC, ∠ PDC = ∠ ABP, ∠ at P are =.

Also AB = DC, $\therefore$ in the △ ABP, DPC there are two ∠ and one side = in each, each to each, and $\therefore$ PB = PD.

Hence PD = PB and PA is common to the △ APD, APB which have R. ∠ at P.

$\therefore$ AD = AB = DC, and BC = AD

$\therefore$ the general form required, is that of Rhombus.

8. Fig. 6. Let OA be a radius of the given ⊙; produce it to B, making AB = AO, and upon OB describe OBC, an equilateral △ meeting in ⊙ in D. With B as centre and distance BA describe a ⊙ meeting BC in E, and ⊙ AO in A. Also with C as centre and distance CD, describe a ⊙ meeting ⊙ AO in D.

Then $\therefore$ CD = CO − OD = BO − AO = BC − BE = CE, the latter ⊙ will meet CB in E.

Now circles whose centres, and the point in which they meet, are in the same straight line, touch in that point; for the line ⊥ to the diameter of one of them at its extremity is also ⊥ diameter of the other, and $\therefore$ touches both circles, *i. e.*, they touch one another.

Hence, ⊙ AE touches ⊙ AD and ⊙ AE

And ⊙ DE touches ⊙ AD and ⊙ AB

Similarly if an equilateral △ be described on OC, a ⊙ may be described, whose diameter = diameter of the given ⊙, and touching that ⊙ and the ⊙ DE, and so on round the ⊙. Hence it is evident there can only be as many circles of the required description, as equilateral △ whose ∠ meet at the centre. Since there are 6 ∠ of an equilateral △ in 4 R. ∠, 6 such circles may be described of same diameter with the given ⊙, and touching it and one another.

9. Fig. 7. Join A, B the centres of the two circles and produce it to C, making BC to AC as the radius of ⊙ B is to that of ⊙ A. Again, in AB take BC′ to AC′ as radius of ⊙ B to radius of ⊙ A. From C draw CN touching ⊙ B in N, (by describing on BC a ⌓ meeting ⊙ B in N,) join BN, and draw AM ∥ BN, and join MN. Again, from C′ draw C′N′ touching ⊙ B in N′, and AM′ ∥ BN′ and join C′M′. CNM, N′C′M′ are straight lines, and touch the circles in N, M; N′, M′ respectively.

∵ BC : AC :: BN : AM, and BN is ∥ AM; CN, NM are in the same straight line.

$$\therefore \angle AMC = \angle BNC = R. \angle$$

$$\therefore CM \text{ touches } \odot\ A \text{ in } M.$$

Again, ∵ BC′ : AC′ :: BN′ : AM′ and BN′ ∥ AM′

$$\therefore C'N' \text{ and } C'M' \text{ are in the same straight line.}$$

$$\therefore \angle AM'C = \angle BN'C = R. \angle.$$

$$\therefore N'M' \text{ touches } \odot\ A \text{ in } M'.$$

∴ Either MN, or M′N′ answer the conditions of the problem.

Otherwise,

Take a ⊙ concentric with ⊙ A whose rad. = rad. of ⊙ A ∓ rad. of ⊙ B according as ⊙ A is > or ∠ ⊙ B, and from B draw a line touching the new ⊙. Produce the rad. to meet ⊙ A, and draw a tangent to ⊙ A at the meeting point; this line shall also touch the other ⊙.

10. Fig. 8. Let P be any point in the equilateral △ ABC; PS, PQ, PM ⊥ to AB, AC, BC respectively, and AN ⊥ BC. AN = PS + PQ + PM.

Through P draw DE ∥ BC meeting AN in T.

Then ∵ DE is ∥ BC, the ∠ D = ∠ B = ∠ C = ∠ E, and the ∠s at S and Q are R. ∠s, ∴ the △s PSD, PQE are equiangular.

$$\therefore PS : PQ :: PD : PE.$$

$$\therefore PS + PQ : PQ :: PD + PE : PE.$$

$$\therefore PS + PQ : DE :: PQ : PE.$$

Now △s ATE, PQE are similar, ∵ the ∠ at E is common to them both, and they have R. ∠s.

$\therefore$ PS + PQ : DE : : PQ : PE :: AT : AE = DE, (for $\angle$ D = $\angle$ B, and $\angle$ E = $\angle$ C.)

$\therefore$ PS + PQ = AT. Now PM, TN are $\perp$ BC and $\therefore$ $\parallel$, also PT is $\parallel$ BC, $\therefore$ PN is a $\square$. $\therefore$ TN = PM.

$$\therefore AN = AT + TN = PS + PQ + PM.$$

Otherwise,

Join PA, PB, PC. Then $\triangle$ ABC = $\triangle$ APB + $\triangle$ APC + $\triangle$ BPC.

$$\therefore \frac{BC \,.\, AN}{2} = \frac{AB \,.\, PS}{2} + \frac{AC \,.\, PQ}{2} + \frac{BC \,.\, PM}{2}$$

But BC = AB = AC

$$\therefore AN = PS + PQ + PM.$$

11. Fig. 9. Take O the centre of the quadrant ADC and join AD.

Then $\because$ $\angle$ AOC = R. $\angle$, $\angle$ ADC, which = supp. of the $\angle$ in the opposite segment = supp. of $\frac{1}{2}$ $\angle$ O = supp. of $\frac{1}{2}$ R. $\angle$. But $\angle$ ADC = supp. of $\angle$ BDA.

$\therefore$ $\angle$ BDA = $\frac{1}{2}$ R. $\angle$. Also $\angle$ B = R. $\angle$

$\therefore$ $\angle$ BAD = $\angle$ BDA

$\therefore$ BA = BD.

Again, let BA be < BC, then the $\angle$ BAC is > $\angle$ BCA.

$\therefore$ $\angle$ BAO is > $\angle$ BCO; but $\because$ $\angle$s B and O are R. $\angle$s $\angle$ BAO + $\angle$ BCO = 2 R. $\angle$s.

$\therefore$ $\angle$ BAO is > R. $\angle$, and $\angle$ BCO < R. $\angle$.

$\therefore$ BA lies without the circumference ADC and BC cuts it.

12. Fig. 10. Let AB passing through the point of contact P of the circles ACP, BDP cut off the circumferences ACP, BDP, or B′D′P.

Let also MN touch both circles in P.

Then $\angle$ APN = $\angle$ in segment ACP.

= $\angle$ in segment B′D′P′.

$\therefore$ When the circles touch internally, AB′P cuts off similar segments ACP, B′D′P.

Again, the $\angle$ MPB = $\angle$ in segment BDP.

But $\angle$ MPB $=$ $\angle$ APN $=$ $\therefore$ $\angle$ in segment ACP.
$\therefore$ $\angle$ in segment ACP $=$ $\angle$ in segment BDP.
$\therefore$ ACP is similar to BDP.

13. Fig. 11. Let $\angle$ BAC be the given vertical $\angle$, take AB $=$ AC $= \frac{1}{2}$ the given perimeter, and from B and C draw BO, CO $\perp$ AB, AC respectively, and meeting in O (they must meet, otherwise AC and AB would be $\parallel$). Join AO, and since AB $=$ AC, and AO is common to the right angled Δ ABO, AOC, BO $=$ CO. $\therefore$ with centre O and distance $=$ BO or CO describe a $\odot$ passing through B and C. It touches AB and AC, $\because$ B and C are R. $\angle$. Again with A as centre, and given $\perp$ altitude as distance describe a $\odot$ meeting AB in D, and AC in E. Then (by 9) draw FE touching the circles A and O in N and M, and meeting AB, AC in F and G respectively. Join AN. AFG is the Δ required.

For FM $=$ FB, and GM $=$ GC, and $\therefore$ FG $=$ FB $+$ GC. $\therefore$ AF $+$ FG $+$ AG $=$ AB $-$ BF $+$ FB $+$ GC $+$ AC $-$ GC.

$=$ AB $+$ AC $=$ given perimeter.

Also AN $=$ the given $\perp$, and $\angle$ FAG $=$ given vertical $\angle$ $\therefore$ Δ FAG is such as was required.

14. Every quadrilateral figure may be divided into two Δ, the sum of whose $\angle$ shall $=$ the sum of the $\angle$ of the figure. But sum of $\angle$ in two Δ $=$ 4 R. $\angle$. $\therefore$ the sum of $\angle$ in the figure $=$ 4 R. $\angle$.

15. Fig. 12. Upon AB the given base describe a segment of a $\odot$ containing an $\angle$ $=$ the given vertical $\angle$. From B draw BC $\perp$ AB, and $=$ the given $\perp$. Draw CD′ $\parallel$ AB, and meeting the $\odot$ in D and D′, and join AD, BD, AD′, BD′; either of the Δ ABD, ABD′ will answer the given conditions, their altitudes, bases, and vertical $\angle$ being $=$ those given respectively.

N.B. When CB $=$ $\perp$ from the middle of AB and meeting the $\odot$, the two Δ coincide. When CB is $>$ this $\perp$, the problem is impossible.

16. Fig. 13. Let A B, C D be two diameters of the given ⊙ ⊥ to one another. Bisect ∠ A O D, B O D by the lines O E, O F respectively, and produce O E to G and O F to H. Join AE, ED, DF, FB, BG, GC, CH, HA, AE DFBGCHA, is the octagon required.

For ∵ in the △ AOE, EOD, $AO = OE$ and $OE = OD$, and $\angle AOE = \angle EOD$, ∴ $AE = ED$, and $\angle OAE = \angle OED$.

$$\therefore \angle AED = \angle OEA + \angle OED = 2.\ \angle OEA.$$
$$= \text{supplement of } \angle AOE.$$

Similarly ∠ E D F may be shewn = supplement of ∠ E O D.

$$\text{But } \angle AOE = \angle EOD.$$
$$\therefore \angle AED = \angle EDF.$$

In same manner $\angle EDF = \angle DFB$, and since $\angle GOB = \angle AOE = \frac{1}{2} R. \angle = \angle BOF$, ∠ DFB may be proved = ∠ FBG, and so on for the other ∠. ∴ the octagon is equiangular.

Also $AE = ED$, and since the △ EOD, DOF, FOB, &c., have two sides, and the included ∠ in each = respectively, their third sides, or DE, DF, FB, &c., are equal. ∴ the octagon is equilateral. It is ∴ such as was required.

17. Fig. 14 a. Lemma 1. To describe a ⊙ passing through two given points, and touching a given ⊙.

Case 1. Let the given points P, Q be without the given ⊙ ABD. Describe a ⊙ passing through P, Q, and cutting the given ⊙ in A, B, points such that BA, PQ being joined shall not be ||. Let BA, PQ be produced to meet in C. From C draw CD touching ⊙ ABD in D (by describing a ⌓ upon the line formed by joining C and the centre of the ⊙ ABD, &c.), and describe a ⊙ PQD passing through P, Q, D. PQD is the ⊙ required.

$$\text{For } CP.\ CQ = CA.\ CB = CD^2.$$

∴ CD touches both the circles PQD, ABD in D.

∴ P Q D touches ABD in D, and it passes through P and Q. It is ∴ such as was required.

The proof is exactly the same for fig. b.

Case 2. Fig. 14. c. Let P and Q be within the given ⊙

Then as before, describe a ⊙ passing through P, Q, and cutting the ⊙ in A B. Let A B, P Q produced meet in C. From C draw C D or C D′ touching the given ⊙ in D or D′, and the ⊙ passing through D or D′ and P and Q will touch the given ⊙ in D or D′, or it will be such as was required. The proof is similar to that of Case 1.

Case 3. When one of the given points is within the given ⊙, and the other without, the problem is impossible.

Fig. 15. Lemma 2. To describe a ⊙ touching two others given in magnitude and position, and passing through a given point.

Draw N M touching the given circles in N, n, respectively (9) and meeting A a produced in M (A, a being the centres).

Let A a produced both ways cut ⊙ A N in D, and ⊙ a n in d. Take $MQ = \dfrac{Md.\,MD}{MP}$ and through P and Q describe a ⊙ O b also touching the ⊙ a n in b. (Lem. 1.) The ⊙ O b also touches the ⊙ A N.

Join M b and produce it to B, and join O B. Also join b d, E F (which are ∥ ∵ M a : M A :: a n : A N :: a b : A E, and ∴ ∠ F A E = ∠ d a b, and ∴ ∠ A F E = ∠ a d b;) then M d : M b :: M F : M E :: M B : M D (∵ M F. M D = M E. M B.)

∴ M b. M B = M d. M D = M Q. M P by construction.

∴ B is a point in the ⊙ P Q b.

Hence O B = O b. ∴ the ∠ O B b = ∠ O b B

= ∠ A E B = ∠ A B E ∵ A E is ∥ a b and A E = A B.

∴ ∠ O B b = ∠ A B E, or O B coincides with A B.

∴ A and O are in same straight line with B.

∴ ⊙ O b touches the ⊙ A N in B, and it also touches a n, and passes through P. It is ∴ such as was required.

N.B. There are two other cases, *viz.*, when the required ⊙ shall be interior with respect to both the given circles, and when it shall be interior with respect to one, but exterior with respect to the other. These cases may easily be deduced from the preceding one.

Otherwise,

Fig. 16. Let a, A be the given circles, P the given point. Through P and a draw Pc cutting ⊙a in b and c, and with c as centre and distance $= \frac{ba. Pc}{pa}$, describe the ⊙ xy. Draw Ee touching ⊙ AE in E and ⊙ xy in e (9). Join PE, Pe meeting the circles A, a in G, g respectively. Then the ⊙ passing through P, G, g is the ⊙ required.

17. Fig. 17. Case 1. Let the required circle be exterior with respect to each of the three given circles A, B, C.

Then supposing C not > A or B, with A as centre and radius = radius of ⊙ A — radius of ⊙ C describe a ⊙ Aa′. Also with B as center and radius = radius of ⊙ B — radius of circle ⊙ C describe a ⊙ Bb′. By lemma 2. describe a ⊙ OC touching the ⊙ Aa′ in a′, and ⊙ Bb′ in b′, and passing through C. Produce OC to c, and with centre O and radius OC describe a ⊙ Oc. This is the ⊙ required.

Join Oa′, Ob′ and produce them to meet the circles A, B in a, b respectively.

$$\begin{aligned} \text{Then } Oa &= Oa' + aa' = OC + aA - Aa' \\ &= OC + Cc \text{ by construction} \\ &= Oc \end{aligned}$$

Similarly Ob = ⊙ c

∴ the points a, b are in ⊙ Oc.

Also the centres O, A are in the same straight line.

∴ ⊙ Oc touches the ⊙ Aa in the point a.

Similarly it may be shewn that the ⊙ Oc touches the ⊙ Bb in b, and the ⊙ Cc in c.

Fig. 18. Case 2. Let two of the given circles be interior, and the other exterior; with respect to the required ⊙.

Let Bb be the exterior ⊙. With centre B and radius = Bb + Cc describe a ⊙ Bb′. With centre A and radius = Aa — Cc, (Cc being not > Aa) describe a ⊙ Aa′. Then, by lemma 2, describe a ⊙ passing through C and touching the circles Aa′, Bb

whose centre let be O. O is the centre of the required ⊙, which may be described as in case 1.

Fig. 19. Case 3. Let only one of the given circles as (C c) be interior with respect to the required ⊙.

With centres A and B and radii $=$ A a′ $+$ C c, B b′ $+$ C c describe the respective circles A a′, B b′. Describe a ⊙ O A touching ⊙ A a′ in a′, and ⊙ B b′ in b′, and passing through c (lemma 2.) Then O is the centre of the circle required.

Fig. 20. Case 4. Let all the circles be exterior with respect to the required ⊙, of which let ⊙ C c be that which is not $>$ ⊙ A a, or ⊙ B b.

With centres A and B and radii A a $+$ C c, B b $+$ C c describe the respective circles A a′, B b′. Describe a ⊙ O C passing through C and touching the circles A a′, B b′ in a′, b′, respectively. O is the centre of the ⊙ required.

N.B. Hence it appears, that when any circles are interior, the radius of that which is not $>$ any of them, is to be subtracted from the radii of the interior circles, and added to the radii of the exterior. When all are exterior, the radius of that which is not $>$ any of them must be added to the radii of the others, in the construction.

The problem (17) may easily be deduced from the above, as it is less general.

18. Fig. 21. The straight line must evidently be less than the diameter of the given ⊙.

Place A B in given ⊙ $=$ the given straight line. From O the centre of the ⊙ draw O C $\perp$ A B, and with centre O and distance O C describe a ⊙ O C. From P or P′ draw a line touching the ⊙ O C in N or N′ (by describing on P O, or P′ O a ⌓, &c.), and let it be terminated by the given ⊙ in M, R or M′, R′. M R and M′ R′ are the lines required.

For being equally distant from the centre with A B, they are equal to A B and $\therefore$ to the given line.

19. Fig. 22. Let ABCD be the given quadrilateral figure, whose sides are bisected in F, H, G, E respectively. Join AC, BD, GF, EH. $AC^2 + BD^2 = 2(GF^2 + EH^2)$

Because $AF : AB :: 1 : 2 :: AE : AD$

$\therefore$ EF is $\parallel$ DB.

Similarly GH may be proved $\parallel$ DB, FH $\parallel$ AC, EG $\parallel$ AC.

$\therefore$ EF is $\parallel$ GH, and FH is $\parallel$ EG

$\therefore$ EFHG is a $\square$.

Now from similar $\triangle$ AEF, ADB, $BD = 2EF \therefore BD^2 = 4EF^2$, similarly $AC^2 = 4EG^2$

$\therefore AC^2 + BD^2 = 4EF^2 + 4EG^2$. It may also be very easily proved that $FG^2 + EH^2 = 2EF^2 + 2EG^2$

$$\therefore AC^2 + BD^2 = 2FG^2 + 2EH^2.$$

20. Fig. 23. The segments are made by a $\perp$ from the $\angle$ opposite the base.

Let ABC be the $\triangle$, CD, BD the segments of the base BC. Then $BC : AB + AC :: AB - AC : BD - DC$

$$\text{For } AD^2 = AB^2 - BD^2 = AC^2 - CD^2$$

$$\therefore AB^2 - AC^2 = BD^2 - CD^2$$

Or $\overline{AB + AC}.\ \overline{AB - AC} = \overline{BD + CD}.\ \overline{BD - CD}$

$\therefore BD - CD : AB + AC :: AB - AC :: BD + CD$ (fig. a) and $BD + CD : AB + AC :: AB - AC :: BD - CD$ fig. b; so that the proposition only holds when the $\perp$ falls within the $\triangle$.

21. Fig. 24. Case 1. Let the $\perp$ Aa intersect the $\perp$ Bb in P, and the $\perp$ Cc in some other point, if possible, P′.

Then from the similar $\triangle$ AaC, BPa (AaC is similar to APb and $\therefore$ similar to BPa).

$$Ba : Aa :: Pa : Ca \text{ also from the similar}$$

$\triangle$ AaB, P′aC, $Aa : Ba :: Ca : P'a$

$$\therefore 1 : 1 :: Pa : P'a$$

$$\therefore Pa = P'a \text{ or } P' \text{ coincides with } P \therefore \&c.$$

Otherwise,

Let Cc, Bb intersect in P, then join AP and produce it to a,

A a is $\perp$ B C. For b and c being R. $\angle$s, it may easily be proved that A P, B C are the diameters of circles passing through b and c. Let these circles be described. Join b, c. Then the $\angle$ b B C $=$ $\angle$ b c C $=$ $\angle$ P A b, and the vertical $\angle$s at P are equal. $\therefore$ $\angle$ B a P $=$ $\angle$ b $=$ R. $\angle$ $\therefore$ &c.

Case 2. Fig. 25. Let A a, B b, C c, be the lines bisecting the sides in a, b, c respectively, and let A a intersect B b in P, and C c in P′. Join b c intersecting A a in Q.

Then $\because$ A C, A B are cut proportionally in b, c, b c is $\parallel$ B C. $\therefore$ b Q : C a :: A b : A C :: 1 : 2.

Also the $\triangle$ B P a is similar to P Q b.

$\therefore$ P a : P Q :: B a ($=$ a C) : b Q :: 2 : 1.

$\therefore$ P a $+$ P Q : P a :: 3 : 2.

$\therefore$ P a $= \frac{2}{3}$ Q a.

Similarly it may be shewn that P′ a $= \frac{2}{3}$ Q a.

$\therefore$ P a $=$ P′a or P′ coincides with P.

$\therefore$ A a, B b, C c intersect in the same point P.

Otherwise,

Let P be the intersection of B b, C c. Join A P, and produce it to a. A a bisects B C.

For from similar $\triangle$s a C : c Q :: a P : P Q :: B a : b Q.

$\therefore$ a C : a B :: c Q : Q b

$\therefore$ a C $+$ a B or B C : a B :: c Q $+$ Q b or c b : Q b

$\therefore$ a B : Q b :: B C : b c :: A C : A b :: 2 : 1

$\therefore$ a B $=$ 2 b Q $=$ a C.

$\therefore$ &c. &c.

Cases 3 and 4. See *Euc.* B. iv. Prop. 4 and 5.

N. B. The points of intersection in the cases 1, 2, and 4, are in the same straight line.

22. Fig. 26. Let O be the centre of the $\odot$ in which is inscribed the equilateral $\triangle$ A B C, join A O and produce it to D meeting B C in E. Join O B, B D.

Then the $\angle$ B O D $=$ $\angle$ O B A $+$ $\angle$ O A B $=$ 2 $\angle$ B A O

$= \angle$ BAC (evident after drawing from O perpendiculars upon AB, AC) $= \angle C = \angle D$. $\therefore$ in the $\triangle$ BOE, BDE,

BE is common, and $\angle BOE = \angle BDE$, and $BO = BD$

$\therefore \angle OEB = \angle DEB = R. \angle$ and $OE = ED$

$\therefore BE = EC$

Now $BO^2 = BE^2 + OE^2 = \overline{\tfrac{1}{2} BC}|^2 + \overline{\tfrac{1}{2} OD}|^2$

$$= \frac{BC^2}{4} + \frac{OB^2}{4}$$

$\therefore 3\,BO^2 = BC^2 \therefore BC = \sqrt{3}.\ BO$

Or $BC : BO :: \sqrt{3} : 1$

23. Fig. 27. Let the $\angle A = 120$, and draw CN $\perp$ BA produced. Then $\angle CAN = 180 - 120 = 60° = \angle$ of an equilateral $\triangle$. Now CN evidently besects the base of that $\triangle$ (22), $\therefore AN = \frac{1}{2} AC$.

$$\text{Again, } BC^2 = AB^2 + AC^2 + 2\,AB\,.\,AN.$$

$$\text{But, } 2\,AB\,.\,AN = 2\,AB\,.\,\frac{AC}{2} = AB\,.\,AC$$

$$\therefore BC^2 = AB^2 + AC^2 + AB\,.\,AC.$$

24. Fig. 28. For the definition, see *Euc.*

Let ABCD be the rhombus. Join AC, BD intersecting in O, AC, BD bisect each other at R. $\angle$ in O.

Because $AB = AD$, the $\angle ABO = \angle ADO$ and AO is common to the $\triangle$ AOB, AOD

$\therefore BO = DO$, and $\angle AOB = \angle AOD = R. \angle$

$\therefore \angle DOC = \angle BOC = R. \angle$.

Again $\because AB = BC$, and BO is common to the right-angled $\triangle$ BOA, BOC. $\therefore AO = CO$, and DO has been proved $= CO$

$\therefore$ AC, BD bisect each other at R. $\angle$.

25. Fig. 29. In the $\odot$ ACD let AC, ED intersect in B. Also let AB be $>$ BD or BC or BE. Join AE, DC. Let O be the centre. Join OB and produce it to F.

Now $\because \angle A = \angle D$ and vertical $\angle$ at B are equal, the $\triangle$ ABE, BCD are similar.

∴ AB : BD : : BE : BC, or the quantities AB, BD, BE, BC are proportionals.

Again ∵ B is a point in the diameter, and AB is > BD, BE and BC, ∴ AB is least remote from the centre. ∴ AC is > ED

or AB + BC > BD + BE.

26. Fig. 30. Let ABC be the given △, PS the ▱. Draw AE ∥ BC and make ∠ DAE = ∠ P. Take AE : AD : : PQ : PR and join BE cutting AC in q; draw qp ∥ BC, and pr, qs ∥ AD. ps is the ▱ required.

The ∠ rpq = ∠ Bpq − ∠ Bpr = ∠ BAE − ∠ BAD = ∠ DAE = ∠ P.

∴ ∠ qsr = ∠ S, and ∠ pqs = supp. of ∠ qpr = supp. of ∠ P = ∠ Q

∴ ∠ prs = ∠ R. or ps is equiangular with PS.

Again, from similar △ pq : AE : : pB : AB : : pr : AD

∴ pq : pr : : AE : AD : : PQ : PR.

∴ ps is similar to PS.

27. Fig. 31. From O the given centre extend the compasses to meet the ⊙ in A; with the distance AO, and centre A describe an arc cutting ⊙ in B, and with the same distance and centres B and C describe arcs cutting ⊙ in the point C and D respectively. Join AO, OD, AD is a straight line, or it is the diameter required.

For ∵ AB = AO = OB = BC, &c., the △ AOB, BOC, COD are equilateral and ∴ equiangular.

∴ ∠ BOA + ∠ BOC + ∠ COD = three ∠ of an equilateral △ = 2 R. ∠.

∴ AO, OD are in the same straight line ∴ AD is a diameter of the ⊙.

28. Fig. 32. Let Aa be ∥ Bb and ∴ in the same plane with it. Let a plane pass through any point Q in Aa ⊥ Aa, and intersecting Bb in R, and Cc, the common section of the two given planes in P.

Now ∵ AQ is ⊥ the plane PRQ, and RB ∥ AQ,

R B is also $\perp$ to the plane. But A a, B b are common sections to the planes C a, B a; C b, B a respectively, $\therefore$ the plane P R Q is $\perp$ to both the planes B c, A c, and $\therefore$ to their common section C c. $\therefore \angle CPQ = R. \angle = \angle AQP$.

$\therefore$ C c is $\parallel$ A a & B b.

Otherwise,

Pass through Q$'$ another plane R$'$ P$'$ Q$'$ $\perp$ A a and meeting B b, C c in R$'$, P$'$ respectively.

Then it may be easily proved that PR is = and $\parallel$ P$'$R$'$ and $\therefore$ that C c is $\parallel$ B b.

29. Fig. 33. Let O be the centre of the inscribed $\odot$, c the R. $\angle$. Join O N, O M.

Then $\therefore$ M, N, C are R. $\angle$, O is a R. $\angle$.

$\therefore CN = OM$, and $CM = ON$.

Hence $AB = AR + BR = AN + BM$.

$= AC - CN + BC - CM$.

$= AC + BC - (OM + ON) = AC + BC - 2$ radius.

$\therefore AB = AC + BC -$ diameter of the inscribed $\odot$.

30. Fig. 34. Let two circles, each pass through A, B, and intersect the given $\odot$, (O) in a, b; a$'$, b$'$ respectively. Join a b, a$'$b$'$, A B, and let B A, b a produced meet in P. Join P a$'$ and produce it to meet $\odot$ (O) in x, and $\odot$ A a$'$ b$'$ in y.

$\therefore Pa'. Px = Pa. Pb = PA. PB = Pa'. Py$.

$\therefore Px = Py$ $\therefore$ x and y coincide in b$'$.

$\therefore$ a$'$b$'$, a b, A B meet in P, and similarly it may be proved for any number of circles.

31. Fig. 35. Let A B C D be the square. With centres B, C and distances B D, C B, describe arcs intersecting in E, also with centres B and A, and same distances, describe arcs intersecting in E$'$. Again, with centres B, E, and distance B E describe arcs intersecting in F, and with B, E$'$ as centres and distance B E$'$, describe arcs intersecting in F$'$. Join B F, B F$'$ intersecting D C, D A in G, G$'$; respectively. Join B G, B G$'$, G G$'$; and B G G$'$ is the Δ required.

Join B E, C E, B E′, A E′, E F, E F′.

Then since in the $\triangle$ B C D, B C E, B E = B D, C E = D C and B C is common, the $\angle$ B C E = $\angle$ B C D = R. $\angle$.

$\therefore$ $\angle$ C B E = $\frac{1}{2}$ R. $\angle$.

$\therefore$ $\angle$ D B E = $\angle$ D B C + $\angle$ C B E = R. $\angle$, and similarly it may be shewn that D B E′ = R. $\angle$.

Now $\angle$ F B E, F′ B E′ being $\angle$ of an equilateral $\triangle$ are each = $\frac{2}{3}$ R. $\angle$, $\therefore$ each of $\angle$ D B G, D B G′ = $\frac{1}{3}$ R. $\angle$.

$\therefore$ $\angle$ G B G′ = $\frac{2}{3}$ R. $\angle$ = $\angle$ of an equilateral $\triangle$.

Again the $\angle$ B G C = $\angle$ G B D + $\angle$ G D B = $\angle$ D B G′ + $\angle$ B D G′ = $\angle$ B G′ A.

And A B = B C and $\angle$ C = $\angle$ A.

$\therefore$ B G = B G′ $\therefore$ $\angle$ B G G′ = $\angle$ B G′ G = $\frac{1}{2}$ supplement of $\angle$ G B G′ $= \dfrac{2\,R.\angle - \frac{2}{3}\,R.\angle}{2} = R.\angle - \dfrac{R.\angle}{3} = \frac{2}{3}\,R.\angle =$ $\angle$ G B G′. $\therefore$ B G G′ is the equilateral $\triangle$ required.

32. Fig. 26. Let the radius O D bisect the chord B C in E, B D = D E.

For joining D C, since O D bisects B C in E, the $\angle$ B E D, C E D are R. $\angle$. And D E is common to the $\triangle$ B E D, C E D, $\therefore$ B D = C D; $\therefore$ arc B D = arc C D.

33. Fig. 36. a. Case 1. Let $\angle$ A at the vertex be obtuse and $\therefore$ $>$ $\angle$ C or $\angle$ B.

Let also B D be the line which is = B A. It must evidently be without the $\triangle$, or two $\angle$ of a $\triangle$ would be $>$ 2 R. $\angle$. Produce C B.

Then $\angle$ E B D = $\angle$ D + $\angle$ C = $\angle$ D A B + $\angle$ C = $\angle$ C + $\angle$ A B C + $\angle$ C = 3 $\angle$ C.

Fig. b. Case 2. When $\angle$ A is acute but greater than either of the equal $\angle$ B C. B D falls within the $\triangle$, and $\angle$ E B D = $\angle$ B D C + $\angle$ C = supplement of $\angle$ A + $\angle$ C ($\because$ B D = B A) = 2 $\angle$ C + $\angle$ C = 3 $\angle$ C.

Fig. (c). Case 3. When $\angle$ A is less than $\angle$ C or $\angle$ B, B D

falls on the other side of the base, and $\angle$ E B D $=$ $\angle$ B C D $+$ $\angle$ D $=$ supp. of $\angle$ C $+$ $\angle$ A.

$= 2$ R. $\angle$ $-$ $\angle$ C $+ 2$ R. $\angle$ $- 2$ $\angle$ C.

$= 4$ R. $\angle$ $- 3$ $\angle$ C.

34. Fig. 37. Let a b, touching the $\odot$ A O in C intersect the tangents A a, B b at the extremity of the diameter A B, in a, b respectively. . Join a o, b o. The $\angle$ a o b is a R. $\angle$.

Join O C. Then $\because$ a C $=$ a A, and a O is common to the right angled $\triangle$ A O a, C O a, the $\angle$ C O a $=$ $\angle$ A O a. Similarly $\angle$ C O b $=$ $\angle$ b O B.

$\therefore$ the $\angle$ a o b $=$ $\angle$ a O C $+$ $\angle$ b O C $= \frac{1}{2}$ ($\angle$ A O C $+$ $\angle$ C O B).

$= \frac{1}{2}$ $\odot$ 2 R. $\angle$ $=$ R. $\angle$.

35. Fig. 38. Case 1. Let A B be $>$ a b, to which add the equal lines, B M, b m.

Take B M : A B :: b x : a b, then $\because$ A B is $>$ a b, B M is $>$ b x.

Also (by *Euc.* B. V. Prop. 18.) A M : A B : a x : a b.

$\therefore$ A B : a b :: A M : a x.

But A M : a x in a less ratio than A M : a m (*Euc.*)

$\therefore$ A B : a b in a less ratio than A M : a m.

Case 2. Let A B be $\angle$ a b. The proof is nearly the same as the preceding one.

36. Fig. 39. Let A B C be the given $\triangle$, and let A E, A F bisect $\angle$ B A C, C A D; B F, B C, B E, are in harmonic progression, or B F : B E :: B F $-$ B C or C F : B C $-$ B E or E C.

In the line B A (produced if necessary) take A G $=$ A H $=$ A C, and join E G, F H.

Then $\because$ in $\triangle$ A E G, A E C, A G $=$ A C, A E is common, and the $\angle$ at A are equal, $\therefore$ G E $=$ E C and $\angle$ A G E $=$ $\angle$ A C E.

Similarly it may be proved that H F $=$ C F, and $\angle$ A H F $=$ $\angle$ A C F.

Again the $\angle$ AGE = $\angle$ ACE = supp. of $\angle$ ACF = supp. of $\angle$ AHF = $\angle$ DHF.

$\therefore$ FH is $\parallel$ GE.

$\therefore$ BF : BE :: HF (= CF) : GE (= CE.)

:: BF − BC : BC − BE.

$\therefore$ BF, BC, BE, are in harmonic progression.

37. The sides when produced will evidently form (n + 4). $\triangle$ whose $\angle$ together are = 2 (sum of exterior $\angle$ of the polygon) + sum of $\angle$ at the points of concourse.

$\therefore$ sum required = sum of $\angle$ of (n + 4) $\triangle$ − 2. 4 R $\angle$.
= 2 (n + 4) R. $\angle$ − 8 R $\angle$.
= 2 n R. $\angle$ + 8 R. $\angle$ − 8 R. $\angle$.
= 2n R. $\angle$.

38. Fig. 40. Let AB, DE be any two opposite sides of the hexagon ABCDEF. AB is $\parallel$ DE.

Join AD, AC, DF.

Then in the $\triangle$ ABC, DFE, $\angle$ at B and E are equal and AB, BC are each = FE, ED, $\therefore$ AC = DF.

Again in $\triangle$ ACD, ADF, AC = DF, CD = AF, and AD is common. $\therefore$ $\angle$ CDA = $\angle$ FAD.

But whole $\angle$ CDE = whole $\angle$ BAF.

$\therefore$ $\angle$ ADE = $\angle$ DAB which are alternate.

$\therefore$ AB is $\parallel$ DE.

39. Fig. 41. Let ABC be the $\triangle$, and AD bisect the base BC in D. $AB^2 + AC^2 = 2\,AD^2 + 2\,BD^2$.

Draw AN $\perp$ BC.

Then $BA^2 = AD^2 + BD^2 \pm 2\,BD.\,DN$ according as AN falls within or without the $\triangle$.

And $CA^2 = AD^2 + CD^2$ (or BD^2) $\mp 2\,CD\,DN$.

$\therefore BA^2 + CA^2$
$= 2\,AD^2 + 2\,BD^2 \pm 2\,BD.\,DN \mp 2\,BD.\,DN.$
$= 2\,AD^2 + 2\,BD^2$.

40. Fig. 42. Let AB be a side of the inscribed hexagon, ab a side of the circumscribed hexagon.

Join O A, O B, O a, O b, (O being the centre of the ⊙). Then ∵ ∠ AOB = $\frac{1}{6}$ of 4 R. ∠, or $\frac{2}{3}$ of R. ∠, and ∠ A = ∠ B, the △ AOB is equilateral.

In nearly the same way it may be shewn that a o b is an equilateral △.

∴ △ a o b is similar to △ AOB

and inscribed hexagon : circumscribed hexagon : : 6 △ AOB : 6 △ a o b

$$
\begin{aligned}
&: : \triangle AOB : \triangle aOb \\
&: : OA^2 : Ob^2 \\
&: : Ob^2 - Ab^2 : Ob^2 \\
&: : Ob^2 - (\tfrac{1}{2} Ob)^2 : Ob^2 \\
&: : Ob^2 - \frac{Ob^2}{4} : Ob^2 \\
&: : 3\, Ob^2 : 4\, Ob^2 \\
&: : 3 : 4
\end{aligned}
$$

41. Fig. 43. Let A B C be the equilateral △. Bisect the arcs A B, A C, in D, G respectively, and join D G intersecting A B in E and A C in F. DE = EF = FG.

Join AD. Then ∵ A B = AC, the arc AB = arc A C.

∴ arc AD = $\frac{1}{2}$ arc AB = $\frac{1}{2}$ arc AC = arc AG

∴ ∠ DAB = ∠ ADG

∴ AE = ED. Similarly AF = FG

Again, ∠ AEG = ∠ ADE + ∠ DAD = 2 ∠ ADG = ∠ B and ∠ EAF = ∠ BAC

∴ ∠ AFE = ∠ C

∴ AE = EF = AF

∴ DE = AE = EF, and EF = AF = FG

∴ DE = EF = FG.

42. Fig. 1. Let ABC be the △, and A its R. ∠. Let A be described on DC, A′ on AC, A″ on DA, DC, AC, AD being their homologous sides, (this is a necessary limitation,) then A = A′ + A″.

Similar figures are to one another in the duplicate ratio of their homologous sides,

$\therefore A' : A'' :: AC^2 : AD^2$

$\therefore A' + A'' : A'' :: AC^2 + AD^2 : AD^2 :: DC^2 : AD^2$
$:: A : A''$

$\therefore A' + A'' : A :: A'' : A'' :: 1 : 1$

$\therefore A = A' + A''$

43. Fig. 44. Let the circles O, P intersect in D, E, and A a passing through D, be $\parallel$ B b passing through E. A a = B b.

Draw through O, P, N M, n m $\perp$ A a

Then $\because$ A a is $\parallel$ B b, and N M, n m both $\perp$ A a, M n is a rectangle.

$\therefore$ N M = n m, and N n = M m.

Again, $\because$ N M, m n are $\perp$ A a, and B b, and drawn through the centres, A D = 2 N D, a D = 2 n D $\therefore$ A a = 2 N D + 2 n D = 2 N n

B E = 2 M E, b E = 2 m E $\therefore$ B b = 2 M E + 2 b m E = 2 M m

But N n = N m.

$\therefore$ A a = 2 N n = 2 M m = B b.

44. Fig. 45. Let the chords A a, B b intersect in P.

$AP^2 + Pa^2 + BP^2 + Pb^2$ = square on the diameter.

From O the centre of the $\odot$ draw O N $\perp$ A a, O M $\perp$ B b, and join O A, O B.

Then A a, B b are bisected in N, M, respectively.

$\therefore AP^2 + Pa^2 = 2\,AN^2 + 2\,PN^2$

and $BP^2 + Pb^2 = 2\,BM^2 + 2\,PM^2$.

$\therefore AP^2 + Pa^2 + BP^2 + Pb^2 = 2AN^2 + 2PM^2 + 2BM^2 + 2PN^2$

$= 2AO^2 + 2BO^2$ (for O N = P M, O M = P N)

$= 4AO^2$ = square on the diameter.

45. Fig. 46. Let the circles touch in the point A, and A B, A b be their diameters.

The lune + less $\odot$ = greater $\odot$.

$\therefore$ 3 less $\odot$ = greater $\odot$.

$\therefore$ greater $\odot$: less $\odot$ :: 3 : 1 :: $Ab^2 : AB^2$

$\therefore Ab : AB :: \sqrt{3} : 1.$

46. Fig. 47. Let $\triangle$ A B C : $\triangle$ a b c :: B C : b c, their altitudes are equal.

Describe the rectangles D C, d c as in the figure.

Then △ A B C : △ a b c : : D C : d c : : D B . B C : d b . b c

But △ A B C : △ a b c : : B C : b c

∴ D B . B C : d b . b c : : B C : b c

or D B : d b : : 1 : 1

∴ D B the altitude of △ A B C = d b the altitude of △ a b c.

47. Fig. 48. Let A B C D be the ▱, P the point without it. Join P A, P B, P D, P C and A C. The △ P A C = △ P A B + △ P A D.

From B, C, D, draw B b, C c, D d respectively ⊥ P A (produced if necessary), and D N ⊥ C c.

Then D N is ∥ b A, and D C is ∥ A B.

Hence ∠ b A B = ∠ N D C, and C D = B A, and ∠ b = ∠ N.

∴ B b = C N. Also N C = D d.

∴ C c = B b + D d.

$$\text{Now } \triangle\ PAC = \frac{Cc.\ PA}{2}$$

$$\triangle\ PBA = \frac{Bb.\ PA}{2}$$

$$\triangle\ PDA = \frac{Dd.\ PA}{2}$$

$$\therefore \triangle\ PBD + \triangle\ PDA = (Bb + Dd).\ \frac{PA}{2} = \frac{Cc.\ PA}{2}$$

$$= \triangle\ PCA.$$

48. Fig. 49. Let the three straight lines A a, B b, C c, not in the same plane be equal and parallel. Join A B, A C, B C, a b, a c, b c. Then the △s A B C, a b c are ∥.

∵ A a is = and ∥ C c, A C is = and ∥ a c.

Similarly A B, B C may be shewn to be = and ∥ a b, b c respectively.

∴ In △s A B C, a b c the sides are = each to each.

∴ △ A B C = △ a b c.

Again ∵ A C, B A; a c, b a meet, and are ∥ not being in the same plane, the plane B A C is ∥ plane b a c (*Euc.* B. XI. Prop. 15.)

$\therefore$ $\triangle$ ABC is both $=$ and $\parallel$ $\triangle$ abc.

49. Fig. 50. If possible let the straight line ABC cut the $\odot$ OB in three points A, B, C. Join OB, OC, and draw ON $\perp$ AB in N.

Then $OC^2 = \overline{NBC}^2 + ON^2$.

But $OC^2 = OB^2 = NB^2 + ON^2$.

$\therefore NB^2 = \overline{NBC}^2$.

$\therefore$ NB $=$ NBC which is absurd, and the same may be proved for any number of points. $\therefore$ AB cannot cut the $\odot$ in more than two points.

50. Fig. 51. Let ABC be the $\triangle$.

Upon AB, BC describe segments of circles containing an $\angle$ $= \frac{1}{3}$ of 4 R. $\angle$s, and let them intersect in P. P is the point required.

Join AP, CP, BP, then $\because$ $\angle$ APC $+$ $\angle$ BPC $+$ $\angle$ APB $=$ 4. R. $\angle$s, and $\angle$ APB $+$ $\angle$ BPC $= \frac{2}{3}$ of 4. R. $\angle$s.

$\therefore$ $\angle$ APC $= \frac{1}{3}$ of 4 R. $\angle$s $=$ $\angle$ BPC, or $\angle$ APB.

ARITHMETIC.

Problem 51. It is here supposed that the floor is rectangular.

Now 10 yds. 2 ft. contain 32 ft.

5 yds. 1 ft. contain 16 ft.

$\therefore$ the number of square feet in the floor $= 32 \times 16 = 512$.

$\therefore$ the price required $= \frac{512}{10}$ £ $=$ £51. 4*s*. ($\because$ 2*s*. $= \frac{1}{10}$ of a £).

52. 17*s*. $= \frac{17}{20}$ of a pound.

and $9\frac{1}{2}d. = \dfrac{19d.}{2} = \dfrac{19}{2 \times 12}s. = \dfrac{19}{2 \times 12 \times 20}$ of a pound.

$\therefore$ 17*s*. $9\frac{1}{2}d. = \frac{17}{20} + \dfrac{19}{24 \times 20} = \dfrac{24 \times 17 + 19}{24 \times 20} = \frac{427}{480}$ £. $=$ .88958333 .... by common division.

53. $17\frac{1}{2}$ ells $= 17 \times 5 + \frac{1}{2} \times 5 = 85 + \frac{5}{2} = \frac{175}{2}$ qrs.

18 yds. $= 18 \times 4 = 72$ qrs.

$\therefore \frac{175}{2}$: £6. 17*s*. $10\frac{3}{4}d.$ :: 72 : cost required $= 72 \times$ (£6. 17. $10\frac{3}{4}d.$) $\cdot \frac{2}{175} = \frac{144}{175} \times$ (£6. 17*s*. $10\frac{3}{4}d.$) $= \dfrac{12 \times 12}{5 \times 5 \times 7} \times$ (£6. 17*s*. $10d\frac{3}{4}$.) $=$ £5. 18*s*. 5*d*. $1q. \frac{33}{35}$ (first multiplying 12 and 12, and then dividing by 5, 5, 7, successively.)

54. $\frac{3}{4} : \frac{1}{4} :: 12 + \frac{2}{3}$ or $\frac{38}{3} : \frac{38}{3} \times \frac{1}{4} \times \frac{4}{3} = \frac{38}{9}$£. $=$ £4. 4*s*. 5*d*. $1q. \frac{1}{3}$ $=$ the cost required.

55. 18s. 1½d. = 435 halfpence.
3s. 7½d. = 87 halfpence.

∴ 87 : 935½ yards, :: 435 : 4677 $\frac{35}{58}$ yards, = the number required.

56. In the 3 per cents. 60 is par, and in the 5 per cents. 100 is par; hence, purchasing in these stocks must be in the ratio of 60 : 100 or of 3 : 5.

∴ 3 : 5 :: £59½ : £99. 3s. 4d. the rate required.

57. Divide by 3, and the result is .285 = decimal required.

58. The number of horses will be evidently and directly as the number of acres, and inversely as the time.

∴ $\frac{16}{20}$: 7 :: $\frac{24}{7}$: $\frac{24}{7} \times 7 \times \frac{20}{16} = \frac{24 \times 20}{16} = 30$ = the number of horses required.

59. $\frac{3}{4}$ of 5s. 6d. = $\frac{16s.\ 6d.}{4}$ = 4s. 1½d.

$\frac{2}{3}$ of $\frac{1}{5}$ of 17s. 2¼d. = $\frac{2}{15}$ × 17s. 2¼d. = $\frac{34s.\ 4\frac{1}{2}d.}{15}$ = 2s. 3½d.

$\frac{5}{7}$ of £5. 6s. 7¼d. = $\frac{£26.\ 13s.\ 0\frac{1}{4}d.}{7}$ = £3. 16s. 1¾d.

∴ the required sum = £4. 2s. 6¾d.

60. $\frac{2}{3}$ of a guinea = $\frac{2}{3}$ of 21s. = 14s. = $\frac{14}{20}$ = $\frac{7}{10}$£ = .7£

61. 9s. 4d. = 112d., £100. = 20 × 12 × 100d. = 24000d.

∴ 112 : $\frac{3}{4}$ yds. :: 24000 : $\frac{3}{4} \times \frac{24000}{112}$
$= \frac{3 \times 375}{7} = \frac{1125}{7} = 160\frac{5}{7}$ yds. = the number required.

62. $\frac{1}{7}$ of $\frac{2}{11}$ of a guinea $= \frac{2}{77}$ of 42 sixpences, $= \frac{2}{77} \times 42$ $= \frac{2 \times 6}{11}$ sixpences, $= \frac{12}{11} \times \frac{1}{5}$ of half-a-crown $= \frac{12}{55}$ of half-a-crown.

Again, $\frac{12}{55}$ of half-a-crown $= \frac{12 \times 30}{55}d. = \frac{12 \times 6}{11}d.$

$= \frac{72}{11}d. = 6d.\ \frac{6}{11}.$

Also, $\frac{6}{11}d. = \frac{24}{11}q. = 2q.\ \frac{2}{11}.$

$\therefore$ the exact value required is $6d.\ 2q.\ \frac{2}{11}.$

63. $2\frac{3}{4}$ yds. $= \frac{11}{4}$ yds. and $7\frac{2}{5}$ yds. $= \frac{37}{5}$ yds.
and $3s.\ 8\frac{1}{4}d. = 44\frac{1}{4}d. = \frac{177}{4}d.$

$$\therefore \frac{11}{4} : \frac{177}{4}d. :: \frac{37}{5} : \frac{37}{5} \times \frac{177}{4} \times \frac{4}{11} = \frac{37 \times 177}{55}$$

$= 119\frac{4}{55}d. = 9s.\ 11d.\ \frac{4}{55} =$ the price required.

64. $12\frac{2}{3} = \frac{38}{3}.$

$\therefore \frac{3}{4} : \frac{1}{4} :: \frac{38}{3} : \frac{38}{3} \times \frac{1}{4} \times \frac{4}{3} = \frac{38}{9}$£. $=$ £4. 4s. 5d. 1q. $\frac{1}{3}$.
$=$ the required cost.

65. $2\frac{1}{3} = \frac{7}{3}$. Now $\frac{7}{3}$ of a Flemish ell $= \frac{7}{3}$ of $\frac{3}{8}$ of an English ell $= \frac{7}{8}$ of an English ell. Also $1\frac{1}{8} = \frac{9}{8}$.

$\therefore \frac{9}{8} : \frac{5}{6} :: \frac{7}{8} : \frac{7}{8} \times \frac{5}{6} \times \frac{8}{9} = \frac{35}{54}$ of a guinea, $= \frac{35}{54} \times 21s.$

$= \frac{35 \times 7}{18}s. = \frac{245}{18}s. = 13s.\ 7d.\ 1q.\ \frac{1}{3}$

66. $6s.\ 8d. = \frac{1}{3}$ of a pound.

$\therefore \frac{2710}{3} =$ £903. 6s. 8d. $=$ the value required.

67. $2\frac{1}{4} = \frac{9}{4} =$ also 2.25. Also $\frac{3}{5} = .6$.
$5\frac{7}{8} = \frac{47}{8} =$ also 5.875.

Hence $\frac{3}{5} : \frac{9}{4} :: \frac{47}{8} : \frac{47}{8} \times \frac{9}{4} \times \frac{5}{3} = \frac{47 \times 15}{32}$

$=$ £22. 0s. $7\frac{1}{2}d.$

Secondly, .6 : 2.25 :: 5.875 : £22.08125 = £22. 0s. $7\frac{1}{2}d$.

$$
\begin{array}{r}
5.875 \\
2.25 \\
\hline
29375 \\
11750 \\
11750 \\
\hline
6\,)\,13.21875 \\
\hline
22.03125 \\
20 \\
\hline
.62500 \\
12 \\
\hline
7.500 \\
4 \\
\hline
2.0 \\
\hline
\end{array}
$$

68. $2\frac{1}{2} = \frac{5}{2}$, $2\frac{9}{10} = \frac{29}{10}$, and 1 cwt. $= \frac{1}{20}$ of a ton.

Now the price of carriage will evidently be as the distance $\times$ the quantity of goods.

$\therefore \frac{29}{10} \times \frac{5}{2} : \frac{3}{40} :: 1 \times \frac{1}{20} : \frac{3}{40 \times 20} \times \frac{20}{29 \times 5} = \frac{3}{200 \times 29}$ of a guinea.

$= \frac{3 \times 21}{200 \times 29}$ of a shilling $= \frac{3 \times 21 \times 12 \times 4}{200 \times 29}$ of a farthing.

$= \frac{378}{725}$ of a farthing.

69. $\sqrt{\frac{9.024016}{25.3009}} = \frac{\sqrt{9.024016}}{\sqrt{25.3009}} = \frac{3.004}{5.03}$ (by extracting them separately) $= .5974155\ldots\ldots$ (by actual division.)

Otherwise.

Take the root of quotient of $\frac{9.024016}{25.3009}$. This method is more operose, except when the numbers are not perfect squares.

70. $1\frac{2}{3}$ of a halfpenny $= \frac{5}{3} \times \frac{1}{2} = \frac{5}{6}$ of a penny.

$\therefore 11 + \frac{5}{6} = \frac{71}{6}$ of a penny $= \dfrac{71}{6 \times 12 \times 20}$ of a pound,

$= \frac{71}{1440} = .04930555 \ldots\ldots$ by division.

71. $3\frac{3}{4}d. = 3.75d.$ Divide by 12.

$\therefore 3.75d. = .3125s.$

$\therefore 3.3125s. = .165625£.$ (dividing by 20.)

72. Make the number of decimal places in the dividend = that in the divisor by adding ciphers to the right, and divide. The quotient will be 24.339 ......

73. 1s. $5\frac{1}{2}d.$ contains 35 halfpence, and there are 60 halfpence in half-a-crown.

$\therefore \frac{3}{7}$ of 35 halfpence $= \dfrac{3 \times 35}{7 \times 60}$ of half-a-crown.

$= \frac{15}{60} = \frac{1}{4}$ of half-a-crown.

74. 100 : $74\frac{1}{2}$ :: £156. 15s. 1d. : purchase money required.

Now $74\frac{1}{2} = \frac{149}{2}$, £100. = 24000d.

and £156. 15s. 1d. = 37621d.

$\therefore 24000 : \dfrac{149}{2} :: 37621 : \dfrac{149 \times 37621}{48000} = \dfrac{5605521}{48000}$ £.

= £116. 15s. 7d. 2q. $\frac{21}{50}$.

75. The number of square feet in the slab = the length × the breadth, expressed in feet.

5 ft. 7 in. $= 5 + \frac{7}{12}$ ft. $= \frac{67}{12}$ ft.

3 ft. 5 in. $= 3 + \frac{5}{12}$ ft. $= \frac{41}{12}$ ft.

$\therefore$ the number of square feet $= \frac{67}{12} \times \frac{41}{12}$, which at 6s. each must =

$\dfrac{67 \times 41 \times 6}{144}s. = \dfrac{67 \times 41}{24}s. =$ £5. 14s. $5\frac{1}{2}d.$

The number of square feet might have been found by duodecimals.

76. $62\frac{1}{2} = \frac{125}{2}$, and £1034. 15s. = £1034.$\frac{3}{4} = \frac{4139}{4}$£.

$\therefore\ 100 : \frac{125}{2} :: \frac{4139}{4} : \dfrac{125 \times 4139}{800} = \dfrac{5 \times 4139}{32} = \dfrac{20695}{32}$£.

= £646. 14s. $4\frac{1}{2}d.$ = the purchase required.

77. 24 ft. 6 in. = $24\frac{1}{2}$ ft. = $\frac{49}{2}$ ft. = $\frac{49}{6}$ yds.

16 ft. 3 in. = $16\frac{1}{4}$ ft. = $\frac{65}{4}$ ft. = $\frac{65}{12}$ yds.

$\therefore$ the number of square yards in the ceiling = $\frac{49}{6} \times \frac{65}{12}$.

and the cost $\therefore = \dfrac{49 \times 65}{6 \times 12} \times 3s. = \dfrac{3185}{24}s.$ = £6. 12s. 8d.

Otherwise by duodecimals.

78.

£.	s.	d.		
4	5	6		= value of 1 cwt.
2	2	9		= ditto of 2 qrs.
0	1	6	$1q.\frac{2}{7}$	= ditto of 2 ℔ = $\frac{1}{28}$ of 2 qrs.
0	0	9	$0\frac{9}{14}$	= ditto of 1 ℔.
∴ 6	10	6	$1\frac{13}{14}q.$	= the value required.

ALGEBRA.

EQUATIONS OF THE FIRST AND SECOND DEGREES.

79. $$\sqrt{x} - 4 = \frac{259 - 10x}{4 + \sqrt{x}}$$

First, $(\sqrt{x} - 4)(\sqrt{x} + 4) = 259 - 10x$

or $x - 16 = 259 - 10x$

$\therefore 11x = 259 + 16 = 275$

$\therefore x = \dfrac{275}{11} = 25.$

80. $$\frac{10}{x} - \frac{14 - 2x}{x^2} = 2\frac{4}{9} = \frac{22}{9}$$

Multiply both sides by $9x^2$, and we have

$90x - 126 + 18x = 22x^2$

$\therefore 22x^2 - 108x = -126$

$\therefore x^2 - \dfrac{54}{11}x = \dfrac{-63}{11}$

$$\therefore x^2 - \frac{54}{11}x + \left(\frac{27}{11}\right)^2 = \frac{27^2}{11^2} - \frac{63}{11} = \frac{27^2 - 11.63}{11^2}$$

$$= \frac{729 - 693}{11^2} = \frac{36}{11^2}$$

$\therefore x - \dfrac{27}{11} = \pm \dfrac{6}{11}$

$\therefore x = \dfrac{27 \pm 6}{11} = \dfrac{33}{11}$ or $\dfrac{21}{11} = 3$ or $\dfrac{21}{11}$

81. $$\frac{42x}{x-2} = \frac{35x}{x-3}$$

First, $\dfrac{42}{x-2} = \dfrac{35}{x-3}$ $\therefore 42x - 126 = 35x - 70$

$$\therefore\ 7x = 126 - 70 = 56$$

$$\therefore\ x = 8$$

82. $\frac{x+1}{2} + \frac{x-2}{3} = 16 - \frac{x+3}{4}$. Multiply by 12,

and $6x + 6 + 4x + 8 = 192 - 3x - 9$

$$\therefore\ 13x = 192 - 23 = 169$$

$$\therefore\ x = \frac{169}{13} = 13.$$

83. $3x^2 - 14x = -15$

$$\therefore\ x^2 - \frac{14}{3}x = -5$$

$$\therefore\ x^2 - \frac{14}{3}x + \frac{49}{9} = \frac{49}{9} - 5 = \frac{49-45}{9} = \frac{4}{9}$$

$$\therefore\ x - \frac{7}{3} = \pm\frac{2}{3}$$

$$\therefore\ x = \frac{7\pm2}{3} = 3 \text{ or } \frac{5}{3}$$

84. $3x^2 - 19x = -16$

$$\therefore\ x^2 - \frac{19}{3}x = -\frac{16}{3}$$

$$\therefore\ x^2 - \frac{19}{3}x + \frac{19^2}{36} = \frac{19^2}{36} - \frac{16}{3} = \frac{19^2 - 12.16}{36}$$

$$= \frac{169}{36}$$

$$\therefore\ x - \frac{19}{3} = \pm\frac{13}{6}$$

$$\therefore\ x = \frac{19\pm13}{6} = \frac{32}{6} \text{ or } \frac{6}{6} = \frac{16}{3} \text{ or } 1.$$

85. $\frac{16}{x} - \frac{100 - 9x}{4x^2} = 3$. Multiply by $4x^2$.

$$\therefore\ 64x - 100 + 9x = 12x^2$$

$$\therefore\ 12x^2 - 73x = -100$$

$$x^2 - \frac{73}{12}x + \frac{73^2}{24^2} = \frac{73^2}{24^2} - \frac{100}{12} = \frac{5329-4800}{24^2}$$

$$= \frac{529}{24^2}$$

$$\therefore\ x - \frac{73}{24} = \pm\frac{23}{24}$$

$$\therefore\ x = \frac{73\pm23}{24} = 4 \text{ or } \frac{25}{12}$$

86. $\left.\begin{aligned} x^2 y + y^2 x &= 30 \\ \frac{1}{x} + \frac{1}{y} &= \frac{5}{6} \end{aligned}\right\}$ The equations being homogeneous, put $x = v y$.

Then $v^2 y^3 + y^3 v = 30$, and $\therefore\ y^3 = \dfrac{30}{v^2 + v}$

and $\dfrac{1}{v y} + \dfrac{1}{y} = \dfrac{5}{6}$, and $y^3 = \left(\dfrac{1}{v} + 1.\ \dfrac{6}{5}\right)^3 = \overline{\dfrac{v+1}{v^3}}^{\,3} . \dfrac{6^3}{5^3}$

$$\therefore\ \overline{\frac{v+1}{v^3}}^{\,3} . \frac{6^3}{5^3} = \frac{30}{v^2 + v}$$

$$\therefore\ \overline{\frac{v+1}{v^2}}^{\,4} . = \frac{30\ .\ 5^3}{6^3} = \frac{5^4}{6^2}$$

$$\therefore\ \overline{v+1}^{\,2} = \frac{5^2}{6}.v$$

$$\text{or } v^2 + 2v + 1 = \frac{25}{6}.v$$

$$\therefore\ v^2 - \frac{13}{6}v = -1$$

$$\text{or } v^2 - \frac{13}{6}v + \frac{13^2}{12^2} = \frac{13^2-12^2}{12^2} = \frac{25}{12^2}$$

$$\therefore\ v - \frac{13}{12} = \pm\frac{5}{12}$$

$$\therefore\ v = \frac{13\pm5}{12} = \frac{3}{2} \text{ or } \frac{2}{3}$$

Hence $y = \dfrac{v+1}{v} . \dfrac{6}{5} = \dfrac{\frac{3}{2}+1}{\frac{3}{2}} . \dfrac{6}{5} = \dfrac{5}{2} . \dfrac{6}{5} = 3$

$$\text{or} = \frac{\frac{2}{3}+1}{\frac{2}{3}} . \frac{6}{5} = \frac{5}{3} . \frac{6}{5} = 2$$

$$\therefore\ x = v y = \frac{2}{3} \times 3 = 2, \text{ or } = \frac{3}{2} \cdot 2 = 3.$$

$\therefore$ the corresponding values of x and y are 2, 3; 3, 2 respectively.

Otherwise,

Divide the first equation by $x^2 y^2$, which, with the aid of the second equation, will give $x y$; divide the first by $x y$, which gives $x + y$, from which and $x y$, $x - y$ may easily be found. Add to, and subtract from $x + y$, the quantity $x - y$, and x, and y will be found.

89. $b = \dfrac{a - \sqrt{a^2 - x^2}}{a + \sqrt{a^2 + x^2}}$. Clearing of fractions

$$ab + b\sqrt{a^2 - x^2} = a - \sqrt{a^2 - x^2}$$

$$\therefore (b+1) \cdot \sqrt{a^2 - x^2} = a - ab = a \cdot \overline{1 - b}$$

$$\therefore\ \sqrt{a^2 - x^2} = a \cdot \frac{1-b}{1+b}$$

$$\therefore\ a^2 - x^2 = a^2 \left(\frac{1-b}{1+b}\right)^2$$

$$\therefore\ x^2 = a^2 - a^2 \left(\frac{1-b}{1+b}\right)^2 = a^2 \cdot \frac{\overline{1+b}|^2 - \overline{1-b}|^2}{\overline{1+b}|^2}$$

$$= a^2 \cdot \frac{4b}{(1+b)^2}$$

$$\therefore\ x = \pm \frac{2 a b^{\frac{1}{2}}}{1 + b}$$

90. $3 x^{\frac{5}{3}} + x^{\frac{5}{6}} = 3104$. Since the index of x in one term is double its index in the other, the equation is of the form of a quadratic.

$$x^{\frac{5}{3}} + \tfrac{1}{3} x^{\frac{5}{6}} = \frac{3104}{3}$$

$$\therefore\ x^{\frac{5}{3}} + \tfrac{1}{3} x^{\frac{5}{6}} + \frac{1}{36} = \frac{3104}{3} + \frac{1}{36} = \frac{37249}{36}$$

$$\therefore\ x^{\frac{5}{6}} + \tfrac{1}{6} = \pm \frac{193}{6}$$

$$\therefore\ x^{\frac{5}{6}} = \frac{\pm 193 - 1}{6} = \frac{192}{6} \text{ or } - \frac{194}{6}$$

$$= 32 \text{ or } - \frac{97}{3}$$

$$\therefore\ x = (32^{\frac{1}{5}})^6 \text{ or } \left(\frac{-97}{3}\right)^{\frac{6}{5}} = 2^6 \text{ or } \left(\frac{-97}{3}\right)^{\frac{6}{5}}$$

$$= 64 \text{ or } \left(\frac{-97}{3}\right)^{\frac{6}{5}}$$

91. $x^3 - (a + p)\,x^2 + (ap + q)\,x - aq = o$

First, $x^3 - p\,x^2 + q\,x - (a\,x^2 - ap\,x + a\,q) = o$

or $x\,(x^2 - p\,x + q) = a\,(x^2 - p\,x + q)$

$\therefore\ x = a$

Also $x^2 - p\,x + q = o$

$$\therefore\ x^2 - p\,x + \frac{p^2}{4} = \frac{p^2}{4} - q$$

$$\therefore\ x - \frac{p}{2} = \pm \frac{\sqrt{p^2 - 4q}}{2}$$

$$\therefore\ x = \frac{p \pm \sqrt{p^2 - 4q}}{2}$$

$\therefore$ the three values of x are a, $\dfrac{p + \sqrt{p^2 - 4q}}{2}$ and $\dfrac{p - \sqrt{p^2 - 4q}}{2}$

92. $\left.\begin{array}{l} x^2 + x\,y = 10 \\ y^2 + y\,z = 21 \\ z^2 + x\,z = 24 \end{array}\right\}$ Let $x = y - v$, $z = y + v$

Then $y^2 + y^2 + y\,v = 21$ or $2\,y^2 + y\,v = 21$

and $y^2 + 2\,y\,v + v^2 + \overline{y - v}\,.\,\overline{y + v}$ or $2\,y^2 + 2\,y\,v = 24$

$$\therefore\ y\,v = 3$$

$$\therefore\ y^2 = \frac{21 - 3}{2} = \frac{18}{2} = 9$$

$$\therefore\ y = \pm 3$$

$$\therefore\ v = \frac{3}{\pm 3} = \pm 1$$

$\left.\begin{array}{l} \therefore\ x = \pm 3 \mp 1 = \pm 2 \\ \quad y = \pm 3 \\ \quad z = \pm 3 \pm 1 = \pm 4 \end{array}\right\}$ Now 2, 3, 4, or −2, −3, −4 being substituted for x, y, z, respectively will satisfy the equations. $\therefore$ they are the true values required.

The assumptions in this problem would not necessarily lead to the solution of all similar problems. A legitimate assumption would have been $x = yv$, $z = yw$, but the resulting equation will be of four dimensions.

93. $$\frac{2x-3}{3} - \frac{10x-4}{18} = -1$$

First, $12x - 18 - 10x + 4 = -18$

$\therefore\ 2x = -4$

$\therefore\ x = -2$

94. $$x^{\frac{6}{5}} + x^{\frac{3}{5}} = 756$$

$\because\ \dfrac{6}{5} = 2.\ \dfrac{3}{5}$ the equation is of the form of a quadratic.

$$\therefore\ x^{\frac{6}{5}} + x^{\frac{3}{5}} + \frac{1}{4} = 756 + \frac{1}{4} = \frac{3025}{4}$$

$$\therefore\ x^{\frac{3}{5}} + \frac{1}{2} = \pm\sqrt{\frac{3025}{4}} = \pm\frac{55}{2}$$

$\therefore\ x^{\frac{3}{5}} = 27$ or -28

$\therefore\ x = (27)^{\frac{5}{3}}$ or $(-28)^{\frac{5}{3}}$

$= 3^5$ or $(-28)^{\frac{5}{3}}$

$= 189$ or $(-28)^{\frac{5}{3}}$.

95. $(x+2)^2 + (x+2) = 20$

First, $(x+2)^2 + (x+2) + \frac{1}{4} = 20 + \frac{1}{4} = \dfrac{81}{4}$

$\therefore\ x + 2 + \frac{1}{2} = \pm\frac{9}{2}$

$\therefore\ x = -\dfrac{5 \pm 9}{2} = 2$ or -7

96. $$\left.\begin{aligned} x + y + \sqrt{xy} &= 19 \\ x^2 + y^2 + xy &= 133 \end{aligned}\right\}$$

First, $x + y = 19 - \sqrt{xy}$

$\therefore\ x^2 + 2xy + y^2 = 19^2 - 38\sqrt{xy} + xy$

$\therefore\ x^2 + xy + y^2 = 361 - 38\sqrt{xy} = 133$

Hence $38\sqrt{xy} = 361 - 133 = 228$

$$\therefore \sqrt{xy} = \frac{228}{38} = \frac{114}{19} = 6$$

$$\therefore xy = 36^2$$

Hence $x^2 - 2xy + y^2 = 133 - 3xy = 133 - 108 = 25$

$$\therefore x - y = \pm 5$$

and $x + y = 19 - \sqrt{xy} = 19 - 6 = 13$

$$\therefore 2x = 18 \text{ or } 8$$

$$2y = 8 \text{ or } 18$$

$$\therefore x = 9 \text{ or } 4 \text{ and } y = 4 \text{ or } 9$$

N. B. When $x = 9$, y must $= 4$, and *vice versâ*. Their interchanging values, is owing to the symmetry of the equations in which they are similarly involved.

97. $3x^{\frac{4}{3}} - \frac{5}{2} x^{\frac{8}{3}} = -592$

First, $\frac{5}{2} (x^{\frac{4}{3}})^2 - 3x^{\frac{4}{3}} = 592$

$$\therefore (x^{\frac{4}{3}})^2 - \frac{9}{5} x^{\frac{4}{3}} = \frac{1184}{5}$$

$$\therefore (x^{\frac{4}{3}})^2 - \frac{6}{5} x^{\frac{4}{3}} + \frac{9}{5^2} = \frac{9}{5^2} + \frac{1184}{5} = \frac{9 + 5920}{5^2} = \frac{5929}{5^2}$$

$$\therefore x^{\frac{4}{3}} - \frac{3}{5} = \pm \frac{77}{5}$$

$$\therefore x^{\frac{4}{3}} = \frac{3 \pm 77}{5} = 16 \text{ or } - \frac{74}{5}$$

$$\therefore x = (16)^{\frac{3}{4}} \text{ or } \left(-\frac{74}{5}\right)^{\frac{3}{4}}$$

$$= 8 \text{ or } \left(-\frac{74}{5}\right)^{\frac{3}{4}}$$

98. $(x - 4)^2 + 2(x - 4) = \frac{2}{x} - 1$

First, $(x - 4).\ (x - 4 + 2) = -\frac{x-2}{x}$

or $(x - 4).\ (x - 2) = -\frac{x-2}{x}$

$$\therefore\ x - 4 = -\frac{1}{x}$$

$$\therefore\ x^2 - 4x = -1$$

$$\therefore\ x^2 - 4x + 4 = 4 - 1 = 3$$

$$\therefore\ x - 2 = \pm\sqrt{3}$$

$$\therefore\ x = 2 \pm \sqrt{3}$$

Equations solved by this artifice are generally, of the form

$$(x-a)^2 + \frac{a}{2}(x-a) = \frac{a}{2x} - 1$$

99. $$2x\sqrt[3]{x} - 3x\sqrt[3]{\frac{1}{x}} = 20$$

First, $x\sqrt[3]{x} = x \,.\, x^{\frac{1}{3}} = x^{\frac{4}{3}}$

$$x\sqrt[3]{\frac{1}{x}} = \frac{x}{x^{\frac{1}{3}}} = x^{\frac{2}{3}}$$

$$\therefore\ 2x^{\frac{4}{3}} - 3x^{\frac{2}{3}} = 20$$

$$\text{or } (x^{\frac{2}{3}})^2 - \frac{3}{2}\,x^{\frac{2}{3}} = 10$$

$$\therefore\ (x^{\frac{2}{3}})^2 - \frac{2}{3}\,x^{\frac{2}{3}} + \frac{9}{16} = 10 + \frac{9}{16} = \frac{169}{16}$$

$$\therefore\ x^{\frac{2}{3}} - \frac{3}{4} = \pm\frac{13}{4}$$

$$\therefore\ x^{\frac{2}{3}} = \frac{3 \pm 13}{4} = 4 \text{ or } -\frac{5}{2}$$

$$\therefore\ x = (4)^{\frac{3}{2}} \text{ or } \left(-\frac{5}{2}\right)^{\frac{3}{2}}$$

$$= 8 \text{ or } \left(-\frac{5}{2}\right)^{\frac{3}{2}}$$

100. $$3x^2 - 3x + 6 = 5\tfrac{1}{3} = \frac{16}{3}$$

First, $x^2 - x = \frac{16}{9} - 2 = -\frac{2}{9}$

$$\therefore\ x^2 - x + \tfrac{1}{4} = \tfrac{1}{4} - \tfrac{2}{9} = \tfrac{1}{36}$$

$$\therefore\ x - \tfrac{1}{2} = \pm\tfrac{1}{6}$$

$$\therefore\ x = \frac{3\pm1}{6} = \tfrac{2}{3} \text{ or } \tfrac{1}{3}$$

101. $\left.\begin{aligned} x^2 + y^2 &= 5 \\ 3xy &= 6 \end{aligned}\right\}$ Let $x = vy$

Then $v^2y^2 + y^2 = 5$

$3vy^2 = 6$

$\therefore \dfrac{v^2+1}{3v} = \frac{5}{6}$

or $v^2 + 1 = \frac{5}{2}v$

$\therefore v^2 - \frac{5}{2}v = -1$

$\therefore v^2 - \frac{5}{2}v + \frac{25}{16} = \frac{25}{16} - 1 = \frac{9}{16}$

$\therefore v - \frac{5}{4} = \pm\frac{3}{4}$

$\therefore v = \dfrac{5 \pm 3}{4} = 2 \text{ or } \frac{1}{2}$

Hence $x = vy = 2y$ or $\frac{1}{2}y$

$\therefore 2y^2$ or $\frac{1}{2}y^2 = 2$ (from 2d equation)

$\therefore y = \pm 1$ or ± 2

and $x = 2y$ or $\frac{1}{2}y = \pm 2$ or ± 1

$\therefore$ the corresponding values of x and y are $+1$, $+2$, or $+2$, $+1$, or -1, -2, or -2, -1.

Otherwise,

Add $2xy$, obtained from the second equation, to the first, and thence get $x + y$; subtract $2xy$ and get $x - y$. The sum and difference of $x + y$, $x - y$ divided by 2 will give x and y respectively.

102. $3x + 2\sqrt{x} - 1 = 0$

First, $(x^{\frac{1}{2}})^2 + \dfrac{2}{3}x^{\frac{1}{2}} = \dfrac{1}{3}$

$\therefore (x^{\frac{1}{2}})^2 + x^{\frac{1}{2}} + \dfrac{1}{9} = \dfrac{1}{9} + \dfrac{1}{3} = \dfrac{4}{9}$

$\therefore x^{\frac{1}{2}} + \dfrac{1}{3} = \pm\dfrac{2}{3}$

$\therefore x^{\frac{1}{2}} = \dfrac{-1 \pm 2}{3} = \dfrac{1}{3} \text{ or } -1$

$\therefore x = \dfrac{1}{9} \text{ or } 1$

103. $$\left.\begin{array}{l} x^2 y^2 + 6\,xy = 16 \\ \sqrt{\dfrac{1}{4x}} = \dfrac{1}{2}\sqrt{\dfrac{1}{y-1}} \end{array}\right\}$$

First, $\frac{1}{4x} = \frac{1}{4} \cdot \frac{1}{y-1}$

$\therefore\ x = y - 1$

Again, $(xy)^2 + 6\,xy = 16$

$\therefore\ (xy)^2 + 6\,xy + 9 = 16 + 9 = 25$

$\therefore\ xy + 3 = \pm 5$

$\therefore\ xy = -3 \pm 5 = 2$ or -8

Hence, $y^2 - y = 2$ or -8

$\therefore\ y^2 - y + \frac{1}{4} = \frac{1}{4} + 2$ or $\frac{1}{4} - 8$

$= \frac{9}{4}$ or $\frac{-31}{4}$

$\therefore\ y - \frac{1}{2} = \pm\frac{3}{2}$ or $\pm\frac{\sqrt{-31}}{2}$

$\therefore\ y = \frac{1 \pm 3}{2}$ or $\frac{1 \pm \sqrt{-31}}{2} = 2$ or -1 or $\frac{1 \pm \sqrt{-31}}{2}$

Hence $x = y - 1 = 1$ or -2 or $\frac{-1 \pm \sqrt{-31}}{2}$ and the corresponding values are 1, 2, or -2, -1, &c.

104. $$1 - \frac{2-x}{x} = \frac{x+2}{2} - 1$$

First, $2x - 4 + 2x = x^2 + 2x - 2x$

Or, $x^2 - 4x = -4$

$\therefore\ x^2 - 4x + 4 = 0$

$\therefore\ x - 2 = 0$

$\therefore\ x = 2.$

In this case the two values of x are equal. The general form of such quadratics is

$$x^2 - a\,x + \frac{a^2}{4} = 0$$

105. $$5\,x = 3 - 2\,x^2$$

First, $x^2 + \frac{5}{2}\,x = \frac{3}{2}$

$\therefore\ x^2 + \frac{5}{2}\,x + \frac{25}{16} = \frac{25}{16} + \frac{3}{2} = \frac{49}{16}$

$\therefore x + \frac{5}{4} = \pm \frac{7}{4}$

$\therefore x = \dfrac{-5 \pm 7}{4} = \frac{1}{2} \text{ or } -3$

106. $3x - \dfrac{x-4}{4} - 4 = \dfrac{5x+14}{3}$

First, $36x - 3x + 12 - 48 = 20x + 56$

$\therefore 13x = 92$

$\therefore x = \frac{92}{13} = 7\frac{1}{13}$

107. $x + \dfrac{24}{x-1} = 3x - 4$

First, $\dfrac{24}{x-1} = 2x - 4$

$\therefore 12 = \overline{x-2} \cdot \overline{x-1} = x^2 - 3x + 2$

$\therefore x^2 - 3x = 10$

Hence, $x^2 - 3x + \frac{9}{4} = 10 + \frac{9}{4} = \frac{49}{4}$

$\therefore x - \frac{3}{2} = \pm \frac{7}{2}$

$\therefore x = \dfrac{3 \pm 7}{2} = 5 \text{ or } -2$

108. $\left.\begin{aligned} x^3 + y^3 &= 72 \\ x + y &= 6 \end{aligned}\right\}$

First, $x^3 + 3x^2 y + 3xy^2 + y^3 = 6^3 = 216$

$\therefore 3x^2y + 3xy^2 \text{ or } 3xy . (x + y) = 216 - 72 = 144$

$\therefore 3xy = \dfrac{144}{6} = 24$

and $xy = \dfrac{24}{3} = 8$

Now $x^2 + 2xy + y^2 = 6^2 = 36$

But $\underline{4xy = 32}$

$\therefore x^2 - 2xy + y^2 = 4$

$\therefore x - y = \pm 2$

and $\underline{x + y = 6}$

$\therefore 2x = 6 \pm 2 = 8 \text{ or } 4$

and $2y = 6 \mp 2 = 4 \text{ or } 8$

$\therefore x = 4 \text{ or } 2$

and $y = 2 \text{ or } 4$

The corresponding values of x and y are 4, 2 or 2, 4. They interchange values because of the symmetry of the equations with respect to them.

109. $\sqrt{x} - 4 = \dfrac{259 - 10x}{\sqrt{x} + 4}$

First, $x - 16 = 259 - 10x$

$\therefore x + 10x = 259 + 16 = 275$

or $11x = 275$

$\therefore x = \dfrac{275}{11} = 25$

110. $\left.\begin{array}{l} xy = 63 \\ \text{and } (x+y)^2 : (x-y)^2 :: 64 : 1 \end{array}\right\}$

First $x + y : x - y :: 8 : 1$

$\therefore 2x : 2y :: 9 : 7$

or $x : y :: 9 : 7$

$\therefore x = \dfrac{9y}{7}$

Hence $\dfrac{9y^2}{7} = 63$

$\therefore y^2 = \dfrac{63 \times 7}{9} = 7^2$

$\therefore y = \pm 7$

and $x = \dfrac{9y}{7} = \dfrac{9}{7}.(\pm 7) = \pm 9$

and the corresponding values of x and y are 7, 9 or -7, -9.

111. $x^2.(x+4) + 2x.(x+4) = 2 - (x+4)$

First $x . \overline{x+4} . \overline{x+2} = 2 - x - 4 = -\overline{x+2}$

$\therefore x . \overline{x+4} = -1$

or $x^2 + 4x = -1$

$\therefore x^2 + 4x + 4 = 4 - 1 = 3$

$\therefore x + 2 = \pm\sqrt{3}$

$\therefore x = -2 \pm \sqrt{3}$

X

112. $$x + y - \frac{\sqrt{x+y}}{\sqrt{x-y}} = \frac{6}{x-y}\Bigg\}$$

$$x^2 + y^2 = 41$$

First $x^2 - y^2 - \sqrt{x+y}\cdot\sqrt{x-y} = 6$

or $x^2 - y^2 - \sqrt{x^2 - y^2} = 6$

$\therefore\ x^2 - y^2 - \sqrt{x^2-y^2} + \frac{1}{4} = 6 + \frac{1}{4} = \frac{25}{4}$

$\therefore\ \sqrt{x^2 - y^2} = \frac{1}{2} \pm \frac{5}{2} = 3 \text{ or } -2$

$\therefore\ x^2 - y^2 = 9 \text{ or } 4$

But $\underline{x^2 + y^2 = 41}$

$\therefore\ 2\,x^2 = 50 \text{ or } 45$

$2\,y^2 = 32 \text{ or } 37$

$\therefore\ x = \pm\sqrt{25} \text{ or } \pm\sqrt{\frac{45}{2}} = \pm\, 5 \text{ or } \pm\sqrt{\frac{45}{2}}$

$y = \pm\sqrt{16} \text{ or } \pm\sqrt{\frac{37}{2}} = \pm\, 4 \text{ or } \pm\sqrt{\frac{37}{2}}$

and the corresponding values of x and y are $5, 4$ or $-5, -4$ or

$\sqrt{\frac{45}{2}}\quad \sqrt{\frac{37}{2}}$ or $-\sqrt{\frac{45}{2}} - \sqrt{\frac{37}{2}}$

N. B. The quantity $\sqrt{x^2 - y^2}$, in the first equation reduced, may be positive or negative. The positive values if x and y are taken when $\sqrt{x^2 - y^2}$ is positive, and their negative values when x and y are negative.

113. $$3\,x^n \sqrt[3]{x^n} + \frac{4\,x^n}{\sqrt[3]{x^n}} = 4 \text{ or } 3\,x^{n+\frac{n}{3}} + 4x^{n-\frac{n}{3}} = 4$$

or $x^{\frac{4n}{3}} + \frac{4}{3}\,x^{\frac{2n}{3}} = \frac{4}{3}$

First $\left(x^{\frac{2n}{3}}\right)^2 + \frac{4}{3}x^{\frac{2n}{3}} + \frac{4}{9} = \frac{4}{3} + \frac{4}{9} = \frac{16}{9}$

$\therefore\ x^{\frac{2n}{3}} = \pm\frac{4}{3} - \frac{2}{3} = \frac{2}{3} \text{ or } -2$

$\therefore\ x = \left(\frac{2}{3}\right)^{\frac{3}{2n}} \text{ or } (-2)^{\frac{3}{2n}}$

114. $$\left.\begin{aligned} x + y + \sqrt{x + y} &= 6 \\ x^2 + y^2 &= 10 \end{aligned}\right\}$$

First $x + y + \sqrt{x + y} + \frac{1}{4} = 6 + \frac{1}{4} = \dfrac{25}{4}$

$$\therefore \sqrt{x + y} = \pm \tfrac{5}{2} - \tfrac{1}{2} = 2 \text{ or } -3$$

$$\therefore x + y = 4 \text{ or } 9$$

$$\therefore x^2 + 2\,xy + y^2 = 16 \text{ or } 81$$

$$\text{or } 2\,xy + 10 = 16 \text{ or } 81$$

$$\therefore 2\,xy = 6 \text{ or } 71$$

Hence $x^2 - 2\,xy + y^2 = 10 - 6 \text{ or } 10 - 71 = 4 \text{ or } -61$

$$\therefore x - y = \pm 2 \text{ or } \pm \sqrt{-61}$$

But $\underline{x + y = 4 \text{ or } 9}$

$$\therefore 2\,x = 6 \text{ or } 2, \text{ or } 9 \pm \sqrt{-61}$$

$$2\,y = 2 \text{ or } 6, \text{ or } 9 \mp \sqrt{-61}$$

$$\therefore x = 3 \text{ or } 1, \text{ or } \frac{9 \pm \sqrt{-61}}{2}$$

$$y = 1 \text{ or } 3, \text{ or } \frac{9 \mp \sqrt{-61}}{2}$$

and the corresponding values of x and y are 3, 1 or 1, 3,

$$\text{or, } \frac{9 + \sqrt{-61}}{2}, \frac{9 - \sqrt{-61}}{2}, \text{ or } \frac{9 - \sqrt{-61}}{2}, \frac{9 + \sqrt{-61}}{2},$$

the two first pairs of values being taken when $\sqrt{x + y}$, in the first equation, is positive, and the two last when $\sqrt{x + y}$ is negative.

115. $$\frac{x}{x + 4} + \frac{4}{\sqrt{x+4}} = \frac{21}{x}$$

First $\dfrac{1}{x + 4} + \dfrac{4}{x}\sqrt{\dfrac{1}{x + 4}} = \dfrac{21}{x^2}$

$$\therefore \frac{1}{x + 4} + \frac{4}{x}\sqrt{\frac{1}{x + 4}} + \frac{4}{x^2} = \frac{21}{x^2} + \frac{4}{x^2} = \frac{25}{x^2}$$

$$\therefore \sqrt{\frac{1}{x + 4}} = \pm \frac{5}{x} - \frac{2}{x} = \frac{3}{x} \text{ or } -\frac{7}{x}$$

$$\therefore \frac{1}{x + 4} = \frac{29}{x^2} \text{ or } \frac{49}{x^2}$$

$\therefore x^2 = 9x + 36$ or $49x + 196$

$\therefore x^2 - 9x = 36$, or $x^2 - 49x = 196$

$\therefore x^2 - 9x + \frac{81}{4} = \frac{81}{4} + 36$ or $x^2 - 49x + \frac{49^2}{4} = \frac{49^2}{4} + 196 = \frac{3185}{4}$

$\therefore x = \frac{9}{2} \pm \frac{15}{2} = 12$ or -3 or $x = \frac{49}{2} \pm \frac{\sqrt{3165}}{2} = \frac{49 \pm \sqrt{3185}}{2}$ the positive and negative values corresponding to like values of $\sqrt{x+4}$ in the original equation.

116. $\left.\begin{array}{l} x + y = a \\ x^3 + y^3 = b \end{array}\right\}$ First, $x^3 + 3x^2y + 3xy^2 + y^3 = a^3$

or $b + 3xy(x + y) = a^3$

Hence $xy = \frac{a^3 - b}{3a}$

$\therefore 4xy = \frac{4}{3a}(a^3 - b)$

Now $x^2 + 2xy + y^2 = a^2$

$\therefore x^2 - 2xy + y^2 = a^2 - \frac{4}{3a}(a^3 - b) = \frac{3a^3 - 4a^3 + 4b}{3a} = \frac{4b - a^3}{3a}$

$\therefore x - y = \pm \sqrt{\frac{4b - a^3}{3a}}$

But $x + y = a$

$\therefore 2x = a \pm \sqrt{\frac{4l - a^3}{3a}} \therefore x = \frac{a}{2} \pm \frac{1}{2}\sqrt{\frac{4b - a^3}{3a}}$

$2y = a \mp \sqrt{\frac{4b - a^3}{3a}} \therefore y = \frac{a}{2} \mp \frac{1}{2}\sqrt{\frac{4b - a^3}{3a}}$

117. $\left.\begin{array}{l} xz = y^2 \\ x + y + z = 21 \\ x^2 + y^2 + z^2 = 189 \end{array}\right\}$

First $x^2 + z^2 + 2xz = 189 + y^2$

$\therefore x + z = \pm\sqrt{189 + y^2} = 21 - y$

$\therefore 189 + y^2 = 441 - 42y + y^2$

$\therefore 42y = 441 - 189 = 252$

$\therefore y = \frac{252}{42} = 6$

Hence $x^2 + z^2 = 189 - 36 = 153$

and $2xz = 2.36 = 72$

$\therefore x - 2xz + z^2 = 153 - 72 = 81$

$\therefore x - z = \pm 9$

and $x + z = 21 - y = 15$

$\therefore 2x = 24$ or 6

$2z = 6$ or 24

$\therefore x = 12$ or 3

$z = 3$ or 12

$y = 6$

The corresponding values of x, y, z, are 12, 3, 6, or 3, 12, 6. x and z being similarly involved in each of the given equations, interchange values.

118. $\sqrt{5 + x} + \sqrt{x} = \frac{15}{\sqrt{5 + x}}$

First, $5 + x + \sqrt{5x + x^2} = 15$

or $\sqrt{5x + x^2} = 10 - x$

$\therefore 5x + x^2 = 100 - 20x + x^2$

$\therefore 25x = 100$

or $x = \frac{100}{25} = 4$

119. $\left.\begin{array}{l} x^2 + y^2 = 34 \\ x^2 - xy = 10 \end{array}\right\}$

Let $x = vy$

Then $v^2y^2 + y^2 = 34$

$v^2y^2 - vy^2 = 10$

$\therefore \frac{v^2 + 1}{v^2 - v} = \frac{34}{10}$

or $10v^2 + 10 = 34v^2 - 34v$

$\therefore\ 24v^2 - 34v = 10$

Hence $v^2 - \frac{17}{12}v = \frac{5}{12}$

$$\therefore\ v^2 - \frac{17}{12}v + \frac{17^2}{24^2} = \frac{5}{12} + \frac{17^2}{24^2} = \frac{529}{24^2}$$

$\therefore\ v = \frac{17}{24} \pm \frac{23}{24} = \frac{40}{24}$ or $-\frac{6}{24} = \frac{5}{3}$ or $-\frac{1}{4}$

Hence $x = \frac{5}{3}y$ or $-\frac{1}{4}y$

$$\therefore\ \frac{25}{9}y^2 + y^2 = 34 \quad \therefore\ y^2 = 9 \text{ and } y = \pm 3$$

Again, $\frac{y^2}{16} + y^2 = 34$ hence $y^2 = 32$ and $y = \pm 4\sqrt{2}$

$\therefore\ x = \frac{5}{3}(\pm 3) = \pm 5$, and x also $= -\frac{1}{4}y = -\frac{1}{4}.(\pm 4\sqrt{2})$
$= \mp\sqrt{2}$

and the corresponding values of x and y are $5, 3$, or $-5, -3$, or $-\sqrt{2}, 4\sqrt{2}$, or $+\sqrt{2}, -4\sqrt{2}$.

120. $a + x + \sqrt{2ax + x^2} = b$

First $\sqrt{2ax + x^2} = \overline{b-a} - x$

$$\therefore\ 2ax + x^2 = \overline{b-a}|^2 - 2\overline{b-a}.\,x + x^2$$
$$= \overline{b-a}|^2 - 2bx + 2ax + x^2$$

or $2bx = \overline{b-a}|^2$

$$\therefore\ x = \frac{\overline{b-a}|^2}{2b}$$

121. $$\left.\begin{aligned} \frac{y}{x} - \frac{x}{x+y} &= \frac{x^2-y^2}{y} \\ \frac{x}{y} - \frac{x+y}{x} &= \frac{y}{x} \end{aligned}\right\}$$

From the second equation we have

$$\frac{x}{y} - 1 - \frac{y}{x} = \frac{y}{x}$$

or $\frac{x}{y} - \frac{2}{\left(\frac{x}{y}\right)} = 1$

$$\therefore\ \left(\frac{x}{y}\right)^2 - \frac{x}{y} = 2$$

$$\therefore \left(\frac{x}{y}\right)^2 - \frac{x}{y} + \tfrac{1}{4} = \tfrac{1}{4} + 2 = \tfrac{9}{4}$$

$$\therefore \frac{x}{y} = \tfrac{1}{2} \pm \tfrac{3}{2} = 2 \text{ or } -1$$

$$\therefore x = 2y \text{ or } -y$$

Hence $\dfrac{y}{2y} - \dfrac{2y}{2y+y} = \dfrac{4y^2 - y^2}{y}$

$$\therefore \tfrac{1}{2} - \tfrac{2}{3} = 3y \quad \therefore y = -\tfrac{1}{18}$$

Reducing the first equation and substituting for x its other value $(-y)$ we obtain $y = o$

Hence $x = 2y$ or $-y = -\frac{1}{9}$ or

The corresponding values of x and y are $-\frac{1}{9}, -\frac{1}{18}$ or (o, o)

122. $$\frac{x}{x+2} - \frac{x-9}{3x-20} = \frac{9}{13}$$

First, $x - \dfrac{x^2 - 7x - 18}{3x - 20} = \dfrac{9}{13}x + \dfrac{18}{13}$

and $13x - \dfrac{13x^2 - 91x - 234}{3x - 20} = 9x + 18$

or $4x - 18 = \dfrac{13x^2 - 91x - 234}{3x - 20}$

$$\therefore 12x^2 - 134x + 360 = 13x^2 - 91x - 234$$

or $x^2 + 43x = 360 + 234 = 594$

$$\therefore x^2 + 43x + \frac{43^2}{4} = 594 + \frac{4^{\ 2}}{4} = \frac{4225}{4}$$

$$\therefore x + \frac{43}{2} = \pm \sqrt{\frac{4225}{4}} = \pm \frac{65}{2}$$

$$\therefore x = \pm \frac{65}{2} - \frac{43}{2} = 11 \text{ or } -54$$

123. $$\left.\begin{aligned} x^4 - x^2 + y^4 - y^2 &= 84 \\ x^2 + x^2y^2 + y^2 &= 49 \end{aligned}\right\}$$

First, $(x^2 + y^2)^2 - 2x^2y^2 - (x^2 + y^2) = 84$

But $-2x^2y^2 = 2(x^2 + y^2) - 98$

$$\therefore (x^2 + y^2)^2 + x^2 + y^2 = 84 + 98 = 182$$

$$\therefore (x^2 + y^2)^2 + x^2 + y^2 + \tfrac{1}{4} = \tfrac{1}{4} + 182 = \frac{729}{4}$$

$\therefore x^2 + y^2 = -\frac{1}{2} \pm \frac{27}{2} = 13$ or -14

Hence $x^2 y^2 = 49 - 13$ or $49 + 14 = 36$ or 63

$\therefore xy = \pm 6$ or $\pm 3\sqrt{7}$

Hence $x^2 - 2xy + y^2 = 13 \mp 12$ or $13 \mp 6\sqrt{7}$, or it $= -14 \mp 12$ or $-14 \mp 6\sqrt{7}$

$\therefore x - y = \pm 1$ or ± 5, or it $= \pm\sqrt{13 \mp 6\sqrt{7}}$, or it $= \pm\sqrt{-26}$ or $\pm\sqrt{-2}$, or it $= \pm\sqrt{-14 \mp 6\sqrt{7}}$

Similarly $x + y = \pm 5$ or ± 1, or it $= \pm\sqrt{13 \pm 6 \cdot \sqrt{7}}$, or it $= \pm\sqrt{-2}$ or $\pm\sqrt{-26}$, or it $= \pm\sqrt{14 \pm 6\sqrt{7}}$

Hence corresponding values of x and y are easily found, which from x and y being similarly involved, will interchange.

124. $$\frac{a}{x} + \frac{\sqrt{a^2 - x^2}}{x} = \frac{x}{b}$$

First, $\sqrt{a^2 - x^2} = \frac{x^2}{b} - a = \frac{x^2 - ab}{b}$

$$\therefore a^2 - x^2 = \frac{x^4 - 2abx^2 + a^2b^2}{b^2}$$

$$\therefore b^2a^2 - b^2x^2 = x^4 - 2abx^2 + a^2b^2$$

$$\therefore x^4 = \overline{2ab - b^2} \cdot x^2$$

$$\therefore x^2 = 2ab - b^2$$

and $x = \pm\sqrt{2ab - b^2}$

125. $$\left.\begin{aligned} x^m y^n &= a \\ x^p y^q &= b \end{aligned}\right\}$$

First, $x^m = \frac{a}{y^n}$ and $x^p = \frac{b}{y^q}$

$$\therefore x = \frac{a^{\frac{1}{m}}}{y^{\frac{n}{m}}} = \frac{b^{\frac{1}{p}}}{y^{\frac{q}{p}}}$$

$$\therefore \frac{a^{\frac{1}{m}}}{b^{\frac{1}{p}}} = y^{\frac{n}{m} - \frac{q}{p}} = y^{\frac{np - mq}{mp}}$$

$$\therefore\ y = \frac{a^{\frac{p}{np-mq}}}{b^{\frac{m}{np-mq}}}$$

$$\text{and } x^m = \frac{a}{y^n} = a \times \frac{b^{\frac{mn}{np-mq}}}{a^{\frac{np}{np-mq}}} = \frac{b^{\frac{mn}{np-mq}}}{a^{\frac{mq}{np-mq}}}$$

$$\therefore\ x = \frac{b^{\frac{n}{np-mq}}}{a^{\frac{q}{np-mq}}}$$

126. $x + \sqrt{x^2 + \sqrt{x^2 + 96}} = 11$

First $\sqrt{x^2 + \sqrt{x^2 + 96}} = 11 - x$

$$\therefore\ x^2 + \sqrt{x^2 + 96} = 121 - 22\,x + x^2$$

$$\therefore\ \sqrt{x^2 + 96} = 121 - 22x$$

$$\therefore\ x^2 + 96 = 14641 - 5324x + 484\,x^2$$

$$\therefore\ 483\,x^2 - 5324\,x = 96 - 14641 = -14545$$

$$\therefore\ x^2 - \frac{5324}{483}\,x = -\frac{14545}{483}$$

$$\therefore\ x^2 - \frac{8 + 324}{483}\,x + \left(\frac{2662}{483}\right)^2 = \frac{61009}{483^2}$$

$$\therefore\ x = \frac{2662}{483} \pm \frac{247}{483} = \frac{2909}{483} \text{ or } 5$$

127. $\left.\begin{aligned} x\,(y + z) &= a \\ y\,(x + z) &= b \\ z\,(x + y) &= c \end{aligned}\right\}$ First $\begin{aligned} x\,y + x\,z &= a. \\ x\,y + y\,z &= b. \\ x\,z + y\,z &= c. \end{aligned}$

$\therefore$ by addition $2\,x\,y + 2\,x\,z + 2\,y\,z = a + b + c$

$$\therefore\ 2\,x\,y = a + b + c - 2\,c = a + b - c$$

$$2\,x\,z = a + b + c - 2\,b = a + c - b$$

$$2\,y\,z = a + b + c - 2\,a = b + c - a$$

Put $\frac{a + b + c}{2} = S$ then $x\,y = \frac{a + b - c}{2} = S - c$

$$x\,z = \frac{a + c - b}{2} = S -$$

$$y\,z = \frac{b + c - a}{2} = S - a$$

Hence, $x^2\,y^2\,z^2 = (S-a).(S-b).(S-c)$

and $x\,y\,z = \pm\sqrt{(S-a).(S-b).(S-c)}$

Hence $x = \pm\dfrac{\sqrt{(S-a).(S-b).(S-c)}}{S-a} = \pm\sqrt{\dfrac{(S-b)(S-c)}{S-a}}$

and $y = \dfrac{x\,y\,z}{x\,z} = \pm\dfrac{\sqrt{(S-a).(S-b).(S-c)}}{S-b} = \pm\sqrt{\dfrac{(S-a).(S-b)}{S-b}}$ c

and $z = \dfrac{x\,y\,z}{x\,y} \pm \dfrac{\sqrt{(S-a).(S-b).(S-c)}}{S-c} = \pm\sqrt{\dfrac{(S-a)(S-b)}{S-c}}$

The corresponding values must have the same sign.

Otherwise. Let $x = v\,y$, $z = w\,y$. Eliminate y^2 in the resulting equations, &c. &c.

128. $\dfrac{x^4+1}{(x+1)^4} = \frac{1}{2}$

First, $2\,x^4 + 2 = x^4 + 4\,x^3 + 6\,x^2 + 4\,x + 1$

$\therefore\ x^4 - 4\,x^3 + 6\,x^2 - 4\,x + 1 = 12\,x^2$

or $(x-1)^4 = 12x^2$

$\therefore\ \overline{x-1}|^2 = \pm\sqrt{3}.2\,x$

or $x^2 - 2\,x + 1 = \pm\,2\sqrt{3}\,x$

$\therefore\ x^2 - 2.(1\pm\sqrt{3})\,x = -1$

$\therefore\ x^2 - 2\,(1\pm\sqrt{3})\,x + (1\pm\sqrt{3})^2 = (1\pm\sqrt{3})^2 - 1$

$= 1 + 3 \pm 2\sqrt{3} - 1$

$= 3 \pm 2\sqrt{3}$

$\therefore\ x - (1\pm\sqrt{3}) = \pm\sqrt{3\pm2\sqrt{3}}$

$\therefore\ x = 1 \pm \sqrt{3} \pm \sqrt{3\pm2\sqrt{3}}$ an expression which contains the four surd values of x.

129. $\sqrt{x^4-1} + \sqrt{x^2-1} = x^3$

First, $\sqrt{x^4-1} = x^3 - \sqrt{x^2-1}$

$\therefore\ x^4 - 1 = x^6 - 2\,x^3\sqrt{x^2-1} + x^2 - 1$

$\therefore\ x^4 = x^6 - 2\,x^3\sqrt{x^2-1} + x^2$

$\therefore\ x^2 = x^4 - 2\,x\sqrt{x^2-1} + 1$

$$\text{Or, } x^4 - x^2 - 2\sqrt{x^4 - x^2} = -1$$

$$\therefore (x^4 - x^2) - 2\sqrt{x^4 - x^2} + 1 = 1 - 1 = 0$$

$$\therefore \sqrt{x^4 - x^2} - 1 = 0$$

$$\therefore \sqrt{x^4 - x^2} = 1$$

$$\text{and } x^4 - x^2 = 1$$

$$\text{Hence, } x^4 - x^2 + \tfrac{1}{4} = 1 + \tfrac{1}{4} = \tfrac{5}{4}$$

$$\therefore x^2 - \tfrac{1}{2} = \frac{\pm\sqrt{5}}{2}$$

$$\therefore x^2 = \frac{1 \pm \sqrt{5}}{2}$$

$$\text{and } x = \pm\sqrt{\frac{1 \pm \sqrt{5}}{2}}$$

ALGEBRA IN GENERAL.

130. For the proof see *Wood.*

```
216 ) 3186 ( 14
      216
      ----
      1026
       864
      ----
       162 ) 216 ( 1
             162
             ---
              54 ) 162 ( 3
                   162
                   ---
                   ...
                   ---
```

$\therefore$ 54 is the greatest common measure of 216 and 3186, which being divided by it, we have $\frac{216}{3186}$ reduced to $\frac{4}{59}$.

131. Put .1666 ... $= S$. Multiply by 10.

Then $1.666\ldots = 10\,S$

Again, put $.666\ldots = S'$ and multiply by 10.

Then $6.66\ldots = 10\,S'$

Or $6 + S' = 10\,S'$.

$\therefore 9\,S' = 6$

$\therefore S' = \frac{6}{9} = \frac{2}{3}$

Hence $1 + \frac{2}{3} = 10\,S$.

$\therefore S = \frac{5}{3} \div 10 = \frac{5}{30} = \frac{1}{6}$

Now $\frac{1}{6}$ of a pound $= \frac{20}{6}s. = 3\,\frac{1}{3}s.$

$\frac{1}{3}s. = \frac{12}{3}d. = 4\,d.$

$\therefore$ the value required is $3s.\ 4d.$

132. $\sqrt[m]{1+x} : 1 = 1 + \frac{1}{m}x + \frac{1}{m}\cdot\frac{\frac{1}{m}-1}{2}x^2 + \&c. : 1$ by the binomial theorem.

Now, since x is very small compared with 1, the terms which involve x^2, x^3, &c., may be neglected.

$\therefore \sqrt[m]{1+x} : 1 = 1 + \frac{x}{m} : 1$ nearly.

It must be supposed that m is a whole number, or a fraction of considerable magnitude, or $\frac{1}{m}\cdot\frac{\frac{1}{m}-1}{2}$ might be as great as x^2 is small, and their product too great to be neglected.

133. For the required proof, see Wood, or Woodhouse, *(Analyt. Cal.)*

$$
\begin{array}{rl}
 & 738.763264\ (\,904 \\
 & 729 \\
3a^2 = 243\,) & 9763, \text{ and } 243 \text{ is not contained in } 97 \\
3\,(a+b)^2 = 24300\,) & 9763264 \text{ which is contained 4 times in } 97632 \\
 & \left.\begin{array}{rl} 97200 & = 3\,.\,(a+b)^2.\,c \\ 4320 & = 3\,.\,(a+b)\,.\,c^2 \\ 64 & = c^3 \end{array}\right\}\text{subtrahend} \\
 & 9763264
\end{array}
$$

$\therefore$ 9.04 is the root required.

N. B. It is evident that when the number is a perfect cube, the first digit in units' place is such as would be produced in the units' place by cubing some one of the numbers 0, 1, 2 . . . 9.

134. Let a and b be the extremes.

Then $\frac{a+b}{2}$ = the arithmetic mean.

and $\sqrt{ab}$ = the geometric mean (since a : geometric mean :: geometric mean : b.)

$\therefore$ the arithmetic mean is > geometric if $\frac{a+b}{2}$ be $> \sqrt{ab}$, or $a + b > 2\sqrt{ab}$, or $a^2 + 2ab + b^2 > 4ab$, or $a^2 - 2ab +$

$b^2 > o$, or $(a - b)^2 > o$, which is the case, since every square is positive, and a is supposed not $=$ to b.

135. Let $x^2 + p\,x + q = o$ be the quadratic.

$\therefore\ x^2 + p\,x = -\,q.$

Now, put $(x + Q)^2 = x^2 + 2\,Q\,x + Q^2 = x^2 + p\,x + Q^2$, and equating the co-efficients of the like powers of x.

$2\,Q = p$

$\therefore\ Q = \frac{p}{2}$

and $Q^2 = \frac{p^2}{4}$

$\therefore$ the quantity to be added to both sides in order to complete the square $=$ the square of half the co-efficient of the second term.

Otherwise (which is better.)

Since $(x + Q)^2 = x^2 + 2\,Q\,x + Q^2$, it is evident that Q^2, the quantity added is the square of half the co-efficient of the second term.

136. (1) Let r be the common ratio.

Then $1, r, r^2, r^3, r^4, 32$ is the series.

Now the last term $=$ first term $\times\ r^5 = r^5$.

or, $r^5 = 32$

$\therefore\ r = 2$

$\therefore$ the required means are 2, 4, 8, 16.

(2) Again, let d be the common difference.

Now, the last term $=$ the first $+\ \overline{n-1}.\,d$ (n being the number of terms).

$\therefore\ 11 = 1 + 4\,d \quad \therefore\ d = \frac{10}{4} = \frac{5}{2}$

and the means are $1 + \frac{5}{2},\ 1 + 5,\ 1 + \frac{15}{2}$

or $\frac{7}{2}$, 6 and $\frac{17}{2}$

137. Let $.123333\ldots\ldots = S.$

$\therefore\ 12.33\ldots\ldots = 100\,S.$

Again let $.33 \ldots . = S'$.

$\therefore 3.33 \ldots . = 10\,S'$.

$\therefore 3 + S' = 10\,S'$.

or $S' = \frac{3}{9} = \frac{1}{3}$

$$\therefore S = \frac{12 + \frac{1}{3}}{100} = \frac{37}{300}$$

Now $\frac{37}{300}\,£ = \frac{37 \times 20}{300} s. = \frac{74}{30} s. \;= 2\frac{7}{15} s.$

and $\frac{7}{15}\,d = \frac{84}{15}\,d = \frac{28}{5}\,d = 5\,\frac{3}{5}\,d.$

and $\frac{3}{5} d. = \frac{12}{5}\,q = 2\,\frac{2}{5}\,q$

$\therefore$ the value required is $2s.\ 5d.\ 2\frac{2}{5}q.$

Otherwise.

Multiply decimally, by 20, 12, and 4 successively, &c.

138. $$a^3 - x^3 \left(a^{\frac{3}{2}} - \frac{x^3}{2a^{\frac{3}{2}}} - \frac{x^6}{8a^{\frac{9}{2}}} - \frac{x^9}{16\,a^{\frac{15}{2}}}\right., \text{ \&c.}$$

$$\frac{a^3}{}$$

$$2a^{\frac{3}{2}} - \frac{x^3}{2a^{\frac{3}{2}}}\Big) - x^3$$

$$- x^3 + \frac{x^6}{4a^3}$$

$$2a^{\frac{3}{2}} - \frac{x^3}{a^{\frac{3}{2}}} - \frac{x^6}{8a^{\frac{9}{2}}}\Big) - \frac{x^6}{4\,a^3}$$

$$- \frac{x^6}{4\,a^3} + \frac{x^9}{8\,a^6} + \frac{x^{12}}{64a^9}$$

$$- \frac{x^9}{8a^6} - \frac{x^{12}}{64a^9}$$

Otherwise.

$$(a^3 - x^3)^{\frac{1}{2}} = a^{\frac{3}{2}}\left(1 - \frac{x^3}{a^3}\right)^{\frac{1}{2}} = a^{\frac{3}{2}}\left\{1 - \tfrac{1}{2}\,\frac{x^3}{a^3} + \left(\tfrac{1}{2} \times - \tfrac{1}{4}\,\frac{x^6}{a^6}\right) - , \text{ \&c.}\right.$$

which will give the same result as before.

139. $b = a\,r^{n-1}$, where r is the common ratio, and n the number of terms.

But $n = 8$

$\therefore b = a\,r^7$

$\therefore r = \left(\frac{b}{a}\right)^{\frac{1}{7}}$

$$\therefore a\left(\frac{b}{a}\right)^{\frac{1}{7}}, a \times \left(\frac{b}{a}\right)^{\frac{2}{7}}, a\left(\frac{b}{a}\right)^{\frac{3}{7}}, a\left(\frac{b}{a}\right)^{\frac{4}{7}}, a\left(\frac{b}{a}\right)^{\frac{5}{7}}, a\left(\frac{b}{a}\right)^{\frac{6}{7}}$$

or, $b^{\frac{1}{7}}.a^{\frac{6}{7}}$, $b^{\frac{2}{7}}.a^{\frac{5}{7}}$, $b^{\frac{3}{7}}.a^{\frac{4}{7}}$, $b^{\frac{4}{7}}.a^{\frac{3}{7}}$, $b^{\frac{5}{7}}.a^{\frac{2}{7}}$, $b^{\frac{6}{7}}.a^{\frac{1}{7}}$, are the means required.

140. Let $x^3 + y^3 = x^2y + y^2x + Q$.

or $(x + y)^3 - (3\,x^2y + 3\,xy^2) = x^2y + y^2x + Q$.

$\therefore \overline{x + y}|^3 - 4\,xy.\,(x + y) = Q$.

$\therefore (x + y)\,.\,(\overline{x + y}|^2 - 4\,xy) = Q$.

But $\overline{x + y}|^2 - 4\,xy = x^2 - 2\,xy + y^2 = \overline{x - y}|^2$

$\therefore Q = (x + y).\,\overline{x - y}|^2$, which is positive.

$\therefore x^3 + y^3$ is $> x^2y + y^2x$ by the quantity $(x + y)\,(x - y)^2$.

141. By *Wood*,

$$\sqrt{a - \sqrt{b}} = \sqrt{\frac{a + \sqrt{a^2 - b}}{2}} - \sqrt{\frac{a - \sqrt{a^2 - b}}{2}}$$

$$\therefore \sqrt{7 - 4\sqrt{3}} = \sqrt{\frac{7 + 1}{2}} - \sqrt{\frac{7 - 1}{2}}$$

$$= \pm 2 \mp \sqrt{3}.$$

142. The last term $100 = 10r^{n-1}$, n being the number of terms.

$\therefore 100 = 10\,r^3$. $\therefore r = 10^{\frac{1}{3}}$

$\therefore 10\,.\,10^{\frac{1}{3}}$, $10\,.\,10^{\frac{2}{3}}$

or $10^{\frac{4}{3}}$, and $10^{\frac{5}{3}}$ are the means required, which may, by first raising them to the 4th and 5th powers respectively, and afterwards extracting their cube roots, be expressed in numbers.

143. Let $x - y$, x, $x + y$ be the numbers, y being their common difference.

Then $x - y + x + x + y = 15$, or $3x = 15$, or $x = 5$.

Also $\overline{x-y}|^3 + x^3 + \overline{x+y}|^3 = 495$

$\therefore x^3 - 3x^2y + 3xy^2 - y^3 + x^3 + x^3 + 3x^2y + 3xy^2 + y^3 = 495$.

or $3x^3 + 6xy^2 = 495$.

$\therefore 3.125 + 6.5.\,y^2 = 495$

$\therefore y^2.\,30 = 495 - 375 = 120$

$\therefore y^2 = 4$ and $y = \pm 2$

$\therefore 5 \mp 2, 5, 5 \pm 2$.

or 3, 5, 7, are the numbers required. The signs merely indicate that the series may descend as well as ascend.

144. Let $.7485353\ldots = S$

Then $748.5353\ldots = 1000\,S$

Again put $.5353\ldots = S'$

Then $53.53\ldots = 100\,S'$

or $53 + S' = 100\,S'$

$$\therefore S' = \frac{53}{99}$$

$$\text{Hence } 1000\,S = 748 + \frac{53}{99} = \frac{74105}{99}$$

$$\therefore S = \frac{74105}{99000} = \frac{14821}{19800}.$$

145. If a be the root, the remainder (r), cannot exceed $2a+1$.

For $(a+1)^2 - a^2 = a^2 + 2a + 1 - a^2 = 2a + 1$

$\therefore$ if $2a + 1$ be the remainder, the quantity to be extracted $\overline{a+1}|^2$ contains a root $a + 1$ which is $> a$, contrary to the supposition.

146. For the investigation, see *Wood.*

$$\sqrt{a + \sqrt{b}} = \sqrt{\frac{a + \sqrt{a^2 - b}}{2}} + \sqrt{\frac{a - \sqrt{a^2 - b}}{2}}$$

Hence $a = o$

$$\sqrt{b} = 2\sqrt{-1} \quad \therefore b = -4 \quad \therefore \sqrt{a^2 - b} = \pm 2$$

$$\text{Hence } \sqrt{2\sqrt{-1}} = \sqrt{\frac{\pm 2}{2}} + \sqrt{\frac{\mp 2}{2}} = \pm 1 \pm \sqrt{-1}.$$

$\therefore$ either $1 + \sqrt{-1}$ or $-1 - \sqrt{-1}$ is the root.

147. Let $.785454 \ldots\ldots = S$

Then $78.5454 \ldots\ldots = 100\, S$

Again put $.5454 \ldots\ldots = S'$

Then $54.54 \ldots\ldots = 100\, S'$

$\therefore 54 + S' = 100\, S'$

$$\therefore S' = \frac{54}{99} = \frac{6}{11}$$

$$\therefore 100\, S = 78 + \frac{6}{11} = \frac{864}{11}$$

$$\therefore S = \frac{864}{1100} = \frac{216}{275}.$$

148. Let $x - 3y$, $x - y$, $x + y$, $x + 3y$ be the required numbers, $2y$ being their common difference.

Then, adding these, $4x = 56$, or $x = 14$

Again $\overline{x - 3y}\big|^2 + \overline{x + 3y}\big|^2 + \overline{x - y}\big|^2 + \overline{x + y}\big|^2 = 864$

or $2\,x^2 + 18y^2 + 2x^2 + 2y^2 = 864$

or $4x^2 + 20y^2 = 864$

or $20y^2 = 864 - 4x^2 = 864 - 784 = 80$

$\therefore y^2 = 4$ and $y = \pm 2$

Hence 14 ∓ 6, 14 ∓ 2, 14 ± 2, 14 ± 6.

$$\left.\begin{matrix} \text{or } 8, & 12, & 16, & 20 \\ 20, & 16, & 12, & 8 \end{matrix}\right\}$$ are the numbers required, the signs only shewing that the series may descend as well as ascend.

149. Let $.726363 \ldots\ldots = S$

Then $72.6363 \ldots\ldots = 100\, S$

Again put $.6363 \ldots\ldots = S'$

Then $63.63 \ldots\ldots = 100\, S'$

or $63 + S' = 100\, S'$

$$\therefore S' = \frac{63}{99} = \frac{7}{11}$$

Hence $72 + \frac{7}{11} = 100\, S$

or $S = \frac{799}{1100}$.

150. Put $.6363\ldots = S$

$\therefore 63.63\ldots = 100\,S$

$\therefore S = \frac{63}{99} = \frac{7}{11}$

Hence $1.6363\ldots = 1 + \frac{7}{11} = \frac{18}{11}$

and $\frac{18}{11}$ of a crown $= \frac{5}{21}$ of $\frac{18}{11}$ of a guinea.

$= \frac{5 \times 6}{7 \times 11} = \frac{30}{77}$ of a guinea, which may be expressed decimally, by actual division.

151. 209) 380 (1
209
—
171) 209 (1
171
—
38) 171 (4
152
—
19) 38 (2
38
—

$\therefore$ 19 is the greatest common measure of 209 and 380, which being divided by it, $\frac{209}{380}$ is reduced to $\frac{11}{20}$.

152. Let d be the common difference.

Then $1 + (1 + d) \quad + (1 + 2d) \quad + \ldots 1 + (n - 1)d = S$

Also $(\overline{1 + n} - 1.d) + (\overline{1 + n} - 2.d) \quad + (1 \overline{+ n} - 3.d) +$, &c. $+ 1 = S$

$2 + \overline{n - 1}\,.\,d) + (2 + \overline{n - 1}.d) + \ldots + (2 + \overline{n - 1}.d)$
$= 2\,S$, by addition.

or $n\,.\,(2 + \overline{n - 1}\,d) = 2\,S$

$$\therefore\ 2n + n.\overline{n-1}\,d = 2\,S$$

$$\text{or } n.\overline{n-1}\,d = 2\,.(S-n)$$

$$\therefore\ d = \frac{2\,.\,(S-n)}{n\,.\,(n-1)}.$$

153. Let $x - 3y, x - y, x + y, x + 3y$ be the numbers.

Then $2y = 3$, or $y = \frac{3}{2}$

$$\text{and } \left(x - \frac{9}{2}\right)\left(x + \frac{9}{2}\right)\left(x - \frac{3}{2}\right)\left(x + \frac{3}{2}\right) = 280$$

$$\text{or } \left(x^2 - \frac{81}{4}\right).\left(x^2 - \frac{9}{4}\right) = 280$$

$$\text{or } x^4 - \frac{45}{2}x^2 + \frac{729}{16} = 280$$

$$\therefore\ x^4 - \frac{45}{2}x^2 + \left(\frac{45}{4}\right)^2 = 280 - \frac{729}{16} + \left(\frac{45}{4}\right)^2$$

$$= \frac{5776}{16}$$

$$\therefore\ x^2 - \frac{45}{4} = \pm\frac{76}{4}$$

$$\therefore\ x^2 = \frac{45 \pm 76}{4} = \frac{121}{4} \text{ or } -\frac{31}{4}$$

$\therefore\ x = \pm\frac{11}{2}$ or $\pm\frac{\sqrt{-31}}{2}$ which latter value, $\frac{\sqrt{-31}}{2}$ being impossible, cannot answer the conditions of the question.

Hence $\pm\frac{11}{2} - \frac{9}{2}$, $\pm\frac{11}{2} - \frac{3}{2}$, $\pm\frac{11}{2} + \frac{3}{2}$, $\pm\frac{11}{2} + \frac{9}{2}$ or,

$\left.\begin{array}{l} 1,\ 4,\ 7,\ 10 \\ -10, -7, -4, -1 \end{array}\right\}$ are the numbers required, negative values answering as well as positive ones.

154. Generally $S = (2a + \overline{n-1}\,d)\frac{n}{2}$, where a is the first term, d the common difference, and n the number of terms.

$$\text{Hence } S = (2 + \overline{n-1})\frac{n}{2} = n + \frac{n^2}{2} - \frac{n}{2} = \frac{n^2}{2} + \frac{n}{2}$$

$$S' = (2 + 2.\overline{n-1})\frac{n}{2} = n^2$$

$$S'' = (2 + 3.\overline{n-1})\frac{n}{2} = n + \frac{3n^2}{2} - \frac{3n}{2} = \frac{3n^2}{2} - \frac{n}{2}$$

$$\therefore S + S'' = = \frac{n^2}{2} + \frac{n}{2} + \frac{3n^2}{2} - \frac{n}{2} = \frac{4n^2}{2} = 2n^2 = 2S'.$$

155. $S = 1 + (1 + m) + (1 + 2m) + \ldots.(1 + \overline{n-2}.m) + (1 + \overline{n-1}.m)$

Also $S = (1 + \overline{n-1}.m) + (1 + \overline{n-2}m) + \ldots. + (1 + m) + 1$

$$\therefore 2S = (2 + \overline{n-1}\,m) + (2 + \overline{n-1}\,m) + \ldots.\ n \text{ terms.}$$

$$= n.(2 + \overline{n-1}\,m) = 2n + mn^2 - mn = mn^2 - \overline{m-2}.n.$$

$$\therefore S = \frac{mn^2 - (m-2)n}{2}.$$

156. Their angular velocities are as 9 to 5.

Let θ be the $\angle$ described by the slower hand.

$\therefore \frac{9}{5}\theta$ = ditto by the swifter.

$= 4\text{R}.\angle + \theta$, since after leaving the slower, it must evidently gain the whole circumference.

Hence $\frac{4}{5}\theta = 4\text{ R}.\angle$.

$\therefore \theta = 5\text{ R}.\angle =$ (in time) 5 hours + $\frac{1}{4}$ of 5 hours.

$= 5 + \frac{5}{4} = 6 + \frac{1}{4}$ hours.

$\therefore$ After 6 hours and a quarter, or 6 hours and 15 minutes they meet again.

157. Let x be the less.

Then $x + 4 =$ the greater.

and $2x.(x + 4) = x^3$

or $2.(x + 4) = x^2$

$\therefore x^2 - 2x = 8$

$\therefore x^2 - 2x + 1 = 9.$

$\therefore x - 1 = \pm 3$

$\therefore x = 1 \pm 3 = 4$ or $- 2$

$\therefore$ 4, 8, or -2, and $+2$ each, answer the conditions of the question.

158.
$$\begin{array}{l} 336\ )\ 720\ (\ 2 \\ 672 \\ \hline 48\ )\ 336\ (\ 7 \\ 336 \\ \hline \dots \end{array}$$

$\therefore$ 48 is the greatest common measure of 336 and 720.

Again
$$\begin{array}{l} 48\ )\ 1736\ (36 \\ 144 \\ \hline \ 296 \\ \ 288 \\ \hline \ \ 8\ )\ 48\ (\ 6 \\ 48 \\ \hline \dots \end{array}$$

Hence 8 is the greatest common measure of 48 and 1736, and $\therefore$ of 336, 720, and 1736.

159. For the proof, see *Wood.*

$$\sqrt{a - \sqrt{b}} = \sqrt{\frac{a + \sqrt{a^2 - b}}{2}} - \sqrt{\frac{a - \sqrt{a^2 - b}}{2}}$$

Hence $a = 5$, $\sqrt{b} = +2\sqrt{6}$ or $b = 24$.

$$\therefore \sqrt{5 - 2\sqrt{6}} = \sqrt{\frac{5+1}{2}} - \sqrt{\frac{5-1}{2}} = \pm\sqrt{3} \mp \sqrt{2}$$

160. Put $.012323 \dots = S$

Then $1.2323 \dots = 100\,S$

Again let $.2323 \dots = S'$

Then $23.23 \dots\ 100\,S'$

$\therefore 23 + S' = 100\,S'$

or $99\,S' = 23$

$$\therefore S' = \frac{23}{99}$$

Hence $100\,S = 1 + \dfrac{23}{99} = \dfrac{122}{99}$

$$\therefore S = \frac{122}{9900} = \frac{61}{4950}$$

Now $\frac{61}{4950}$ £ $= \frac{20 \times 61}{4950}s. = \frac{122}{495}s. = \frac{122 \times 12}{495}d.$

$= \frac{1464}{495}d. = 2\frac{474}{495}d.$

Also $\frac{474}{495}d. = \frac{1896}{495}q. = 3\frac{411}{495}q. = 3\frac{137}{165}q.$

$\therefore$ the value required is 2d. $3\frac{137}{165}q$.

Otherwise.

$$\begin{array}{r} .012323 \\ 20 \\ \hline .24646\ldots \\ 12 \\ \hline 2.95757\ldots. \\ 4 \\ \hline 3.83030\ldots. \end{array}$$

Put $.83030\ldots. = S$

$\therefore 8.3030\ldots. = 10\,S$

Again let $.3030\ldots = S'$

$\therefore 30 + S' = 100\,S'$

$\therefore S' = \frac{30}{99} = \frac{10}{33}$

Hence $S = \dfrac{8 + \frac{10}{33}}{10} = \frac{274}{330} = \frac{137}{165}$

$\therefore$ the value is as before.

161. Let $.2121\ldots. = S$

Then $21.2121\ldots. = 100\,S$

or $100\,S = 21 + S$

$\therefore S = \frac{21}{99} = \frac{7}{33}$

162. Let x be the digit of units, y that of tens.

Then $10y + x =$ the number.

and by the question $\frac{10y + x}{y - x} = 21$

and $\frac{10y + x}{y + x} + 17 = 10x + y$

Hence $10y + x = 21y - 21x$

$\therefore 11y = 22x$

or $y = 2x$

$\therefore \frac{20x + x}{2x + x} + 17 = 10x + 2x$

or $\frac{21}{3} + 17 = 12x$

$\therefore 12x = 17 + 7 = 24$

and $x = 2$

Hence $y = 2x = 4$

$\therefore$ the number required, is 42.

163. Let $\sqrt{2 - 4\sqrt{-2}} = x - y$ \
Then $\sqrt{2 + 4\sqrt{-2}} = x + y$ $\Big\}$ $\therefore x^2 - y^2 = \sqrt{4 + 32} = \pm 6$

Again $2 - 4\sqrt{-2} = x^2 - 2xy + y^2 \therefore x^2 + y^2 = 2$

Hence $x^2 = \frac{\pm 6 + 2}{2} = 4$ or -2, $\therefore x = \pm 2$ or $\pm\sqrt{-2}$

$y^2 = \frac{2 \mp 6}{2} = -2$ or 4, $\therefore y = \pm\sqrt{-2}$ or ± 2

$\therefore \sqrt{2 - 4\sqrt{-2}} = x - y = \pm 2 \mp \sqrt{-2}$ or it $=$ $\pm\sqrt{-2} \mp 2$.

164. Let x be the number.

Then $3x + 2x^{\frac{1}{2}} = 1$

$\therefore x + \frac{2}{3}x^{\frac{1}{2}} = \frac{1}{3}$

$\therefore x + \frac{2}{3}x^{\frac{1}{2}} + \frac{1}{9} = \frac{1}{3} + \frac{1}{9} = \frac{4}{9}$

$\therefore x^{\frac{1}{2}} + \frac{1}{3} = \pm\frac{2}{3}$

$\therefore x^{\frac{1}{2}} = \pm\frac{2}{3} - \frac{1}{3} = \frac{1}{3}$ or -1

$$\therefore x = \frac{1}{9} \text{ or } 1.$$

N. B. The negative value of $x^{\frac{1}{2}}$ must be taken, because, in the operation the positive one is not introduced.

165. For the proof, see *Wood.*

$$\left.\begin{aligned} &\text{Let } \sqrt{1 - 4\sqrt{-3}} = x - y \\ &\text{Then } \sqrt{1 + 4\sqrt{-3}} = x + y \end{aligned}\right\} \therefore x^2 - y^2 = \sqrt{1 + 48} = \pm 7$$

Also $1 - 4\sqrt{-3} = x^2 - 2xy + y^2$, $\therefore x^2 + y^2 = 1$

Hence $x^2 = \dfrac{\pm 7 + 1}{2} = 4 \text{ or} -3$, $\therefore x = \pm 2 \text{ or } \pm\sqrt{-3}$

$$y^2 = \frac{1 \mp 7}{2} = -3 \text{ or } 4, \therefore y = \pm\sqrt{-3} \text{ or } \pm 2$$

$$\therefore \sqrt{1 - 4\sqrt{-3}} = x - y = \pm 2 \mp \sqrt{-3} \text{ or } \pm\sqrt{-3} \mp 2$$

166. For the proof, see *Wood.*

$$\left.\begin{aligned} &\text{Let } \sqrt{1 + 4\sqrt{-3}} = x + y \\ &\text{Then } \sqrt{1 - 4\sqrt{-3}} = x - y \end{aligned}\right\} \text{Hence, as in the preceding}$$

solution, $x = \pm 2 \text{ or } \pm\sqrt{-3}$

$$y = \pm\sqrt{-3} \text{ or } \pm 2$$

and $\sqrt{1 + 4\sqrt{-3}} = x + y = \pm 2 \pm \sqrt{-3}.$

167. Let x and y be the two numbers.

Then $xy = 24$

$x^2 + y^2 = 52$

$\therefore 2xy = 48$

$\therefore x^2 + 2xy + y^2 = 100$

and $x + y = \pm 10$

Also $x^2 - 2xy + y^2 = 4$

$\therefore x - y = \pm 2$

$\therefore 2x = \pm 12$ or $x = \pm 6$

$2y = \pm 8$ or $y = \pm 4$

168. Let $.346565\ldots = S$

Then $34.6565\ldots = 100\,S$

Again, let $.6565\ldots = S'$

Then $65.65\ldots = 100\,S'$

$\therefore\ 65 + S' = 100\,S'$

or $S' = \dfrac{65}{99}$

Hence $100\,S = 34 + \dfrac{65}{99} = \dfrac{3431}{99}$

$\therefore\ S = \dfrac{3431}{9900}$ the answer.

169. Let $.124343\ldots = S$

Then $12.4343\ldots = 100\,S$

Again, let $.4343\ldots = S'$

Then $43.43\ldots = 100\,S'$

or $43 + S' = 100\,S'$

$\therefore\ S' = \dfrac{43}{99}$.

Hence $100\,S = 12 + \dfrac{43}{99} = \dfrac{1231}{99}$

$\therefore\ S = \dfrac{1231}{9900}$.

170. Their angular velocities are as 5 to 60, or as 1 to 12. Let θ be the angular distance of the hour hand described from 12 o'clock. Then the $\angle$ described by the minute hand $= 12 \times \theta$.

And $12\,\theta = 2 \times 4$ R. $\angle + \theta$, (since the minute hand must gain two complete revolutions upon the other.)

Hence $11\,\theta = 8$. R. $\angle$

$\therefore\ \theta = \dfrac{8\,.\,\text{R.}\,\angle}{11} = \dfrac{24}{11}$ hours (in time, since R. $\angle = 3$ hours)

$= 2\dfrac{2}{11}$ hours

Now $\dfrac{2}{11}$h. $= \dfrac{120}{11}$min. $= 10\dfrac{10}{11}$ min.

and $\dfrac{10}{11}$ min. $= \dfrac{600}{11}$ sec. $= 54\dfrac{6}{11}$.

$\therefore$ they meet $10'\ 54''\ \frac{6}{11}$ after two o'clock.

171. Let x and y be the digits of units and tens respectively.

Then the number $= 10y + x$

and $\frac{10y + x}{x} = 27 + \frac{2}{x} = \frac{27x + 2}{x}$

$\frac{10y + x}{9} = 3x + \frac{2}{9} = \frac{27x + 2}{9}$

$\therefore 10y + x = 27x + 2 = 10y + x$

Hence there is only one independent equation for two unknown quantities, and any integral values, less than 10, of x and y which will satisfy the equation $10y + x = 27x + 2$, will give the answer required.

To find which values, $10y = 26x + 2$, $\therefore y = 2x + \frac{6x + 2}{10}$

Put $\frac{6x + 2}{10} = w \quad \therefore 6x + 2 = 10w$ and $x = w + \frac{2w - 1}{3}$

Put $2w - 1 = 3w'$.

$\therefore w = w' + \frac{w' + 1}{2}$. Put $w' + 1 = 2w''$. $\therefore w' = 2w'' - 1$.

Let $w'' = 1$, then $w' = 1$, $w = 1 + 1 = 2$, and $x = 3$ and $y = 8$.

All other values of w'' give $y > 10$.

Hence 83 is the only answer to the question.

172. $b = a + ar + ar^2$, when r is the common ratio.

$\therefore r^2 + r = \frac{b}{a} - 1$

$\therefore r^2 + r + \frac{1}{4} = \frac{b}{a} - 1 + \frac{1}{4} = \frac{4b - 3a}{4a}$

$\therefore r + \frac{1}{2} = \pm \sqrt{\frac{4b - 3a}{4a}}$

And $r = \pm \frac{2}{1} \sqrt{\frac{4b - 3a}{a}} - \frac{1}{2}$

$= \pm \frac{\sqrt{4b - 3a} - \sqrt{a}}{2\sqrt{a}}$.

173. The interior $\angle$ + 4 R. $\angle$. = 2. as many R. $\angle$ as the figure has sides.

Let x be the number of sides.

Then $120° + 125 + 130 + \ldots . = 2x \times 90 - 4 \times 90$

Now (since there are as many $\angle$ as sides), the number of terms in the series $= x$.

Hence $180\,x - 360 = (2\,a + \overline{n-1}\,b)\frac{n}{2}$ (a is first term, b the common term, &c.

$$=(240 + \overline{x-1}.5)\frac{x}{2}$$

$$= \frac{240x}{2} + \frac{5x^2}{2} - \frac{5x}{2}$$

$$\text{or } 5\,x^2 + 235x - 360x = -720$$

$$\text{or } x^2 - 25x = -144$$

$$\therefore x^2 - 25x + \frac{25^2}{4} = -144 + \frac{25^2}{4} = \frac{49}{4}$$

$$\therefore x - \frac{25}{2} = \pm\frac{7}{2}$$

$$\therefore x = \frac{25 \pm 7}{2} = 16 \text{ or } 9$$

When $x = 16$, some of the $\angle$ are salient or $> 180°$.

174. For the proof, see *Wood.*

Let $\sqrt{-2n\sqrt{-m^2}} = x - y$

Then $\sqrt{2n\sqrt{-m^2}} = x + y$

Hence $\sqrt{4n^2 m^2} = x^2 - y^2$

Also $-2n\sqrt{-m^2} = x^2 + y^2 - 2xy$

$$\therefore x^2 + y^2 = 0$$

$$\text{and } x^2 - y^2 = \pm 2mn$$

$$\therefore 2x^2 = \pm 2mn$$

$$2y^2 = \mp 2mn$$

$$\text{or } x = \pm\sqrt{\pm nm}$$

$$\text{and } y = \pm\sqrt{\mp nm}$$

Hence $\sqrt{-2n\sqrt{-m^2}} = \pm\sqrt{\pm nm} \mp \sqrt{\mp nm}$. i. e.,

the root required is either $\sqrt{nm} - \sqrt{-nm}$, or $-\sqrt{nm} + \sqrt{-nm}$.

175. Let m and n be the extremes.

Then $\frac{m+n}{2} = a$

$\sqrt{mn} = b$ (for $m : b :: b : n$)

$\frac{2mn}{m+n} = c$ (for $m : n :: m - c : c - n$)

Now, a is $> b$, if $\frac{m+n}{2}$ be $> \sqrt{mn}$ or $\overline{m+n}|^2$ be $> 4mn$ or $m^2 - 2mn + n^2$ be > 0 or $\overline{m-n}|^2$ be > 0. But $\overline{m-n}|^2$ being a perfect square is > 0.

$\therefore a$ is $> b$.

Again, b is $> c$, if $\sqrt{mn}$ be $> \frac{2mn}{m+n}$, or if $m+n$ be $> 2\sqrt{mn}$, or if $\frac{m+n}{2}$ be $> \sqrt{mn}$, or if an arithmetic be $>$ a geometric mean. But this has been proved to be the case.

$\therefore b$ is $> c$.

Otherwise,

Take in the same straight line two lines to represent the extremes, and upon their sum, as a diameter, describe a ⊙. From the point where the lines meet, draw a line ⊥ to them, meeting the circumference of the ⊙. Also from that point inflect a line = rad. of the ⊙, and produce it to meet the ⊙ on the other side of the diameter.

Then it may be shewn that the radius inflected $= a$, the ⊥ $= b$, and the part produced of the radius $= c$; and hence that a is $> b$, and b is $> c$.

176. Let a, $a+n$, $a + 2n$, &c., be the quantities wherein x, their common difference, is inconsiderable with respect to themselves.

Now $\frac{a + nx}{a + (n-1)x} = \frac{\left(\frac{a^n + n\,a^{n-1}x}{a^{n-1}}\right)}{\left(\frac{a^{n-1} + (n-1)\,a^{n-2}x}{a^{n-2}}\right)}$

$$= \frac{\frac{(a+x)^n}{a^{n-1}}}{\frac{(a+x)^{n-1}}{a^{n-2}}}, \text{ nearly} = \frac{a+x}{a} \text{ nearly.}$$

$\therefore$ any two consecutive terms of the quantities have the same ratio as $a + x$ and a; or the quantities have a common ratio, and are $\therefore$ in geometric progression.

177. The number of combinations, taken 2 and 2 together $= n \cdot \frac{n-1}{2}$

$$\ldots\ldots 3 \text{ and } 3 \ldots\ldots = n \cdot \frac{n-1}{2} \cdot \frac{n-2}{3}$$

$$\ldots\ldots 4 \text{ and } 4 \ldots\ldots = n \cdot \frac{n-1}{2} \cdot \frac{n-2}{3} \cdot \frac{n-3}{4}$$

$$\&c. = \&c.$$

$$\ldots\ldots\ldots n \text{ and } n = 1$$

And when taken separately, the number $= n$

$\therefore$ the total number required $= n + n \cdot \frac{n-1}{2} + n \cdot \frac{n-1}{2} \cdot \frac{n-2}{3} + \ldots\ldots n \cdot \frac{n-1}{2} + n + 1 = (1+1)^n - 2 + 1 = 2^n - 1.$

In the application, as there are six different coins, the number of different sums = total number of combinations that can be formed of six things $= 2^6 - 1 = 64 - 1 = 63$.

178. $\frac{.7584}{316} = \frac{7584}{10000 \times 316} = \frac{7584}{3160000} =$ (by actual division) $.0024$

Again, let $.72323\ldots\ldots = S$

Then $7.2323\ldots\ldots = 10 \cdot S$

Also, put $.2323\ldots\ldots = S'$

Then $23.23\ldots\ldots = 100\,S'$

Or $23 + S' = 100\,S'$

$\therefore\ S' = \frac{23}{99}$

Hence $10\,S = 7 + \frac{23}{99} = \frac{716}{99} \quad \therefore\ S = \frac{716}{990} = \frac{358}{495}$

$\therefore$ the sum required $= 5 + \dfrac{358}{495}$

179. $x - a)\, x^3 - nax^2 + na^2x - a^3\, (x^2 - \overline{n-1}\,ax + a^2$

$$\begin{array}{l} x^3 - ax^2 \\ \hline -(n-1)\,ax^2 + n\,a^2x \\ -(n-1)\,.\,ax^2 + (n-1)\,a^2x. \\ \hline \qquad a^2x - a^3 \\ \qquad a^2x - a^3 \\ \hline \qquad \cdot \quad \cdot \cdot \\ \hline \end{array}$$

180. The number of combinations when taken 4 and 4 together, $= n\,.\,\dfrac{n-1}{2}.\,\dfrac{n-2}{3}.\,\dfrac{n-3}{4}$, and the number when taken 2 and 2 together $= n\,.\,\dfrac{n-1}{2}$.

Hence $n.\dfrac{n-1}{2}.\,\dfrac{n-2}{3}.\,\dfrac{n-3}{4} : n\,.\,\dfrac{n-1}{2} :: 15 : 2$

Or $\overline{n-2}.\ \overline{n-3} : 12 :: 15 : 2$

$\therefore 2\,.\,\overline{n-2}.\ \overline{n-3} = 180$

$\therefore n^2 - 5n + 6 = 90$

$\therefore n^2 - 5n = 90 - 6 = 84$

$\therefore n^2 - 5n + \dfrac{25}{4} = 84 + \dfrac{25}{4} = \dfrac{361}{4}$

$\therefore n - \dfrac{5}{2} = \pm\dfrac{19}{2}$

$\therefore n = \dfrac{\pm 19 + 5}{2} = 12$ or -7.

Since 12 is supposed positive by the nature of the question, 12 is the only answer.

181. Let x be the greater number, y the less.

Then $\left.\begin{array}{l} \dfrac{2}{3}y + \dfrac{x}{4} = 7 \\ \text{and } y - \dfrac{x}{3} = 2 \end{array}\right\}$

From the first equation

$y + \dfrac{3x}{8} = \dfrac{21}{2}$

and $y - \dfrac{x}{3} = 2$

$\therefore \dfrac{17}{24}x = \dfrac{17}{2}$

$\therefore x = \dfrac{24}{2} = 12$

and $y = \frac{x}{3} + 2 = 4 + 2 = 6$

182. Let $\sqrt{-3 + 6\sqrt{-2}} = x + y$

Then $\sqrt{-3 - 6\sqrt{-2}} = x - y$

and $\sqrt{9 + 72} = x^2 - y^2$

Also $-3 + 6\sqrt{-2} = x^2 + 2xy + y^2$

$\therefore\ x^2 + y^2 = -3$

and $x^2 - y^2 = \pm 9$

$\therefore\ x = \pm\sqrt{3}$ or $\pm\sqrt{-6}$

$y = \pm\sqrt{-6}$ or $\pm\sqrt{3}$

$\therefore$ the root required $= \pm\sqrt{3} \pm \sqrt{-6}$

183. First, $(a \pm b\sqrt{-1}) \pm (a' \pm b'.\sqrt{-1}) = (a \pm a') \pm (b \pm b')\sqrt{-1}$, which is of the form $A \pm B\sqrt{-1}$. This proves the cases of addition and subtraction.

Again, $(a \pm b\sqrt{-1}).(a' \pm b'\sqrt{-1}) = aa' \pm ab'\sqrt{-1} \pm a'b\sqrt{-1} - bb' = aa' - bb' \pm (ab' + a'b)\sqrt{-1}$, which is also of the form $A \pm B\sqrt{-1}$.

Again, $\frac{a \pm b\sqrt{-1}}{a' \pm b'\sqrt{-1}} = \frac{(a \pm b\sqrt{-1}) \times (a' \mp b'\sqrt{-1})}{(a' \pm b'\sqrt{-1}) \times (a' \mp b'\sqrt{-1})}$

$$= \frac{aa' - bb' \mp (ab' - a'b)\sqrt{-1}}{a'^2 + b'^2}$$

$= \frac{aa' - bb'}{a'^2 + b'^2} \mp \frac{ab' - a'b}{a'^2 + b'^2}\sqrt{-1}$, which is of the form $A \pm B\sqrt{-1}$.

The same process will evidently apply, whatever be the number of factors in the sum, difference, product, or quotient. Hence, since involution is nothing more than the continued multiplication of a quantity by itself, the proposition is true also for involution.

184. $$2a - x)\ 1\ (\frac{1}{2a} + \frac{x}{4a^2} + \frac{x^2}{8a^3} + \ldots\ldots$$

$$\frac{1 - \frac{x}{2a}}{\quad\frac{x}{2a}}$$

$$\frac{\frac{x}{2a} - \frac{x^2}{4a^2}}{\qquad\frac{x^2}{4a^2}}$$

$$\frac{\frac{x^2}{4a^2} - \frac{x^3}{8a^3}}{\qquad\frac{x^3}{8a^3}}$$

The n^{th} term in the quotient is evidently

$\frac{x^{n-1}}{(2a)^n}$ and the corresponding remainder $\frac{x^n}{(2a)^n}$

Otherwise,

$\frac{1}{2a - x} = (2a - x)^{-1} = (2a)^{-1} \times \left(1 - \frac{x}{2a}\right)^{-1}$ which being expanded by the binomial theorem, will give the above result.

185. For the proof of the rule, see *Wood.*

$$x^4 - y^4)\ x^8 + x^6 y^2 + x^2 y + y^3\ (x^4 + x^2 y + y^3$$
$$x^8 - x^4 y^4$$
$$x^6 y^2 + x^4 y^4 + x^2 y + y^3 \text{ dividing by } y$$
$$x^6 y + x^4 y^3 + x^2 + y^2$$
$$x^6 y - x^2 y^5$$
$$x^4 y^3 + x^2\ (y^5 + 1) + y^2$$
$$x^4 y^3 - y^7$$
$$x^2 \,.\, (y^5 + 1) + y^2 .\, (y^5 + 1)$$
$$x^2 + y^2$$

which, dividing $x^4 - y^4$ without a remainder is the greatest common measure required.

186.

$$\begin{array}{r|l}
 & \dot{3}1\dot{5}\,.\,\dot{2}7\dot{1}\ (\ 17.753\ldots\ldots \\
 & 1 \\
27\ ) & 215 \\
 & 189 \\
347\ ) & 2627 \\
 & 2429 \\
3545\ ) & 18810 \\
 & 17725 \\
35503\ ) & 108500 \\
 & 106509 \\
 & 1991
\end{array}$$

The greatest number of n figures is 999 .... n terms.

least . . . . . 1000 .... n terms.

Now, $(999 \ldots\ldots n \text{ terms})^2 = (10^n - 1)^2 = 10^{2n} - 2.10^n + 1 = 100 \ldots (2n + 1)$ terms $-$ a quantity greater than unity.

$\therefore$ $(99 \ldots\ldots n \text{ terms})^2$, cannot consist of so many as $(2n + 1)$ terms; or, if there be n figures in the root, there cannot be more than $2n$ figures in the power.

Again, $(100 \ldots\ldots n \text{ terms})^2 = 100 \ldots\ldots (2\,n - 1)$ terms.

Or the square of the least number consisting of n figures consists of $(2n - 1)$ figures. $\therefore$ if there be n figures in the root, there cannot be less than $(2n - 1)$ figures in the power.

187. Let a and b represent the two quantities, A the arithmetic, G the geometric, and H the harmonic mean.

Then $\dfrac{a+b}{2} = \mathrm{A}$

$\sqrt{ab} = \mathrm{G}$ $(\because a : \mathrm{G} :: \mathrm{G} : b)$

$\dfrac{2\,ab}{a+b} = \mathrm{H}$ $(\because a : b :: a - \mathrm{H} : \mathrm{H} - b)$

Now $\dfrac{a+b}{2} : \sqrt{ab} :: \sqrt{ab} : \dfrac{2\,ab}{a+b}$

or A : G :: G : H, $\therefore$ G is a mean proportional between A and H.

Now, let $\dfrac{a+b}{2} = \sqrt{ab} + x$

$$\therefore x = \frac{a+b}{2} - \sqrt{ab} = \frac{a+b-2\sqrt{ab}}{2}$$

$$= \frac{(a^{\frac{1}{2}} - b^{\frac{1}{2}})^2}{2} = \text{a positive quantity.}$$

$$\therefore \frac{a+b}{2} \text{ is } > \sqrt{ab} \text{ or A is } > \text{G}$$

and since A is to G as G is to H, G is > H.

$\therefore$ *à fortiori*, A is > H.

$\therefore$ the arithmetic mean is the greatest.

188. $(a^2 - b^2)^{\frac{1}{2}} = a.\left(1 - \frac{b^2}{a^2}\right)^{\frac{1}{2}}$

$$= a\left(1 - \tfrac{1}{2}\cdot\frac{b^2}{a^2} + \tfrac{1}{2}\cdot\frac{\frac{1}{2}-1}{2}\frac{b^4}{a^4} - \tfrac{1}{2}\cdot\frac{\frac{1}{2}-1}{2}\,\frac{\frac{1}{2}-2}{3}\,\frac{b^6}{a^6} + \ldots\right.$$

$$= a\left(1 - \tfrac{1}{2}\cdot\frac{b^2}{a^2} - \frac{1}{8}\cdot\frac{b^4}{a^4} - \frac{1}{16}\cdot\frac{b^6}{a^6} - \&c.\right)$$

$$= a - \tfrac{1}{2}\cdot\frac{b^2}{a} - \frac{1}{8}\cdot\frac{b^4}{a^3} - \frac{1}{16}\cdot\frac{b^6}{a^5} - \&c.$$

189. For the proof, see *Wood.*

$$\sqrt{7 - 2\sqrt{3}} = \sqrt{\frac{a + \sqrt{a^2 - b}}{2}} + \sqrt{\frac{a - \sqrt{a^2 - b}}{2}}$$

where $a = 7$, and $\sqrt{b} = -2\sqrt{3}$

Now $a^2 - b = 49 - 12 = 37$, which not being a perfect square, the root cannot be expressed by a binomial surd.

190. For the investigation, see *Wood.*

Let $\sqrt{2 + 2\sqrt{1 - a^2}} = x + y$, then $\sqrt{2 - 2\sqrt{1 - a^2}} = x - y$

and $\sqrt{4 - 4 + 4a^2} = x^2 - y^2$.

Also $2 + 2\sqrt{1 - a^2} = x^2 + y^2 + 2xy$

$\therefore 2 = x^2 + y^2$

and $\pm 2a = x^2 - y^2$

$$\left.\begin{array}{l} \therefore x = \pm\sqrt{1 \pm a} \\ \quad y = \mp\sqrt{1 \mp a} \end{array}\right\} \therefore \text{ the root required} = \pm\sqrt{1 \pm a} \pm \sqrt{1 \mp a}$$

where such signs may be taken, as give, when the root is squared, the original binomial.

191. Let $\sqrt{4\,mn + 2.(m^2 - n^2)\sqrt{-1}} = x + y$

Then $\sqrt{4\,mn - 2\,(m^2 - n^2).\ \sqrt{-1}} = x - y$

Hence $\sqrt{16\,m^2n^2 + 4\,(m^2 - n^2)^2} = x^2 - y^2 = \sqrt{4m^4 + 4n^4 + 8m^2n^2}$

and $4\,mn + 2.\,(m^2 - n^2)\sqrt{-1} = x^2 + y^2 + 2xy$

$\therefore\ 4\,mn = x^2 + y^2$

and $\pm\,2m^2 \pm 2n^2 = x^2 - y^2$

$\therefore\ x^2 = \pm\,m^2 + 2\,mn \pm n^2 = (m + n)^2$ or $-\,(m - n)^2$

$y^2 = \mp\,m^2 + 2mn \mp n^2 = -\,(m - n)^2$ or $(m + n)^2$

$\therefore\ x = \pm\,(m + n)$ or $\pm\,(m - n).\sqrt{-1}$

$y = \pm\,(m - n)\sqrt{-1}$ or $\pm\,(m + n)$

Hence the root required $= x + y = (m + n) + (m - n)\sqrt{-1}$, or it $= -\,(m + n) - (m - n)\sqrt{-1}$. The other two values correspond to $\sqrt{-1}$ in the binomial being taken negatively.

192. $\sqrt[4]{-a} \times \sqrt[4]{-b} = a^{\frac{1}{4}} \times (-1)^{\frac{1}{4}} \times (-1)\,b^{\frac{1}{4}} = (ab)^{\frac{1}{4}} \times (-1)^{\frac{1}{2}}$ which is impossible.

Again $\sqrt[3]{-a} \times \sqrt[3]{-b} = a^{\frac{1}{3}} \times (-1)^{\frac{1}{3}} \times (-1)^{\frac{1}{3}} \times b^{\frac{1}{3}} = (ab)^{\frac{1}{3}} \times (-1)^{\frac{2}{3}} = -\,(ab)^{\frac{1}{3}}$ which is possible.

$\therefore$ the former is impossible, and the latter possible.

193. $x - a)\ x^3 - px^2 + qx - r\ (x^2 - (p - a)x + (q - a.\overline{p - a}) + \&c.$

$$x^3 - ax^2$$
$$\overline{\qquad - (p - a)\,x^2 + qx \qquad}$$
$$- (p - a)\,x^2 + (p - a)\,.\,a\,x$$
$$\overline{\qquad (q - a\,.\,\overline{p - a})\,x - r \qquad}$$
$$(q - a\,.\,\overline{p - a})\,x - a\,.\,(q - a\,.\,\overline{p - a.})$$
$$\overline{\qquad\qquad a\,.\,(q - a\,.\,\overline{p - a}) - r \qquad}$$

194. If $a : b :: c : d$

Then $a : c :: b : d :: \pm\,m\,b : \pm\,m\,d$

$\therefore a : \pm mb :: c : \pm md$

$\therefore a \pm mb : \pm mb :: c \pm md : \pm md$

$\therefore a \pm mb : c \pm md :: \pm mb : \pm md :: b : d$

$\therefore a \pm mb : b :: c \pm md : d$

The number of combinations of the letters in the word *Baccalaureus* is evidently the same as that of the divisors in the product

$$B \times a^2 \times c^2 \times l \times u^2 \times r \times \overset{e}{\cancel{a}} \times s,$$

or it is (*Barlow's Theory of Numbers*,)

$(1 + 1) \times (3 + 1) \times (2 + 1) \times (1 + 1) \times (2 + 1) \times (1 + 1) \times (1 + 1) \times (1 + 1) - 5$, or 1147.

196. Let a, $a + d$, $a + 2d$ be the three quantities.

Then $\frac{a + d}{a}$ is $> \frac{a + 2d}{a + d}$ if $\frac{a^2 + 2ad + d^2}{a . (a + d)}$ be $> \frac{a^2 + 2ad}{a . (a + d)}$

or if $a^2 + 2ad + d^2$ be $> a^2 + 2ad$, which is the case by d^2.

$\therefore a + d$ has to a greater ratio than $a + 2d$ has to $a + d$.

Otherwise.

A ratio of greater inequality $\frac{a + d}{a}$ is diminished, by adding the same quantity (d) to both its terms.

197. Let $\frac{m}{A^2}$, n A be the two quantities, m and n being constant.

Then $\frac{m}{A^2} - n A = \frac{m - n A^3}{A^2}$ (a)

and $\frac{m}{A^2} + n A = \frac{m + n A^3}{A^2}$ (b)

Hence, it is evident that when A increases in the remainder (a), (a) decreases in the numerator, and increases in the denominator; it $\therefore$ decreases from the numerator, as well as from the denominator, *i. e.* it decreases in a higher ratio than the inverse square of A. Also when A decreases, the numerator of (a) is in-

creased, and the denominator diminished, *i.e.* (a) is increased both by the variation in the numerator and denominator, or it increases in a higher ratio than the inverse square of A. $\therefore$ a varies in a higher ratio, &c.

The same kind of reasoning will apply in the remaining case (b).

198. Let $.1234141 \ldots = S$

Then $123.4141 \ldots = 1000\, S$

Again, put $.4141 \ldots = S'$

Then $41 + S' = 100\, S'$

$$\therefore S' = \frac{41}{99}$$

$$\text{Hence } S = \frac{123 + \frac{41}{99}}{1000} = \frac{12218}{99000} = \frac{6109}{49500}$$

$$\text{Now } \frac{6109}{49500}£. = \frac{6109 \times 20}{49500}s. = 2\,\frac{1159}{2475}s. = 2s.\ 5d.\ 2\,\frac{394}{825}q.$$

which is the value required.

Otherwise.

$$\begin{array}{r} .123414141 \ldots \\ 20 \\ \hline 2.46828282 \ldots \\ 12 \\ \hline 5.61939393 \ldots \\ 4 \\ \hline 2.47757575 \ldots \end{array}$$

Hence the value is $2s.\ 5d.\ 2\ .4775757q. \ldots$ which is $=$ the former.

199. The reciprocals of quantities in harmonical progression are in arithmetic progression.

Let $\therefore \frac{1}{a}, \frac{1}{a+d}, \ldots \frac{1}{a+md}, \frac{1}{a+(m+1)d} \ldots$ be the harmonic progression.

Then $\frac{1}{a} \times \frac{1}{a+d} : \frac{1}{a+md} \times \frac{1}{a+(m+1)d} :: \frac{1}{a} - \frac{1}{a+d} :$ $\frac{d}{(a+md) \times (a+\overline{m+1}.d)}$, &c.

Now $\frac{1}{a+md} - \frac{1}{a+(m+1)d} = \frac{d}{(a+md) \times (a+\overline{m+1d})}$

$\therefore \frac{1}{a} \times \frac{1}{a+d} : \frac{1}{a+md} \times \frac{1}{a+\overline{m+1}d} :: \frac{1}{a} - \frac{1}{a+d} :$

$\frac{1}{a+md} - \frac{1}{a+\overline{m+1}d}$ which answers the conditions of the problem.

200. The sum of an arithmetic series = the sum of the first and last terms multiplied by half the number of terms.

Hence $20 \times \frac{n+2}{2} - 20 = 10n =$ sum of the n means

and $(2 + (n-2)d)\frac{n-1}{2} - 1 = n-2 + \frac{(n-1)(n-2)}{2}d =$ sum of the first $(n-2)$ of them (d being the common difference.)

$\therefore$ by problem $\dfrac{10n}{n-2+\frac{(n-1).(n-2)}{2}d} = \frac{5}{3}$

or $60n = 10n - 20 + 5(n-1).(n-2)d$

or $d = \frac{50n+20}{5.(n-1).(n-2)} = \frac{10n+4}{(n-1).(n-2)}$

$\therefore$ the means required are

$1 + \frac{10n+4}{(n-1).(n-2)},\ 1 + 2 \times \frac{10n+4}{(n-1).(n-2)}$, &c.

or $\frac{n^2+7n+6}{(n-1).(n-2)},\ \frac{n^2+17n+10}{(n-1).(n-2)}$, &c.

And the p^{th} mean $= \frac{n^2+(10p-3)n+2+4p}{(n-1).(n-2)}$

201. Let a, ar, ar^2, ar^3, be the quantities in geometrical progression.

Then $a + ar^3$ is $>$ $ar + ar^2$

if $1 + r^3$ be $> r + r^2$, or $r.(1+r)$

or if $1 - r + r^2 > r$

or if $1 - 2r + r^2 = (1-r)^2$ be > 0

But every square is positive, and $\therefore > 0$

$\therefore a + ar^3$ is $> ar + ar^2$, &c.

Otherwise,

Assume $r = 1 + x$

$$\left.\begin{aligned}\text{Then } a + ar^3 &= a + a + 3ax + 3ax^2 + ax^3\\ ar + ar^2 &= a + ax + a + 2ax + ax^2\end{aligned}\right\}$$ when it is evident the former is the greater by $2ax^2 + ax^3$.

202. Let x, y, z, w, be the respective numbers required.

$$\left.\begin{aligned}\text{Then } x+\frac{y}{2} &= 12\\ y + \frac{z}{3} &= 12\\ z + \frac{w}{4} &= 12\\ \text{and } w - \frac{y}{2} - \frac{z}{3} - \frac{w}{4} &= 12\end{aligned}\right\}$$

$$\left.\begin{aligned}\text{Hence } x + y + z + w &= 48\\ 2x + y &= 24\\ 3y + z &= 36\\ 4z + w &= 48\end{aligned}\right\}$$

$$\begin{aligned}\text{Again, } x - 3z + y &= 0\\ 2x \qquad + y &= 24\\ \hline \therefore x + 3z &= 24\end{aligned} \qquad \begin{aligned}\text{and } 6x + 3y &= 72\\ z + 3y &= 36\\ \hline \therefore 6x - z &= 36\end{aligned}$$

$$\begin{aligned}\text{Also } 6x + 18z &= 144\\ \text{and } 6x - z &= 36\\ \hline 19z &= 108\end{aligned}$$

$$\therefore z = \frac{108}{19} = 5\frac{18}{19}$$

$$x = \frac{36 + z}{6} = \frac{41\frac{18}{19}}{6} = 6\frac{18}{19}$$

$$y = 24 - 2x = 24 - 12\frac{36}{19} = 10\frac{2}{19}$$

$$w = 48 - 4z = 48 - 20\frac{52}{19} = 25\frac{5}{19}$$

Otherwise,

Let $x =$ the number D had.

Then $12 - \frac{x}{4} =$ C's number.

$$12 - \frac{12 - \frac{x}{4}}{3} = 8 + \frac{x}{12} = \text{B's number.}$$

$$12 - \frac{8 + \frac{x}{12}}{2} = 8 - \frac{x}{24} = \text{A's number.}$$

But it is evident that the whole number given is $= 48$.

$$\therefore x + 12 - \frac{x}{4} + 8 + \frac{x}{12} + 8 - \frac{x}{24} = 48$$

$$\text{or } x + \frac{x}{12} - \frac{x}{4} - \frac{x}{24} = 48 - 28 = 20$$

$\therefore 24x + 2x - 6x - x = 480$

or $19x = 480$

$\therefore x = 25\frac{5}{19}$, and hence the other numbers are easily found.

203. Put $\sqrt{ab - d^2 + 4c^2 \pm 2\sqrt{4abc^2 - abd^2}} = x \pm y$

Then $\sqrt{ab - d^2 + 4c^2 \mp 2\sqrt{4abc^2 - abd^2}} = x \mp y$

Hence $\sqrt{(ab - d^2 + 4c^2)^2 - 4(4abc^2 - abd^2)} = x^2 - y^2$

or $\sqrt{a^2b^2 + d^4 + 16c^4 - 8c^2d^2 - 8abc^2 + 2abd^2} = x^2 - y^2$

or $\pm(ab + d^2 - 4c^2) = x^2 - y^2$

Again, $ab - d^2 + 4c^2 \pm 2\sqrt{4abc^2 - abd^2} = x^2 \pm 2xy + y^2$

$$\left.\begin{aligned} \therefore ab - d^2 + 4c^2 &= x^2 + y^2 \\ \text{and } \pm ab \pm d^2 \mp 4c^2 &= x^2 - y^2 \end{aligned}\right\}$$

$$\left.\begin{aligned} \therefore x^2 &= ab, \text{ or } 4c^2 - d^2 \\ y^2 &= 4c^2 - d^2, \text{ or } ab \end{aligned}\right\}$$

$$\therefore\ x = \pm\sqrt{ab},\ \text{or}\ \pm\sqrt{4c^2 - d^2}$$

$$y = \pm\sqrt{4c^2 - d^2},\ \text{or}\ \pm\sqrt{ab}$$

and the root required is $\sqrt{ab} \pm \sqrt{4c^2 - d^2}$, or $-\sqrt{ab} \mp \sqrt{4c^2 - d^2}$.

or $\pm\sqrt{ab} + \sqrt{4c^2 - d^2}$, or $\mp\sqrt{ab} - \sqrt{4c^2 - d^2}$.

204. Let a be the first term, a the common difference, $2n$ the number of terms.

Then the sum S, of $2n$ terms $= (2a + (2n-1).a)\dfrac{2n}{2} = n.(2n+1) \times a$

S'..... n .... $= (2a + (n-1)a)\dfrac{n}{2} = \dfrac{n}{2} \times (n+1) \times a$

and the last term of $S' = na$.

Now $n.(2n+1).a = n.(n+1).a + n^2 a$

$$= 2 \times \frac{n}{2}(n+1) + n^2 a$$

$$= 2S' + n^2 a = 2S' + 2S' - \frac{n}{2}a$$

$= 4S' - \frac{1}{2}$ the last term.

$\therefore S$ (the sum of $2n$ terms) $= 4S'$ (the sum of n terms) $- \frac{1}{2}$ the last term.

205. For the investigation, see *Wood*.

```
177 ) 2982 ( 16
      177
      ----
      1212
      1062
      ----
       150 ) 177 ( 1
             150
             ---
              27 ) 150 ( 5
                   135
                   ---
                    15 ) 27 ( 1
                         15
                         --
                         12 ) 15 ( 1
                              12
                              --
                               3 ) 12 ( 4
                                   12
                                   --
                                    .
```

Hence 3 is the greatest common measure of the two quantities.
∴ their least common multiple

$$= \frac{177 \times 2982}{3} = 177 \times 994 = 175938.$$

206. Let p be the first term, d the common difference.
Then $p + (m-1)d$, $p + (n-1)d$, are the m^{th} and n^{th} terms respectively.

$$\therefore \left.\begin{array}{l} p + (m-1)d = a \\ p + (n-1)d = b \end{array}\right\} \text{ whence } d = \frac{a-b}{m-n}, \text{ and } p = a - \frac{m-1}{m-n}$$

$$\times (a-b) = \frac{(1-n)a - (1-m)b}{m-n}$$

Hence the x^{th} term $= p + (x-1).d$

$$= \frac{(1-n)a-(1-m)b+(x-1).(a-b)}{m-n} = \frac{(x-n)a-(x-m)b}{m-n}$$

207. Let $.qpp\ldots. = S$

Then $q.pp\ldots. = 10^m S'$ (since $.q = \frac{q}{10^m}$ from the nature of decimals.)

Again, put $.pp\ldots. = S'$

Then $p.pp\ldots. = 10^n S'$

Or $p + S' = 10^n S'$

$$\therefore S' = \frac{p}{10^n - 1}$$

$$\text{Hence } S = \frac{q+S'}{10^m} = \frac{q + \frac{p}{10^n - 1}}{10^m} = \frac{(10^n-1)q+p}{10^m.(10^n-1)}$$

$$= \frac{(999\ldots. n \text{ terms})\, q + p}{100\ldots.(m+1) \text{ terms} \times 999\ldots., n \text{ terms}}$$

208. Let $\sqrt{a + x + \sqrt{2ax + x^2}} = v + y$ ($\sqrt{2ax + x^2}$ is considered a surd)

$$\text{Then } \sqrt{a + x - \sqrt{2ax + x^2}} = v - y$$

$\therefore \sqrt{(a+x)^2 - (2ax+x^2)} = v^2 - y^2$

or $\pm a = v^2 - y^2$

Again, $a + x + \sqrt{2ax + x^2} = v^2 + 2vy + y^2$

$\therefore a + x = v^2 + y^2$ (equating rationals)

$\pm a = v^2 - y^2$

$$\left.\begin{aligned} \therefore v^2 &= a + \frac{x}{2} \text{ or } \frac{x}{2} \\ y^2 &= \frac{x}{2} \text{ or } a + \frac{x}{2} \end{aligned}\right\} \quad \begin{aligned} \therefore v &= \pm\sqrt{\frac{2a+x}{2}} \text{ or } \pm\sqrt{\frac{x}{2}} \\ y &= \pm\sqrt{\frac{x}{2}} \text{ or } \pm\sqrt{\frac{2a+x}{2}} \end{aligned}$$

and the root required is $\pm\sqrt{\frac{2a+x}{2}} \pm \sqrt{\frac{x}{2}}$ i. e. $\sqrt{\frac{2a+x}{2}}$ $+ \sqrt{\frac{x}{2}}$ or $-\sqrt{\frac{2a+x}{2}} - \sqrt{\frac{x}{2}}$.

209. Let a be the first, or least term; d the common difference.

Then (S) the sum of n terms $= (2a + (n-1)\,a)\frac{n}{2} = \frac{n.(n+1)}{2}.a$ $= (n+1) \times \frac{na}{2} = (n+1) \times \frac{n^{\text{th}} \text{ term}}{2} = (n+1) \times (\frac{1}{2} n^{\text{th}} \text{ term})$.

210. Since A $\propto$ B, and B $\propto$ C

B : b :: C : c :: A : a

$\therefore m$B : mb :: nC : nc

Hence, mB : nC :: mb : nc

and mB $\pm n$C : mB :: $mb \pm nc$: mb

$\therefore m$B $\pm n$C : $mb \pm nc$ :: mB : mb :: B : b :: A : a

$\therefore$ A $\propto$ mB $\pm n$C

211. $\left\{\sqrt{a+b\sqrt{-1}} + \sqrt{a-b\sqrt{-1}}\right\}^2 = (a+b\sqrt{-1})$ $+ 2\sqrt{a+b\sqrt{-1}} \times \sqrt{a-b\sqrt{-1}} + (a-b\sqrt{-1}) = 2a +$ $2\sqrt{(a+b\sqrt{-1}) \times (a-b\sqrt{-1})} = 2a + 2\sqrt{a^2+b^2}$, since $\sqrt{-1} \times \sqrt{-1} = -1$

212. The coefficient of the p^{th} or general term

$$= \frac{n \,.\, \overline{n-1} \,.\, \overline{n-2} \ldots . n-p+2}{1 \,.\, 2 \,.\, 3 \ldots . n-p+1}$$

Now, when n is an even number, the middle term must be the $\left(\frac{n}{2}+1\right)^{\text{th}}$. Hence, substituting $\frac{n}{2}+1$ for p in the coefficient of the general term, we get the coefficient of the middle

$$\text{term} = \frac{n \,.\, (n-1) \,.\, (n-2) \ldots . \frac{n}{2}+1}{1 \,.\, 2 \,.\, 3 \ldots . \frac{n}{2}}$$

Again, $n \,.\, (n-1) \,.\, (n-2) \ldots . \frac{n}{2}+1 \times \frac{n}{2} \ldots . 4 . 3 . 2 . 1 = (n-1) \,.\, (n-3) \times \ldots . 5 \times 3 \times 1 \times (n-2) \,.\, (n-4) \ldots . 6 \times 4 \times 2 = 1 \times 3 \times 5 \ldots . (n-1) \times 2^{\frac{n}{2}} \times 1 . 2 . 3 \ldots . \frac{n}{2}$

$$\therefore n \,.\, (n-1) \ldots . \left(\frac{n}{2}+1\right) = 1 \times 3 \times 5 \ldots . (n-1) \times 2^{\frac{n}{2}}$$

Also, the index of x in the $\left(\frac{n}{2}+1\right)^{\text{th}}$ term is $\frac{n}{2}$

$$\therefore \text{the middle term of } (1+x)^n, \text{ \&c.} = 2^{\frac{n}{2}} . \frac{1 . 3 \ldots . (n-1)}{1 . 2 . 3 \ldots . \frac{n}{2}} . x^{\frac{n}{2}}$$

213. The reciprocals of quantities in harmonic progression, are in arithmetic.

Let $\therefore$ the means required be $\frac{1}{\frac{1}{3}+d}$, $\frac{1}{\frac{1}{3}+2d}$

Hence $\frac{1}{12} = \frac{1}{3} + 3d$

$$\therefore 3d = \frac{1}{12} - \frac{1}{3} = \frac{-3}{12}$$

and $d = -\frac{1}{12}$

Hence, the means are $\frac{1}{\frac{1}{3}-\frac{1}{12}}$ and $\frac{1}{\frac{1}{3}-\frac{1}{6}}$ or 4 and 6

214. $n^3 = 2n^2\,\frac{n}{2} = 2n.(n-1+1).\frac{n}{2} = (2n\,.\,\overline{n-1} + 2n)\frac{n}{2}$

$= (2n.(n-1) + 2 + (n-1).2)\frac{n}{2} =$ an arithmetic series whose first term is $n.(n-1)+1$, common difference 2, and number of terms n. Now $n.(n-1)+1$ is an odd number, since whether n is even or odd it is of the form $2m+1$.

$\therefore n^3 = \overline{n.(n-1)+1} + \overline{n.(n-1)+3} + \ldots\ldots n.(n-1)+1+(n-1).2$

$= \overline{n^2-n+1} + \overline{n^2-n+3} + \ldots\ldots \overline{n^2+n-1}$

215. The ratio required $= \frac{1}{2} \times \frac{\sqrt{32}}{25} = \sqrt{\frac{32}{4}} \times \frac{1}{25}$

$= \frac{\sqrt{8}}{25} = \frac{2\sqrt{2}}{25}.$

216. Generally $S = (2a + (n-1)\,b)\frac{n}{2}$ $\left(\begin{array}{l} a = \text{first term,} \\ b = \text{common difference,} \\ n = \text{number of terms.}\end{array}\right.$

Hence $5 = (22 + (n-1)(-5))\,\frac{n}{2}$

$= 11n - \frac{5n^2}{2} + \frac{5n}{2} = \frac{27n - 5n^2}{2}$

$\therefore 5n^2 - 27n = -10, \quad \therefore n^2 - \frac{27}{5}n = -2$

$\therefore n^2 - \frac{27}{5}n + \left(\frac{27}{10}\right)^2 = \left(\frac{27}{10}\right)^2 - \frac{200}{10^2} = \frac{529}{10^2}$

$\therefore n = \frac{27 \pm 23}{10} = 5$ or $\frac{2}{5}$. 5 is the only answer, $\frac{2}{5}$ not being a whole number, according to the conditions of the problem.

217. Let a be the first term, d the common difference.

Then the p^{th} term $\left.\begin{array}{l} P = a + (p-1)d \\ Q = a + (q-1)d \end{array}\right\}$ p and q are supposed given.

Hence $(p-q)d = P-Q$

$$\therefore d = \frac{P-Q}{p-q}$$

and $a = P - (p-1)d = P - (p-1).\,\dfrac{P-Q}{p-q}$

$$= \frac{(p-1).Q-(q-1).P}{p-q}$$

Now, the sum of n terms of the series $= (2a + \overline{n-1}d)\dfrac{n}{2}$

$$= \frac{(2p-1)\,Q - (2q-1)\,P + (n-1)\,P - (n-1)\,Q}{p-q} \times \frac{n}{2}$$

$$= \frac{(2p-n-1)\ Q - (2q-n-1)\,P}{p-q} \times \frac{n}{2}$$

218. Let $x =$ the prime cost in pounds.

Then $100 : 20 :: x : \dfrac{x}{5} =$ the whole gain.

Hence $x + \dfrac{x}{5} = 1000$

$\therefore 6x = 5000$

or $x = 833.333 \ldots .\ £$

$$\begin{array}{lll} £. & s. & d. \\ = 833. & 6. & 8. \end{array}$$

219. In the solution of the general quadratic $x^2 + px + q = 0$ we arrive at the form $x^2 + \dfrac{p}{2} = \pm\dfrac{\sqrt{p^2-4q}}{2}$, whence it appears, that there being two values is owing to the property of the square root of a quantity being positive or negative.

Otherwise.

Let a be a root.

$$\left.\begin{array}{l}\text{Then } a^2 + pa + q = 0 \\ \text{Also } x^2 + px + q = 0\end{array}\right\} \therefore x^2 - a^2 + p.(x-a) = 0$$

$\therefore x + a + p = 0$, or $x = -(p+a)$ which being substituted in the equation for x gives a result $= 0$.

$\therefore -(p+a)$ is also a root.

220. Let $\sqrt{\frac{a^2 c}{b} - cf + 2ac\sqrt{\frac{-f}{b}}} = x+y \ (m)$

Then $\sqrt{\frac{a^2 c}{b} - cf - 2ac\sqrt{\frac{-f}{b}}} = x - y$

$\therefore \sqrt{\left(\frac{a^2 c}{b} - cf\right)^2 + \frac{4a^2 c^2 f}{b}} = x^2 - y^2$

or $\pm \left(\frac{a^2 c}{b} + cf\right) = x^2 - y^2$

Also $\frac{a^2 c}{b} - cf = x^2 + y^2$ squaring m and equating rationals.

$$\left.\begin{aligned} \therefore x^2 &= \frac{a^2 c}{b} \text{ or } -cf \\ y^2 &= -cf \text{ or } \frac{a^2 c}{b} \end{aligned}\right\} \begin{aligned} \therefore x &= \pm a.\sqrt{\frac{c}{b}} \text{ or } \pm \sqrt{-cf} \\ y &= \pm\sqrt{-cf} \text{ or } \pm a\sqrt{\frac{c}{b}} \end{aligned}$$

$\therefore$ the root required is $a\sqrt{\frac{c}{b}} + \sqrt{-cf}$ or $-a\sqrt{\frac{c}{b}} - \sqrt{-cf}$

221. Let $\sqrt{14-8\sqrt{3}} = x + y \ (m)$

Then $\sqrt{14+8\sqrt{3}} = x - y$

$\therefore \sqrt{14^2 - 8^2.3} = x^2 - y^2$

or $\pm 2 = x^2 - y^2$

Also $14 = x^2 + y^2$ squaring (m) and equating rationals.

$$\left.\begin{aligned} \therefore x^2 &= 8 \text{ or } 7 \\ y^2 &= 7 \text{ or } 8 \end{aligned}\right\} \begin{aligned} \therefore x &= \pm 2\sqrt{2} \text{ or } \pm\sqrt{7} \\ y &= \pm\sqrt{7} \text{ or } \pm 2\sqrt{2} \end{aligned}$$

Hence the root required is $\pm (2\sqrt{2} + 7)$

222. Let a = the first term

b = the last

S = the sum.

x = the common difference.

y = number of terms.

Then (*Wood*) $b = a + (y - 1).\, x = a + xy - x$

And $S = (2a + (y-1)\,x)\,\frac{x}{2} = ax + \frac{x^2y}{2} - \frac{x}{2}$

Hence $\frac{b\,x}{2} = \frac{ax}{2} + \frac{x^2y}{2} - \frac{x^2}{2}$

And $S = ax + \frac{x^2y}{2} - \frac{x^2}{2}$

$\therefore S - \frac{bx}{2} = \frac{ax}{2}$

$\therefore x.(a+b) = 2S$

or $x = \frac{2S}{a+b}$

223. $x^4 - 2x^3 + \frac{3}{2}x^2 - \frac{x}{2} + \frac{1}{16}\,(x^2 - x + \frac{1}{4}$

x^4

$2x^2 - x) - 2x^3 + \frac{3}{2}x^2$

$-2x^3 + x^2$

$2x^2 - 2x + \frac{1}{4}\Big)\,\frac{x^2}{2} - \frac{x}{2} + \frac{1}{16}$

$\frac{x^2}{2} - \frac{x}{2} + \frac{1}{16}$

. . . .

N.B. The power evidently is the expansion of $(x - \frac{1}{2})^4$

Whence the root required $= (x - \frac{1}{2})^2 = x^2 - x + \frac{1}{4}$

224. .151636363 . . . .
20

3.03272727 . . . .
12

.39272727 . . . .
4

1.57090909 . . . .

Hence, the value required is 3*s*. 0*d*. 1*q*. 570909 ....

Otherwise.

Let the decimal $= S$, and multiply it by 1000. Again put .6363 .... $= S'$ and multiply by 100. Hence we get S' and afterwards S, in the form of a vulgar fraction, which can easily be reduced as required.

225. $\left(x - \frac{1}{x}\right)^{-\frac{1}{2}} = x^{-\frac{1}{2}}\left(1 - \frac{1}{x^2}\right)^{-1}$

$= x^{-\frac{1}{2}}\left\{1 - \left(-\frac{1}{2}\right)\frac{1}{x}^2 + \left(-\frac{1}{2}.\frac{-\frac{1}{2}-1}{2}\right)\frac{1}{x^4} - \left(-\frac{1}{2}\times\right.\right.$
$\left.-\frac{\frac{1}{2}-1}{2}.\frac{-\frac{1}{2}-2}{3}\frac{1}{x^6} + \&c.\right.$

$= x^{-\frac{1}{2}}\left(1 + \frac{1}{2}\cdot\frac{1}{x^2} + \frac{3}{8}\cdot\frac{1}{x^4} + \frac{5}{160}\cdot\frac{1}{x^6} +, \&c.\right)$

$= \frac{1}{x^{\frac{1}{2}}} + \frac{1}{2x^{\frac{5}{2}}} + \frac{3}{8.x^{\frac{9}{2}}} + \frac{5}{16\ x^{\frac{13}{2}}} + \&c. \ldots.$

226. Let n be the number of things.

Then the total number of odd combinations $= n + \frac{n.n-1.n-2}{2.3}$

$+ \frac{n.n-1.n-2.n-3.n-4.}{2.\ \ 3.\ \ 4.\ \ 5.} + \ldots.(A)$

and the total number of even combinations $= \frac{n.n-1}{2}$

$+ \frac{n.n-1.n-2.n-3}{2\ .\ 3\ .\ 4\ .} + \frac{n.n-1.n-2.n-3.n-4.n-5}{2\ .\ 3\ .\ 4\ .\ 5\ .\ 6\ .} + \ldots. (B)$

Hence $A - B = n - \frac{n\ .\ n-1}{2} + \frac{n.n-1.n-2}{2\ .\ 3} -$

$\frac{n.n-1.n-2.n-3}{2.\ \ 3.\ \ 4.} + \frac{n.n-1.n-2.n-3.n-4}{2\ .\ 3\ .\ 4\ .\ 5\ .} -$, &c., $\pm\ 1$

(according as n is odd or even)

Now $(1 - 1)^n = 1 - n + \frac{n.n-1}{2} - \frac{n.n-1.n-2}{2\ .\ 3} + \ldots \pm 1 = 0$

$\therefore A - B = 1 - (1-1)^2 = 1$
or A is greater by unity than B. *i. e.* &c. &c.

227. Let x and y be the distances travelled by A and B respectively, before they meet.

Then A's rate : B's :: $x : y$ (since the time is given)

Also A's rate : B's :: $\frac{y}{16} : \frac{x}{36}$ (since $S \propto T \times V \propto \therefore V \propto \frac{S}{T}$)

$\therefore$ (A's rate)^2 : (B's)^2 :: $xy \times 36 : xy.16 :: 9 : 4$

$\therefore$ A's rate : B's :: $3 : 2$

Again, let z = A's time in hours,
Then $z + 20$ = B's time . . . . } But $T \propto \frac{1}{V}$ when S is given, $\therefore z : z + 20 :: 2 : 3$

$\therefore 3z = 2z + 40$

$\therefore z = 40$ = A's time in hours.

and $z + 20 = 60$ = B's time.

LOGARITHMS.

228. This is best done by the form log. $(n+m) = \log. m + \frac{2}{p}\left\{\frac{n}{2m+n} + \frac{1}{3}\cdot\left(\frac{n}{2m+n}\right)^3 + \frac{1}{5}\cdot\left(\frac{n}{2m+n}\right)^5 +, \&c.\right\}$ which form may be thus proved to be true.

Put $\frac{1+x}{1-x} = 1 + \frac{n}{m} = \frac{m+n}{m}$

$\therefore\ x = \frac{n}{2m+n}$, and $\log. \frac{1+x}{1-x} = \log. (m+n) - \log. m$

Now $\log. (1+x) = \frac{1}{p}\left(x - \frac{x^2}{2} + \frac{x^3}{3} -, \&c.\right)$

$\log. (1-x) = \frac{1}{p}\left(-x - \frac{x^2}{2} - \frac{x^3}{3} -, \&c.\right)$

$\therefore\ \log. (m+n) - \log. m = \log. (1+x) - \log. (1-x) = \frac{2}{p}\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \ldots\right)$

$\therefore\ \log. (m+n) = \log. m + \frac{2}{p}\left\{\frac{n}{2m+n} + \frac{1}{3}\cdot\frac{n^3}{(2m+n)^3} + \frac{1}{5}\cdot\frac{n^5}{(2m+n)^5} + \ldots\right\}$

Here, (supposing the common logarithm is required) we have $p = 2.30258569\ldots$, and $m + n = 101$.

Now, as the log. 100 is $= 2$, put $(m) = 100$; then $(n) = 1$.

$\therefore\ \log. 101 = 2 + \frac{2}{p}\left\{\frac{1}{201} + \frac{1}{3\times(201)^3} + \frac{1}{5\,.\,(201)^5} + \ldots\right\}$

But $\frac{2}{p} = \frac{2}{2.30258\ldots} = .8685889638\ldots$

and $\frac{1}{201} = .004975124$

$\therefore$ log. 101 $= 2 + \frac{2}{p} \cdot \frac{1}{201}$ nearly $= 2.00432134$ nearly.

If $p = 1$, we have the hyperbolic logarithm, and, generally, by substituting for (p) the reciprocal of the modulus of any system, we obtain the logarithm according to that system.

229. $216 = 8 \times 27 = 2^3 \times 3^3$

$\therefore$ log. $216 =$ log. $(2^3) +$ log. (3^3)

$= 3$ log. $2 + 3$ log. 3.

If therefore log. $2 = a$, and log. $3 = b$, we have log. $216 = 3a + 3b$.

230.
$$81.213 = 8.1213 \times 10$$
$$812.13 = 8.1213 \times 100$$
$$.81213 = \frac{8.1213}{10}$$
$$\text{and } .081213 = \frac{8.1213}{100}$$

Now log. of a product $=$ sum of the logarithms of its factors, and that of quotient $=$ log. of the dividend $-$ log. of the divisor.

Also log. $10 = 1$, and log. $100 = 10^2 = 2$ log. $10 = 2$.

$\therefore$ log. $(81.213) =$ log. $(8.1213) + 1 = 1.9096256$

log. $(812.13) =$ log. $(8.1213) + 2 = 2.9096256$

log. $(.81213) =$ log. $(8.1213) - 1 = \bar{1}.9096256$

and log. $(.081213) =$ log. $(8.1213) - 2 = \bar{2}.9096256$

The justness of the operation may be proved thus:

By the definition of logarithms, if $N = a^n$, $n =$ log. N in that system whose base is (a).

Now $N \times N' = a^n \times a^{n'} = a^{n+n'}$, $\therefore n + n' =$ log. $(N \times N')$ or log. $(N \times N') =$ log. N $+$ log. N′. Again, $\frac{N}{N'} = \frac{a^n}{a^{n'}} = a^{n-n'}$

$\therefore n - n' =$ log. $\frac{N}{N'}$, or log. $\frac{N}{N'} =$ log. N $-$ log. N′.

$\therefore$ log. of a product = sum of logarithms of its factors, and log. of a quotient = log. (dividend) − log. (divisor) upon which truths the operation evidently depends.

231. $\quad$ Log. 360 = log. (20 × 30) = log. 20 + log. 30 ?

$$= 1.30103000 + 1.47712125$$
$$= 2.77815125$$

232. The hyp. log. $(z + x)$ = hyp. log. $\left(z \,.\, \overline{1 + \frac{x}{z}}\right)$ = hyp. log. z + hyp. log. $\left(1 + \frac{x}{z}\right)$

Similarly, hyp. log. $(z - x)$ = hyp. log. z + hyp. log. $\left(1 - \frac{x}{z}\right)$

But hyp. log. $\left(1 + \frac{x}{z}\right) = \frac{x}{z} - \frac{x^2}{2z^2} + \frac{x^3}{3z^3} - \frac{x^4}{4z^4} + \&c.$

And hyp. log. $\left(1 - \frac{x}{z}\right) = -\frac{x}{z} - \frac{x^2}{2z^2} - \frac{x^3}{3z^3} - \&c.$

$\therefore$ hyp. log. $\frac{z + x}{z - x}$ = hyp. log. $\left(1 + \frac{x}{z}\right)$ + hyp. log. z − hyp. log. $\left(1 - \frac{x}{z}\right)$ − hyp. log. z.

$$= \text{hyp. log. } \left(1 + \frac{x}{z}\right) - \text{hyp. log. } \left(1 - \frac{x}{z}\right) = 2 \times \left(\frac{x}{z} + \frac{x^3}{3z^3} + \frac{x}{5z^5} + \ldots\ldots\right)$$

which is Cotes's Series. (See *Harmonia Mensurarum.*)

Let $z = 2$, and $x = 1$

$$\therefore \text{hyp. log. } 3 = 2\left(\frac{1}{2} + \frac{1}{3 \times 8} + \frac{1}{5 \times 32}\right) \text{ nearly}$$
$$= 1.095833 \ldots\ldots \text{ nearly.}$$

233. The modulus of a system whose base is (a) =

$$\frac{1}{(a - 1) - \frac{(a-1)^2}{2} + \frac{(a-1)^3}{3} - \ldots\ldots} = \frac{1}{\text{hyp. log. } (a)}$$

Let a, ar, ar^2, ar^3, &c. be the successive bases.
m, m_1, m_2, m_3, ... the corresponding moduli.

$$\therefore \; m = \frac{1}{\text{hyp. log. } (a)}$$

$$m_1 = \frac{1}{\text{hyp. log. } (ar)} = \frac{1}{\text{hyp. log. } (a) + \text{hyp. log. } (r)}$$

$$m_2 = \frac{1}{\text{hyp. log. } (ar^2)} = \frac{1}{\text{hyp. log. } (a) + 2 \text{ hyp. log. } (r)}$$

$$\text{\&c.} = \text{\&c.} = \text{\&c.}$$

Whence it appears that the reciprocals of m, m_1, m_2, ... are in an increasing arithmetical progression, hyp. log. (r) being the common difference. $\therefore$ m, m_1, m_2, ... decrease in harmonical progression.

234. Since A : B :: C : D

$$AD = BC$$

$$\therefore \log. A + \log. D = \log. B + \log. C$$

$$\therefore \log. D = \log. B + \log. C - \log. A.$$

235. If the logarithm of any number be expressed in terms of that number, it will appear that this expression consists of two factors,—one independent of the base of the system, and $\therefore$ invariable with regard to all systems,—the other a function of the base, but independent of the number, and $\therefore$ varying in different systems. Let L, L′ be logarithms of the same number in two systems, and let C, C be the invariable, M, M′ the variable, factors of their equivalent expressions respectively.

$$\text{Then } L = C \times M$$

$$L' = C \times M'$$

$\therefore L = \frac{M}{M'} \times L'$, whence it appears that when we know the logarithm of a number according to one system, we can find the logarithm according to any other, by multiplying the given logarithm by the expression $\frac{M}{M'}$. Now, in the Naperian system, if L′ be the logarithm, $M' = 1$.

$\therefore$ L $=$ M $\times$ L$'$, or the Naperian logarithm is changed to another by multiplying by M. M in this case is called the modulus.

To determine this MODULUS.

If $x =$ log. N to base (a)

Then (by definition) $a^x =$ N. $\therefore$ x hyp. log. $(a) =$ hyp. log. N

$$\therefore \text{ log. N} = \frac{\text{hyp. log. N}}{\text{hyp. log. }(a)} = \frac{1}{\text{hyp. log. }(a)} \times \text{hyp. log. N.}$$

$\therefore$ The logarithm of any number in a system whose base is (a) may be derived from the hyperbolic or Naperian logarithm of that number by multiplying by the quantity $\dfrac{1}{\text{hyp. log. }(a)}$ which is $\therefore$ the modulus.

For the form of the modulus, see *Woodhouse's Trigonometry.*

N. B. It is evident that, generally, the logarithm of a number, according to any system, may be found from the logarithm to the same number in any other system, by multiplying the given logarithm by the reciprocal of the logarithm (according to the latter system) of the base of the former system.

For, if L be the characteristic in the former system,
L$'$ in the latter,
a the base in the former,
and x the required logarithm,

Then $\because$ $a^x =$ N we have

$$x \times \text{L}'(a) = \text{L}'(\text{N})$$

$$\therefore \text{L (N)} = x = \frac{\text{L}'\text{ (N)}}{\text{L}'\text{ }(a)}$$

236. The decimal part of the logarithms whose numbers are given, being one next less, and the other next greater than that of the logarithm whose number is required, we must add to the less number the quantity $\dfrac{.5787836 - .5787767}{.5787882 - .5787767}$

$$= \frac{69}{115} = .51$$

Now the index is 6 $\therefore$ the integral part must consist of 7 digits.

Hence the required number $= 5785251$.

For the investigation of the rule, See *Woodhouse.*

N. B. The logarithms given in the enunciation are not the tabular ones to the given numbers.

237. Let x and y be the quantities, x being the greater, and d their invariable difference.

$\therefore d = x - y$

$\therefore x = d + y$

Now $\log. x - \log. y = \log. \dfrac{x}{y} = \log. \dfrac{d+y}{y}$

But when (y) increases the ratio $\dfrac{d+y}{y}$ being one of greater inequality, must decrease, and $\therefore$ its logarithm.

$\therefore$ as x and y increase, $\log. x - \log. y$ decreases.

238. The MODULUS $= \dfrac{1}{\text{hyp. log. } (a)} = \dfrac{1}{\text{hyp. log. } 10}$

But $\text{hyp. log. } (10) = \text{hyp. log. } \dfrac{1+\frac{9}{11}}{1-\frac{9}{11}} = \text{hyp. log. } \left(1+\dfrac{9}{11}\right)$

$- \text{hyp. log. } \left(1 - \dfrac{9}{11}\right)$

Also $\text{hyp. log. } \left(1+\dfrac{9}{11}\right) = \dfrac{9}{11} - \dfrac{9^2}{2.11^2} + \dfrac{9^3}{3.11^3} - \ldots.$

and $\text{hyp. log. } \left(1-\dfrac{9}{11}\right) = -\dfrac{9}{11} - \dfrac{9^2}{2.11^2} + \dfrac{9^3}{3.11^3} - \ldots.$

$\therefore \text{hyp. log. } 10 = 2\left(\dfrac{9}{11} + \dfrac{9^3}{3.11^3} + \dfrac{9^5}{5.11^5} + \ldots.\right)$

$= 2.30258509 \ldots.$

Hence, unity being divided by $2.30258509 \ldots..$ we obtain M $= .43429 \ldots..$

SIMPLE INTEREST.

239. If P be the principal put out to interest for n years, at the rate r per pound for one year, then the interest (I) $= n r$ P, and $\therefore$ the amount (M) $=$ P $+ n r$ P. (See *Wood.*)

Here $P = 115$

$$n = 5\frac{1}{2} = \frac{11}{2}$$

$$r = \frac{4\frac{1}{2}}{100} = \frac{9}{200}$$

$$\therefore \ I = \frac{11}{2} \times \frac{9}{200} \times 115 = \frac{99 \times 23}{80} = 28\frac{37}{80}£ = £28.\ 9s.\ 3d.$$

240. Here $P = 555$

$$n = 2\frac{1}{2} = \frac{5}{2}$$

$$r = \frac{4\frac{3}{4}}{100} = \frac{19}{400}$$

$$\therefore \ I = \frac{5}{2} \times \frac{19}{400} \times 555 = \frac{19 \times 111}{32} = 65\frac{29}{32} = 65£\ 18s.\ 1\tfrac{1}{2}d.$$

241. Here $P = £.\,873\ 15s. = 873\frac{3}{4} = \frac{3495}{4}$

$$n = 2\frac{1}{2} = \frac{5}{2}$$

$$r = \frac{4\frac{3}{4}}{100} = \frac{19}{400}$$

$$\therefore I = \frac{5}{2} \times \frac{19}{400} \times \frac{3495}{4} = \frac{19 \times 699}{128} = 103£\ 15s.\ 1\tfrac{1}{4}d.\ 3\tfrac{1}{2}q.$$

242. From the above equation $n = \dfrac{I}{rP}$

But $I = 15s. = \dfrac{3}{4}£.$

$$r = \frac{4\frac{1}{2}}{100} = \frac{9}{200}$$

and $P = 1$

$\therefore n$ the time required $= \dfrac{\frac{3}{4}}{\frac{9}{200}} = \dfrac{\frac{1}{1}}{\frac{3}{50}} = \dfrac{50}{3} = 16\dfrac{2}{3}$ years $= 16$ years 8 months.

243. Here $P = 3£\ 15s.\ 4d. = 3 + \dfrac{3}{4} + \dfrac{1}{60} = \dfrac{180 + 45 + 1}{60} = \dfrac{226}{60} = \dfrac{113}{30}$

$$n = 1$$

$$r = \frac{3\frac{1}{2}}{100} = \frac{7}{200}$$

$$\therefore I = 1 \times \frac{7}{200} \times \frac{113}{30} = \frac{791}{6000}£ = \frac{791}{300}s. = 2s.\ 7d.\ 2\tfrac{14}{25}q.$$

244. Here $P = 315£\ 10s. = 315\dfrac{1}{2} = \dfrac{631}{2}$

$$n = 1$$

$$r = \frac{4\frac{3}{4}}{100} = \frac{19}{400}$$

$$\therefore I = 1 \times \frac{19}{400} \times \frac{631}{2} = \frac{19 \times 631}{800}£ = 14£\ 19s.\ 8d.\ 2\tfrac{4}{5}q.$$

245. Here $P = 12s.\ 4d. = 12s.\ \dfrac{1}{3} = \dfrac{37}{3}s. = \dfrac{37}{60}s.$

$n = 1$ (I suppose)

$$r = \frac{4\frac{1}{2}}{100} = \frac{9}{200}£ = \frac{9}{10}s.$$

$$\therefore I = 1 \times \frac{9}{10} \times \frac{37}{60} = \frac{111}{200}s. = 6d.\ 2\tfrac{16}{25}q.$$

246. Here $P = 1£\ 1s.\ 6d. = 1 + \frac{1}{20} + \frac{1}{40} = \frac{43}{40}£$

$n = 1$

$r = \frac{5}{100} = \frac{1}{20}$

$\therefore I = 1 \times \frac{1}{20} \times \frac{43}{40} = \frac{43}{800}\ £ = \frac{43}{40}s. = 1s.\ 0d.\ 3\frac{1}{5}q.$

247. Here $P = 350£\ 15s. = 350\frac{3}{4} = \frac{1403}{4}$, $P' = 450\frac{3}{4} = \frac{1803}{4}$

$n = 1$ $\qquad n' = 1$

$r = \frac{4\frac{1}{2}}{100} = \frac{9}{200}$, $\qquad r' = \frac{3\frac{1}{2}}{100} = \frac{7}{200}$

$\therefore I : I' :: \frac{9}{200} \times \frac{1403}{4} : \frac{7}{200} \times \frac{1803}{4} :: 9 \times 1403 : 7 \times 1803$

$:: 4209 : 4207 :: 1 : 1 - \frac{2}{4209}$

$\therefore$ they are nearly equal.

248. Here $P = 260$

$n = 18$ months $= 1\frac{1}{2}$ year $= \frac{3}{2}$

$r = \frac{4\frac{1}{2}}{100} = \frac{9}{200}$

$\therefore I = \frac{3}{2} \times \frac{9}{200} \times 260 = \frac{27 \times 13}{20}£ = 17£\ 11s.$

249. Here $P = 547\frac{3}{4}£ = \frac{2191}{4}$

$n = 3$

$r = \frac{3}{100}$

$\therefore I = 3 \times \frac{3}{100} \times \frac{2191}{4} = 49£\ 6s.\ 11d.\ 1\frac{3}{5}q.$

250. Since $I = n r P$

$$r = \frac{I}{nP} = \frac{3£\ 18s.\ 9d.}{25 \times (3\frac{1}{2})}$$

Now £.3 18s. 9d. $= 3 + \frac{18}{20} + \frac{9}{12 \times 20} = \frac{240+72+3}{80} = \frac{315}{80}$

$= \frac{63}{16}$. Hence $r = \frac{63 \times 2}{16 \times 25 \times 7} = \frac{9}{8 \times 25}$, which is the rate per pound.

$\therefore$ the rate per cent. $= 100\, r = \frac{9 \times 100}{200} = \frac{9}{2} = 4\frac{1}{2}$.

251. Here $P = 400$

$$n = \frac{3}{4}$$

$$r = \frac{4\frac{3}{4}}{100} = \frac{19}{400}$$

$$\therefore I = \frac{3}{4} \times \frac{19}{400} \times 400 = \frac{57}{4} = 14£\ 5s.$$

252. Here $P = 115$

$$n = 5\frac{1}{2} = \frac{11}{2}$$

$$r = \frac{4\frac{1}{2}}{100} = \frac{9}{200}$$

$$\therefore I = \frac{11}{2} \times \frac{9}{200} \times 115 = \frac{99 \times 23}{80}£ = 28£\ 9s.\ 3d.$$

253. Here $P = 120\frac{1}{2} = \frac{241}{2}$

$$n = 2\frac{1}{2} = \frac{5}{2}$$

$$r = \frac{4\frac{3}{4}}{100} = \frac{19}{400}$$

$$\therefore I = \frac{5}{2} \times \frac{19}{400} \times \frac{241}{2} = \frac{19 \times 241}{320}£ = 14£\ 6s.\ 2\tfrac{1}{4}d.$$

Hence, the amount required $= 120£\ 10s. + 14£\ 6s.\ 2\frac{1}{4}d.$ $= 134£\ 16s.\ 2\frac{1}{4}d.$

254. Here $P = 56£\ 13s.\ 4d. = 56\frac{2}{3}£. = \frac{170}{3}$

$$n = 5\frac{1}{3} = \frac{16}{3}$$

$$r = \frac{6}{100} = \frac{3}{50}$$

$$\therefore I = \frac{16}{3} \times \frac{3}{50} \times \frac{170}{3} = \frac{16 \times 17}{3 \times 5}£. = 18£.\ 2s.\ 8d.$$

Hence the amount $= \left.\begin{array}{rrr} 56' & 13 & 4 \\ +\ 18 & 2 & 8 \end{array}\right\} = 74£\ 16s.$

255. Here $\left.\begin{array}{r} P = 500 \\ n = \frac{3}{4} \\ \text{and } I = 20 \end{array}\right\}$ Now $I = n r P$

$\therefore r = \frac{I}{nP} = \frac{20}{\frac{3}{4} \times 500} = \frac{4}{3 \times 25}$, which is the rate per pound.

$\therefore$ the rate per cent. $= 100 \times r = \frac{4 \times 100}{3 \times 25} = \frac{16}{3} = 5\frac{1}{3}$.

256. Here $P = 70$

$$n = 3$$

$$r = \frac{3\frac{1}{2}}{100} = \frac{7}{200}$$

$$\therefore I = 3 \times \frac{7}{200} \times 70 = \frac{147}{20} = 7£\ 7s.$$

$\therefore$ the amount $= 77£\ 7s.$

COMPOUND INTEREST.

257. $M = P.R^n$ (where $M =$ the amount of P pounds for n years, R being the amount of one pound at the end of one year). See *Wood.*

Here, the amount must $= 2 \times$ the principal

or $P. R^n = 200$

or $100. R^n = 200$

$\therefore R^n = 2$

Now $R = \left(1 + \frac{1}{20}\right) = \frac{21}{20}$£ (allowing 5 per cent)

$\therefore \left(\frac{21}{20}\right)^n = 2$

$\therefore n \log. \frac{21}{20} = \log. 2$

$$\therefore n = \frac{\log. 2}{\log. 21 - \log. 20} = \frac{.30103}{1.32222 - 1.30103} = \frac{30103}{2119}$$

$= 14$ years $80 \frac{94}{2119}$ days.

Hence it appears that *any principal* will double itself at 5 per cent. compound interest in 14 years $80 \frac{94}{2119}$ days.

n may be found very nearly without logarithms thus:

Since $2 = R^n = \left(1 + \frac{1}{20}\right)^n = 1 + \frac{n}{20} + \frac{n. (n-1)}{2.20^2} + \ldots.$

Neglect the terms after the third, and find n from the solution of a quadratic equation.

258. $PR^n = M$

Hence $R = 1 + \frac{1}{20} = \frac{21}{20}$

$$P = 1$$
$$\text{and } M = 100$$
$$\therefore \left(\frac{21}{20}\right)^n = 100$$
$$\therefore n\,(\log. 21 - \log. 20) = \log. (100) = 2$$
$$\therefore n = \frac{2}{\log. 21 - \log. 20}$$

Now $\log.21 = \log.\dfrac{1050}{50} = \log. 1050 - \log.50 = \log.1050 - \log.\dfrac{100}{2}$

$$= \log. 5050 - \log. 100 + \log.2 = 3.0211893 - 2 + \log.2$$
$$= 1.0211893 + \log. 2$$

and $\log. 20 = \log. 10 + \log. 2 = 1 + \log. 2$

$$\therefore \log. 21 - \log. 20 = .0211893$$
$$\therefore n = 2 \div (.0211893) =$$
$$= 94 \text{ years and } 141.4 \text{ \&c. days.}$$

259. Here $R = \left(1 + \dfrac{r}{m}\right)$ r being the rate per cent. due every year, and $m =$ the number of moments in a year.

$$n = t \text{ years} = (365\tfrac{1}{4}) \times 24 \times 60 \times 60 \times t \text{ moments}$$
$$= 31557600\, t \text{ moments} = mt$$

$\therefore M = PR^n = \left(1 + \dfrac{r}{m}\right)^{mt}$ where logarithms may be used for expediting the calculation.

To obtain an approximate form for M, we have

$$M = 1 + tr + \frac{t.(mt-1)}{2m} r^2 + \frac{t.(mt-1).(mt-2)}{2 \;.\; 3m^2} r^3 + \text{\&c.}$$

$= 1 + tr + \dfrac{t^2r^2}{2} + \dfrac{t^3r^3}{2.3} +$, &c., nearly since mt is very great compared with 1, 2, &c.

$= e^{tr}$ (where e is the hyperbolic base)

260. Here $m = p\,R^n$

Assume $M = p\,R^x$ where x is the time required.

Now $R^n = \dfrac{m}{p}$

$\therefore n \log. R = \log. m - \log. p$

Similarly $x \log. R = \log. M. - \log. p$

$\therefore \dfrac{x}{n} = \dfrac{\log. M - \log. p}{\log. m - \log. p}$

$\therefore x = n \left(\dfrac{\log. M - \log. p}{\log. m - \log. p}\right)$ the time required.

261. Here, $10\,P = M = PR^n = P.\left(1 + \dfrac{1}{20}\right)^n$

$\therefore 10 = \left(\dfrac{21}{20}\right)^n$

$\therefore n\,(\log. 21 - \log. 20) = \log. 10 = 1$

Now, $\log. 21 = \log. \dfrac{105}{5} = \log. 105 - \log. \left(\dfrac{10}{2}\right) = \log. 105 - 1 + \log. 2.$

and $\log. 20 = \log. 10 + \log. 2 = 1 + \log. 2$

$\therefore \log. 21 - \log. 20 = \log. 105 - 2 = 2.0211893 - 2 = .0211893$

$\therefore n = 1 \div .0211893 = 47$ years and $70.7 \ldots.$ days

ANNUITIES.

262. The amount (M) of an annuity (A) for (n) years at compound interest $= \dfrac{R^n - 1}{R - 1} \cdot A$, where $R =$ the amount of one pound for one year. See *Wood.*

Hence, if S be the sum required,

we have $S = \dfrac{(\frac{21}{20})^3 - 1}{\frac{21}{20} - 1} \times 100 = \dfrac{21^3 - 20^3}{21 - 20} \times \dfrac{100}{20^2}$

$$= \frac{21^3 - 20^3}{4} = \frac{1261}{4} = 315£\ 5s.$$

263. First $\dfrac{140}{4} = 35£$ the sum paid at the end of the first quarter.

Also the rate per pound for a quarter $= \dfrac{5}{150} \times 4 = \dfrac{1}{80}$

$\therefore \dfrac{35}{80} =$ the interest for the first quarter, and $2 \times 35 =$ principal at that, &c.

Hence the whole interest $= \dfrac{35}{80} + \dfrac{2 \times 35}{80} + \dfrac{3 \times 35}{80} + \ldots$ $\dfrac{11 \times 35}{80} = (1 + 2 + 3 \ldots + 11)\dfrac{35}{80} = \dfrac{12 \times 11}{2} \times \dfrac{35}{80} = \dfrac{66 \times 35}{80}$ $= 27£.\ 12s.\ 6d.$

$\therefore$ the amount required $= 12£ \times 35 + 27£\ 12s.\ 6d.$

$= 420£ + 27£\ 12s.\ 6d.$

$= 447£\ 12s.\ 6d.$

264. Here $A = \frac{R^n - 1}{R - 1} P$ (See *Wood.*)

$\therefore AR - A = PR^n - P$

$\therefore R^n = \frac{A.(R-1)+P}{P}$

$\therefore n \log. R = \log. (A.\overline{R-1} + P) - \log. P$

$\therefore n = \frac{\log. (A.\overline{R-1} + P) - \log. P}{\log. R}$ the number of years required.

If the rate per cent. were 5., R would $= \frac{21}{20}$, and $\therefore n =$ $\frac{\log. (A + 20.P) - \log. 20 - \log. P}{\log. 21 - \log. 20}$.

265. Let $n =$ the number of years required.

Then $2 R^{n-1} =$ the amount of the first investment up to that time.

$4 R^{n-2} =$ 2^{nd}.

&c. $=$, &c.

$2^n =$ n^{th}

and $PR^n =$ the amount of the sum lent.

$\therefore PR^n = 2R^{n-1} + 4R^{n-2} + \ldots\ldots + 2^n = \frac{2^{n+1} - 2R^n}{2-R}$ since it is a geometric series, whose common ratio $= \frac{2}{R}$, and number of terms $= n$)

Hence $(P.\overline{2-R} + 2) R^n = 2^{n+1}$

$\therefore \log. (P.\overline{2-R} + 2) + n \log. R = (n+1) \log. 2 = n \log. 2 + \log. 2$

$\therefore n.(\log. 2 - \log. R) = \log. (P \overline{2-R} + 2) - \log. 2$

$\therefore n = \frac{\log. (P.\overline{2-R} + 2) - \log. 2}{\log. 2 - \log. R.}$

PRESENT WORTH, AND DISCOUNT.

266. The present worth of any sum, due after a certain time, is a sum such that being put out to interest, it would amount to the given sum in that time.

The discount of any sum, due after a certain time, is equal to the difference between that sum and its present worth; or it is equal to the interest of its present worth for that time.

Hence, if (P) be the present worth of a sum (A) due after (n) years, we have

$$PR^n = A \quad \therefore P = \frac{A}{R^n} \text{ (R = the amount of 1£ in one year)}$$

$$\text{Hence the discount (D)} = A - \frac{A}{R^n} = \frac{A}{R^n}(R^n - 1)$$

Here $A = p$

$$n = 1\frac{1}{2} = \frac{3}{2}$$

$$R = 1 + \frac{4}{20} = \frac{21}{20}$$

$$\therefore D = \frac{p}{\left(\frac{21}{20}\right)^{\frac{3}{2}}} \times \left(\left(\frac{21}{20}\right)^{\frac{3}{2}} - 1\right) = p - \frac{20^{\frac{3}{2}}}{21^{\frac{3}{2}}}p = \frac{p}{21^{\frac{3}{2}}}\left(21^{\frac{3}{2}} - 20^{\frac{3}{2}}\right)$$

We may obtain (D) sufficiently accurate for practical purposes, thus:

$$D = p - \frac{p}{\left(1 + \frac{1}{20}\right)^{\frac{3}{2}}} = p - \frac{p}{1 + \frac{3}{40}} \text{ nearly.}$$

$$= \frac{3p}{43} \text{ nearly.}$$

267. The present worth of an annuity (A) $= \dfrac{1 - \frac{1}{R^n}}{R-1} \times A$ (*Wood.*)

Here $A = 100$

$R = 1 + \dfrac{8}{100} = 1 + \dfrac{2}{25} = \dfrac{27}{25}$

$n = 20$

$\therefore P = \dfrac{1-(\frac{25}{27})^{20}}{\frac{2}{25}} \times 100 = \left(1 - \left(\dfrac{25}{27}\right)^{20}\right) \times 1250 =$ 983£. 3*s*. 3*d*. $2\,\dfrac{21238}{22699}q$.

N. B. To find the value $\left(\dfrac{25}{27}\right)^{20}$ use logarithms.

268. The present value of a perpetuity $P = \dfrac{A}{R-1}$ where R is the amount of one pound for one year, and A the annuity (*Wood.*)

Now in this case R must equal the amount of one pound in five years, and A the sum paid every five years.

$$\therefore P = \frac{20}{R^5 - 1} = \frac{20}{\frac{21^5}{20^5} - 1} = \frac{20^6}{21^5 - 20^5} = \frac{64000000}{884101}£$$

$= 72£.\ 7s.\ 9d.\ 2\,\dfrac{285106}{884101}q.$

269. Here $D = 7£\ 10s. = 7\frac{1}{2}£ = \dfrac{15}{2}£$

$A = 100$

$n = 1\dfrac{1}{2} = \dfrac{3}{2}$

$\therefore \dfrac{15}{2} = 100 - \dfrac{100}{R^{\frac{3}{2}}}$

$\therefore \dfrac{100}{R^{\frac{3}{2}}} = 100 - \dfrac{15}{2} = \dfrac{185}{2}$

$$\therefore R^{\frac{3}{2}} = \frac{200}{185} = \frac{40}{37}$$

$$\therefore \frac{3}{2} \log. R = \log. 40 - \log. 37.$$

$$\therefore \log. R = \frac{2}{3} (\log. 40 - \log. 37)$$

Hence R or $1 + \frac{r}{100} = 1.05335£$ (by the logarithmic tables.)

$\therefore r = 5.335£ = 5£\ 6s.\ 8d.\ 1.6q.$, which is the rate per cent. required.

Otherwise, allowing Simple Interest.

If P be the present worth of (A) due after (n) years, 100 r per cent. simple interest being allowed, we have $P + nr\ P = A$.

$$\therefore P = \frac{A}{1+nr}$$

Hence $D = A - \frac{A}{1+nr}$ or $\frac{15}{2} = 100 - \frac{100}{1+\frac{3}{2}r} = 100 - \frac{200}{2+3r}$

$$\therefore 15 = 200 - \frac{400}{2+3r}$$

$$\therefore 185 \cdot (2 + 3r) = 400, \text{ or } 555r = 400 - 370 = 30$$

$$\therefore r = \frac{30}{555} = \frac{6}{111} = \frac{2}{37}$$

$\therefore$ the required rate $= \frac{100 \times 2}{37} = \frac{200}{37} = 5£\ 8s.\ 1d.\ 1q.\ \frac{7}{37}$

270. $P = \frac{1 - \frac{1}{R^n}}{R - 1} \times A$ (*Wood.*)

Hence, if the annuity were to continue ($n + t$) years, we should have $P = \frac{1 - \frac{1}{R^{n+t}}}{R - 1} \times A$

Now, it is evident, that the present value required, is equal to the difference between that for $(n+t)$ years, and that for (n) years.

$$\therefore \text{ it} = \frac{1 - \frac{1}{R^{n+t}}}{R-1} \times A - \frac{1 - \frac{1}{R^n}}{R-1} \times A$$

$$= \frac{A}{R-1} \times \left(\frac{1}{R^n} - \frac{1}{R^{n+t}}\right) = \frac{A}{R^{n+t}} \times \frac{R^t - 1}{R-1}$$

271. Let A be the annual rent of the estate.

Then its present value $= \frac{A}{R-1}$ (*Wood.*) $= \frac{A}{1+\frac{1}{25}-1} = 25\,A$

$\therefore$ the estate is 25 years' purchase.

272. As the time is less than a year, simple interest must be allowed.

$$\therefore D = A - \frac{A}{1+nr} = 400 - \frac{400}{1+\frac{3}{4}\cdot\frac{19}{400}} = \frac{22800}{1657}\pounds$$

$$= 13\pounds\ 15s.\ 2d.\ 2\tfrac{30}{1657}q.$$

273. Discount is an allowance made on a bill or any other debt not yet become due, in consideration of present payment.

Hence, it is evident, that the discount on any sum must equal the difference between that sum and its present worth.

Let (A) be the sum discounted.

(*n*) the number of years after which it becomes due.

(*r*) the interest of one pound for one year.

(R) the amount of ——————

(D) the discount.

Then (simple interest being allowed) $D = A - \frac{A}{1+nr}$ (*a*)

For $P + nrP = A$ (P being the present value of A)

$\therefore$ P must amount in the given time to (A)

$$\therefore\ P = \frac{A}{1 + nr}$$

Also (compound interest being allowed), $D = A - \dfrac{A}{R^n}$ (*b*)

For $PR^n = A$

$$\therefore\ P = \frac{A}{R^n}$$

In the example, we have $A = 100$, $n = \dfrac{1}{4}, r = \dfrac{1}{20}$

$$\therefore\ D = 100 - \frac{100}{1 + \frac{1}{4} \times \frac{1}{20}} = 100 - \frac{8000}{81} = \frac{100}{81}£.$$

$$= 1\,£.\ \ 4s.\ \ 8d.\ \ 1\tfrac{5}{27}q.$$

N.B. The time being less than a year, only simple interest can be allowed.

274. Let (P) be the present worth required.

Then $\because$ it must be such that it will amount, in 5 years, to 100£.

$PR^5 = 100$, (R) being the amount of a pound for one year.

$\therefore\ P = \dfrac{100}{R^5}$ the present value required.

If the rate per cent. were 5; R would $= \dfrac{21}{20}$

and $\therefore$ P would $= \dfrac{100 \times 20^5}{21^5}, = \dfrac{320000000}{4084101}$

$$= 78£\ \ 7s.\ \ 0d.\ \ 2\,\tfrac{2090982}{4084101}q.$$

275. First, (allowing Simple Interest) $D = A - \dfrac{A}{1+nr}$

$$= 260 - \frac{260}{1+\frac{3}{2}\times\frac{9}{200}}$$

$$= 260 - \frac{260 \times 400}{427} = \frac{7020}{427}$$

$$= 16£\ 8s.\ 9d.\ 2\frac{286}{427}q.$$

Secondly, (at Compound Interest) $D = A - \frac{A}{R^n} = 260$

$$- \frac{260}{(1+\frac{9}{200})^{\frac{3}{2}}}$$

$$= 260 - \frac{260}{(1.045)^{\frac{3}{2}}}$$

$$= 260 - \frac{260}{1.06825} \text{ (by logarithms)}$$

$$= 260£ - 243£\ 7s.\ 9d.\ 1\frac{715}{4273}q.$$

$$= 16£\ 12s.\ 2d.\ 2\frac{3558}{4273}q.$$

276. The present worth required, is evidently = that of 20£ a year to continue for ever, — that of 20£ a year to continue for two years.

$$\text{It} \therefore = \frac{A}{R-1} - \frac{1-\frac{1}{R^n}}{R-1} \times A \text{ (See } \textit{Wood.}) = \frac{A}{R^n(R-1)}$$

$$= \frac{20}{(\frac{21}{20})^2(\frac{21}{20}-1)} = \frac{20^4}{21^2} = \frac{£160000}{441} = 362£\ 16s.\ 2d.\ 3q.\frac{41}{147}$$

277. If Simple Interest be allowed,

$$P = \frac{A}{1+nr} = \frac{75}{1+\frac{5}{4}\times\frac{1}{20}}$$

$$= \frac{16\times 75}{17}£ = 70£\ 11s.\ 9d.\ \frac{3}{17}$$

If Compound Interest be allowed, $P = \frac{A}{R^n} = \frac{75}{(1+\frac{1}{20})^{\frac{5}{4}}}$

$$= 75 \times \left(\frac{20}{21}\right)^{\frac{5}{4}}$$

$$= 70£\ 11s.\ 8d. \text{ (either by}$$

taking 3 terms of the expansion of $\left(1+\frac{1}{20}\right)^{\frac{5}{4}}$ or by logarithms,)

278. The PRESENT WORTH of a sum, due after a certain time, is such a sum as would amount in that time to the given sum exactly.

Hence, if (P) be the present worth of the sum (A) due after (n) years, we have

$$P + nr\,P = A, \quad \therefore P = \frac{A}{1 + nr} \text{ (simple interest.)}$$

$$PR^n = A \quad \therefore P = \frac{A}{R^n} \text{ (at compound interest.)}$$

In the example, $A = 430$

$$n = \frac{3}{4}$$

$$r = \frac{4\frac{1}{2}}{100} = \frac{9}{200}$$

$$\therefore P = \frac{430}{1+\frac{3}{4}\times\frac{9}{200}} = \frac{430\times 800}{827} = 415£\ 19s.\ 2d.\ 2\frac{706}{827}q.$$

N. B. The time being less than a year, compound interest cannot be allowed.

279. Here $P' = \dfrac{a}{1+\frac{b}{12}\times r} = \dfrac{12a}{12+b\,r}$

$$P'' = \frac{c}{1+\frac{d}{12}\times r} = \frac{12\,c}{12+dr}$$

$$\therefore P = P' + P'' = \frac{12a}{12+br} + \frac{12c}{12+dr}$$

$$\therefore r^2 + \frac{12.(b+d)}{bd}\,r + \frac{144}{bd} = \frac{144.(a+c)}{bd\,P} + \frac{12(ad+bc)}{bd\,P}\,r$$

$$\therefore r^2 - \frac{12.(ad+bc-b-d)}{bd\,P}\,r = \frac{144(a+c-P)}{bd\,P}$$

$$\therefore r = \frac{6}{bd\,P}\left((a-1)d+(c-1)b \pm \sqrt{(\overline{a-1}.d + \overline{c-1}.b)^2 + 4bd\,P.(a+c-P)}\right)$$ by the solution of a quadratic equation.

280. The amount of (P) at the end of $(m+n)$ years, since it is not due until the end of (m) years is, $P + n\,r\,P$,

Again, the present value of (P), due at the end of (m) years is $P - \frac{P}{1+mr}$, whose amount for ($m + n$) years is

$$\left(P - \frac{P}{1+mr}\right) + \left(P - \frac{P}{1+mr}\right)(m+n)r = \frac{mrP}{1+mr} \times (1 + \overline{m+n}\,r)$$

$\therefore$ the difference required is $P + nrP - \frac{mrP}{1+mr} \times (1+mr+nr)$

$$= \frac{1 + nr - m^2 r^2}{1+mr} \times P$$

281. The present value of an estate of 100 a year, not subject to the payment of A as stated in the problem, $= \frac{100}{\frac{21}{20}-1} = 2000£.$

(see *Wood*, for the form $\frac{A}{R-1}$).

Hence, it appears that the present value of the sum (A) paid every two years, = 1000£.

Now, if R be the amount of one pound in one year, R^2 is its amount in two years, and the form $\frac{A}{R-1}$ for the present value of a perpetuity becomes $\frac{A}{R^2-1}$ when the payment becomes due every two years instead of every one year.

$\therefore$ we have $\frac{A}{R^2-1} = 1000$

$$\therefore A = 1000 \times (R^2 - 1) = \frac{21^2}{20^2} \times 1000 - 1000$$

$$= \frac{441000}{400} - 1000 = 1102£\ 10s. - 1000£ = 102£\ 10s.$$

282. Here $D = A - \frac{A}{1+nr} = 125£\ 10s. - \frac{125\frac{1}{2}}{1+3\times\frac{9}{200}}£$

Now $\frac{125\frac{1}{2}}{1+\frac{27}{200}}£. = \frac{50200}{454}£. = 110£\ 11s.\ 5d.\ 1\frac{177}{227}q.$

283. Let (n) be the whole number of years, after which

the annuity will cease, (R) the amount of 1£ in 1 year at the given rate, and A the annuity.

Then the present value of (A) for the whole term

$$= \frac{1 - \frac{1}{R^n}}{R - 1} \times A.$$

Also, the present value of the first half of (n) years

$$= \frac{1 - \frac{1}{R^{\frac{n}{2}}}}{R - 1} \times A. \quad \therefore \text{ the present value for the latter half}$$

$$= \frac{1 - \frac{1}{R^n}}{R - 1} \times A - \frac{1 - \frac{1}{R^{\frac{n}{2}}}}{R - 1} A = \frac{A}{R - 1} \times \left(\frac{1}{R^{\frac{n}{2}}} - \frac{1}{R^n}\right)$$

Hence, (by the question) $\frac{1 - \frac{1}{R^n}}{R - 1} \times A = m \times \frac{A}{R - 1} \times \left(\frac{1}{R^{\frac{n}{2}}} - \frac{1}{R^{\frac{n}{1}}}\right)$

$$\therefore 1 - \frac{1}{R^n} = m \times \left(\frac{1}{R^{\frac{n}{2}}} - \frac{1}{R^n}\right)$$

$$\therefore R^n = m\, R^{\frac{n}{2}} - m\ 1$$

$$\therefore R^n - m\, R^{\frac{n}{2}} = 1 - m \text{ which is a quadratic.}$$

$$\therefore R^{\frac{n}{2}} = \frac{m}{2} \pm \frac{m - 2}{2} = m - 1 \text{ or } 1$$

$$\therefore \frac{n}{2} \log. R = \log. (m - 1) \text{ or } \log. 1\ (= 0)$$

$$\therefore n = 2\, \frac{\log. (m - 1)}{\log. R} \text{ or } 0$$

the number of years required.

284. Here $D = A - \frac{A}{1 + nr} = 100 - \frac{100}{1 + \frac{1}{20}} = 100 - \frac{2000}{21} = \frac{100}{21}$£. $= 4$£ $15s.\ 2d.\ 3\frac{3}{7}q.$

Again, the interest of $\frac{100}{21}$£. at 5 per cent. for one year

$$= \frac{100}{21} \times \frac{1}{20} = \frac{5}{21} = 4s.\ 9\tfrac{1}{7}d.$$

285. If the annuity be (a) pounds, the sum due every instant, $= \dfrac{a}{(365\frac{1}{4}) \times 24 \times 60 \times 60} = \dfrac{a}{m}$

Where $m = 3600 \times 6 \times 1461 = 31557600$

Let R be the amount of 1£. for 1 instant.

Then $1 : R :: A : RA =$ the amount at the end of the 2nd instant. $\therefore A + RA = (1 + R)A =$ sum due at the end of the 2nd instant; in the same manner,

$R.(1 + R).A =$ amount at the end of the 3rd instant.

and $(1 + R + R^2).A =$ sum due at the end of 3d instant.

Similarly $(1 + R + R^2 + R^3 + \ldots . R^{p-1})A =$ the sum due at the end of (p) instants $= \dfrac{R^p - 1}{R - 1} \times A$. Now the number of instants in (n) years $= n\,m$. Let, therefore, (P) be the present value required.

Then, $\because R^m =$ the amount of one pound in a year, the amount of (P) in (n) years $= P.R^{mn}$.

$$\therefore P R^{mn} = \frac{R^{mn} - 1}{R - 1} \times A.$$

$$\therefore P = \frac{1 - \dfrac{1}{R^{mn}}}{R - 1} \times A.$$

APPROXIMATE SOLUTION.

Let r be the interest of one pound for one year.

Then $R = 1 + \dfrac{r}{m}$ (m being the number of instants in a year.)

$\therefore R^m = \left(1 + \frac{r}{m}\right)^m = 1 + r + \frac{m-1}{2m} r^2 + \frac{\overline{m-1}.\,\overline{m-2}}{2.\,3.\,m^2} r^3$
$+$ &c.

$= 1 + r + \frac{r^2}{2} + \frac{r^3}{2.3} + \ldots\ldots$ nearly, when m is very large $= e^r$ (e being the base of hyperbolic logarithms.)

$\therefore R = e^{\frac{r}{m}}$

Hence $P = \dfrac{1 - \dfrac{1}{e^{rn}}}{e^{\frac{r}{m}} - 1} \times A$ nearly.

EQUATION OF PAYMENTS.

286. Let A, B, C, ... be the sums due after a, b, c ..... years respectively, (r) the interest of a pound for one year, and (R) its amount for the same time.

Now, it is evident, that the equated time ought to be such, that the present value of the whole sum for that time shall equal the sum of the present values of A, B, C ..... for the times a, b, c..... respectively.

First, let simple interest be allowed.

Then, the present value of (A) $= \dfrac{A}{1 + ar}$

$$\text{———————} \; B = \frac{B}{1 + br}$$

$$\&c. = \&c.$$

And that of (A+B+C+.....) for (x) years $= \dfrac{A+B+C+\ldots}{1 + xr}$

(x being the equated time.)

$$\therefore \frac{A}{1 + ar} + \frac{B}{1 + br} + \ldots\ldots = \frac{A+B+\ldots\ldots}{1 + xr},$$ whence (x) may be found.

Again, let compound interest be allowed.

Then, the present value of (A) $= \dfrac{A}{R^a}$

$$\text{———————} \; B = \frac{B}{R^b}$$

$$\&c. = \&c.$$

And that of (A+B+C+....) $= \dfrac{A.+B+C+\ldots\ldots}{R^x}$ (x being the equated time.)

$$\therefore \frac{A}{R^a} + \frac{B}{R^b} + \frac{C}{R^c} + \ldots\ldots = \frac{A+B+C+\ldots\ldots}{R^x} = \frac{S}{R^x}$$

$$\text{Hence } R^x = \frac{S R^s}{AR^{s-a} + BR^{s-b} + \ldots\ldots} \quad (s = a + b + c +, \text{ \&c.})$$

$$\therefore\ x = \frac{\log. S + s \log. R - \log. (AR^{s-a} + BR^{s-b} + CR^{s-c} + \ldots.)}{\log. R}$$

$$\begin{aligned} \text{Here } A &= S \\ B &= s \\ a &= T \\ b &= t \end{aligned}$$

$\therefore \dfrac{S}{1 + Tr} + \dfrac{s}{1 + tr} = \dfrac{S + s}{1 + xr}$, from which we obtain

$$x = \frac{ST + st + trT.(S + s)}{S + s + r.(St + sT)}$$

287. Here $A = 40$, $a = \dfrac{1}{2}$ year $= 6$ months

$$B = 60,\ b = 1$$

$$C = 80,\ c = 1\frac{1}{4} = \frac{5}{4}$$

$\therefore$ if (x) be the equated time, we have (allowing simple interest)

$$\frac{40}{1 + \frac{1}{2}r} + \frac{60}{1 + r} + \frac{80}{1 + \frac{5}{4}r} = \frac{180}{1 + xr}, \text{ whence}$$

$$x = \frac{24 + 40r + 15r^2}{48 + 84r + 34r^2} \text{ years}$$

If $r = \dfrac{1}{20}$ or the rate be 5 per cent.

$$x = \frac{10415}{10457} \text{ years} = 363 \text{ days nearly.}$$

Allowing compound interest, we have

$$\frac{40}{R^{\frac{1}{2}}} + \frac{60}{R} + \frac{80}{R^{\frac{5}{4}}} = \frac{180}{R^x}$$

$$\therefore\ 2R^{\frac{3}{4}} + 3R^{\frac{1}{4}} + 4 = 9R^{\frac{5}{4} - x}$$

$$\text{Hence } x = \frac{2 \log. 3 + \frac{5}{4} \log. R - \log. (2R^{\frac{3}{4}} + 3R^{\frac{1}{4}} + 4)}{\log. R}$$

Whence R is the amount of one pound in one year.

TRANSFORMATION OF EQUATIONS.

288. Let $y = \frac{1}{p-x}$

Then $yp - yx = 1 \quad \therefore x = \frac{py-1}{y} = p - \frac{1}{y}$

$$\therefore \quad x^3 = p^3 - \frac{3p^2}{y} + \frac{3p}{y^2} - \frac{1}{y^3}$$

$$-px^2 = -p^3 + \frac{2p^2}{y} - \frac{p}{y^2}$$

$$+\ qx = pq - \frac{q}{y}$$

$$-r = -r$$

$$\therefore pq - r - \frac{p^2+q}{y} + \frac{2p}{y^2} - \frac{1}{y^3} = 0$$

$$\therefore y^3 - \frac{p^2+q}{pq-r}y^2 + \frac{2p}{pq-r}y - \frac{1}{pq-r} = 0$$

in which equations the values of (y) being $\frac{1}{p-a}$, $\frac{1}{p-b}$, $\frac{1}{p-c}$, or $\frac{1}{b+c}$, $\frac{1}{a+c}$, $\frac{1}{a+b}$, it is the equation required.

Otherwise.

Let $y^3 + Py^2 + Qy + R = 0$ be the transformed equation.

Then $p = \frac{1}{p-a} + \frac{1}{p-b} + \frac{1}{p-c}$

$$= \frac{p^2-(b+c)p+bc+p^2-(a+c)p+ac+p^2-(a+b)p+ab}{p^3-(a+b+c)p^2+(ab+ac+bc)p-abc}$$

$$= \frac{3p^2-2p^2+q}{p^3-p^3+pq-r} = \frac{p^2+q}{pq-r}$$

Also $R = \frac{1}{p-a} \cdot \frac{1}{p-b} \cdot \frac{1}{p-c} = \frac{1}{(p-a)\,.\,(p-b)\,.\,(p-c)} = \frac{1}{pq-r}$

and $\frac{Q}{R} = p-a+p-b+p-c = 2p$, $\therefore Q = \frac{2p}{pq-r}$

$\therefore y^3 - \frac{p^2+q}{pq-r}y^2 + \frac{2p}{pq-r}y - \frac{1}{pq-r} = 0$ is the equation.

289. Let $y^3 + Py^2 + Qy + R = 0$ be the transformed equation.

Then $(y-a^2)\,.\,(y-b^2)\,.\,(y-c^2) = 0 = (y^{\frac{1}{2}}-a)\,.\,(y^{\frac{1}{2}}+a)\,.$ $(y^{\frac{1}{2}}-b) \times (y^{\frac{1}{2}}+b)\,.\,(y^{\frac{1}{2}}-c)\,.\,(y^{\frac{1}{2}}+c) = (y^{\frac{1}{2}}-a)\,.\,(y^{\frac{1}{2}}-b)\,.$ $(y^{\frac{1}{2}}-c) \times (y^{\frac{1}{2}}+a)\,.\,(y^{\frac{1}{2}}+b)\,.\,(y^{\frac{1}{2}}+c) = (y^{\frac{3}{2}}-py+qy^{\frac{1}{2}}-r)$ $\times (y^{\frac{3}{2}}+py+qy^{\frac{1}{2}}+r) = y^3-(p^2-2q)\,y^2+(q^2-2pr)\,y-r^2$

$$\therefore P = -(p^2-2q)$$
$$Q = (q^2-2pr)$$
$$R = -r^2$$

$\therefore y^3-(p^2-2q)\,y^2+(q^2-2pr)\,y-r^2 = 0$ is the equation required.

N.B. The shortest way of multiplying factors of the form $y^{\frac{3}{2}}-py+qy^{\frac{1}{2}}-r$, and $y^{\frac{3}{2}}+py+qy^{\frac{1}{2}}+r$ is transform them thus. The product $= \left\{(y^{\frac{3}{2}}+qy^{\frac{1}{2}})-(py+r)\right\}\left\{(y^{\frac{3}{2}}+qy^{\frac{1}{2}})+(py+r)\right\}$ $= (y^{\frac{3}{2}}+qy^{\frac{1}{2}})^2-(py+r)^2 = y^3+2qy^2+q^2y-p^2y^2-$ $2pry-r^2 = y^3-(p^2-2q)\,y^2+(q^2-2pr)\,y-r^2$

Otherwise.

$$x^3-px^2+qx-r=0$$
$$\therefore x^3+qx = px^2+r$$
$$\therefore x^6+2qx^4+q^2x^2 = p^2x^4+2prx^2+r^2$$
$$\therefore x^6-(p^2-2q)\,x^4+(q^2-2pr)\,x^2-r^2 = 0$$

Let $y = x^2$

Then $y^3-(p^2-2q)\,y^2+(q^2-2pr)\,y-r^2 = 0$, in which equation the values of y are a^2, b^2, c^2. It is $\therefore$ the equation required.

Otherwise.

Let $y^3 + Py^2 + Qy + R = 0$ be the equation.

$\therefore P = -(a^2 + b^2 + c^2)$

Now $(a + b + c)^2 = a^2 + b^2 + c^2 + 2ab + 2ac + 2bc$

or $p^2 = -P + 2q$

$\therefore P = -(p^2 - 2q)$

Again, $Q = a^2 b^2 + a^2 c^2 + b^2 c^2$

Now $(ab+ac+bc)^2 = a^2 b^2 + a^2 c^2 + b^2 c^2 + 2a^2 bc + 2b^2 ac + 2c^2 ab = Q + 2abc.(a + b + c) = Q + 2rp$

$\therefore Q = q^2 - 2rp$

Also $R = -a^2 b^2 c^2 = -r^2$

$\therefore$ &c... as before.

290. Diminish the roots by $\frac{p}{n} = \frac{12}{2} = 6$

$\therefore$ Let $y = x - 6$

Hence $x^2 = y^2 + 12y + 36$

$-12x = -12y - 72$

$+5 = +5$

$= y^2 - 31$ is the transformed equation.

291. Let $y = \frac{1}{p-x}$ $\therefore x = \frac{py-1}{y} = p - \frac{1}{y}$

$$\therefore x^n = p^n - n.\frac{p^{n-1}}{y} + n.\frac{n-1}{2}.\frac{p^{n-2}}{y^2} - \ldots\ldots\ldots \mp \frac{1}{y^n}$$

$$-px^{n-1} = -p^n + (n-1).\frac{p^{n-1}}{y} - (n-1).\frac{n-2}{2}.\frac{p^{n-2}}{y^2} - \ldots\ldots \mp \frac{p}{y^{n-1}}$$

$$+qx^{n-2} = qp^{n-2} - (n-2)q.\frac{p^{n-3}}{y} + (n-2).\frac{n-3}{2}q.\frac{p^{n-4}}{y^2} - \ldots\ldots\ldots \mp \frac{q}{y^{n-2}}$$

$$-rx^{n-3} = -rp^{n-3} + (n-3)r.\frac{p^{n-4}}{y} - (n-3).\frac{n-4}{2}r.\frac{p^{n-5}}{y^2} + \ldots\ldots \mp \frac{r}{y^{n-3}}$$

&c. = &c.

$\therefore$ by addition we have

$(qp^{n-2} - rp^{n-3} + sp^{n-4} - \ldots.) - (p^{n-1} + (n-2)\,q.p^{n-3} - (n-3).$

$rp^{n-4} + \ldots.) \frac{1}{y} + \left((n-1)p^{n-2} + \frac{\overline{n-2}.\overline{n-3}}{2} qp^{n-4} - \frac{\overline{n-3}.\overline{n-4}}{2}\right.$

$\left. \times rp^{n-5} + \ldots.\right) \frac{1}{y^2} + \ldots.$ &c.

$$\therefore\ y^n - \frac{p^{n-1} + \overline{n-2}\,.\,qp^{n-3} - \overline{n-3}\,rp^{n-4} + \ldots.}{qp^{n-2} - rp^{n-3} + sp^{n-4} - \ldots\ldots} y^{n-1}$$

$$\frac{\overline{n-1}\,p^{n-2} + \frac{\overline{n-2}\,.\,\overline{n-3}}{2} qp^{n-4} - \frac{\overline{n-3}\,.\,\overline{n-4}}{2} rp^{n-5} + \ldots.}{qp^{n-2} - rp^{n-3} + sp^{n-4} - \ldots.} \times y^{n-2} -$$

&c. $= 0$ is the equation required.

292. Let $y^n - Py^{n-1} + Qy^{n-2} - \ldots. = 0$ be the transformed equation.

Then $P = ma + mb + mc + \ldots. = m.(a + b + c + ..) = mp$

$Q = m^2.(ab + ac + \ldots.) = m^2 q$

$R = m^3 (abc + abd + \ldots) = m^3 r$

&c. = &c.

$\therefore\ y^n - mpy^{n-1} + m^2 qy^{n-2} - m^3 ry^{n-3} \ldots. = 0$ is the equation required.

293. $x^3 - 2x^2 + 1 = 0$

$\therefore\ x^3 + 1 = 2x^2$

$\therefore\ x^9 + 3x^6 + 3x^3 + 1 = 8x^6$

$\therefore\ x^9 - 5x^6 + 3x^3 + 1 = 0$

Let now $y = x^3$

Then $y^3 - 5y^2 + 3y + 1 = 0$, which is an equation whose roots are the cubes of the roots of the given equation, $\because\ y = x^3$.

294. Let $y^3 - Py^2 + Qy - R = 0$ be the required equation.

Then $P = (a+b) + (b+c) + (a+c) = 2\,(a+b+c) = 2p$

$P^2 - 2Q = (p-a)^2 + (p-b)^2 + (p-c)^2 = p^2 - 2ap + a^2 + p^2 - 2bp + b^2 + p^2 - 2cp + c^2 = 3p^2 - 2p.(a + b + c) + a^2 + b^2 + c^2 = p^2 + p^2 - 2q$

$\therefore\ 2Q = P^2 - 2p^2 + 2q = 4p^2 - 2p^2 + 2q = 2p^2 + 2q$

$\therefore\ Q = p^2 + q$

Also $R = (p-a).(p-b).(p-c)$

$= p^3 - (a+b+c)p^2 + (ab+ac+bc)p - abc$

$= p^3 - p^3 + qp - r$

$= qp - r$

$\therefore$ the required equation is $y^3 - 2py^2 + (p^2+q)y - qp + r = 0$.

Otherwise.

Let $y = p - x$ and substitute $(p-y)$ for (x) in the given equation; the result, properly arranged, will be the equation required.

295. Let $y = \frac{r}{x}$. Then $x = \frac{r}{y}$, which being substituted in the given equation, the result is

$$\frac{r^3}{y^3} - p\frac{r^2}{y^2} + \frac{qr}{y} - r = 0$$

$\therefore\ y^3 - qy^2 + pr\,y - r^2 = 0$, in which equation the roots are $\frac{abc}{a}, \frac{abc}{b}, \frac{abc}{c}$, or bc, ac, ab, as required.

Otherwise.

Let $y^3 - Py^2 + Qy - R = 0$ be the equation required.

Then $P = ab + ac + bc = q$

$R = ab \,.\, ac \,.\, bc = a^2 b^2 c^2 = r^2$

$$\frac{Q}{R} = \frac{1}{ab} + \frac{1}{bc} + \frac{1}{ac} = \frac{c+a+b}{abc} = \frac{p}{r}$$

$$\therefore\ Q = \frac{Rp}{r} = \frac{r^2 p}{r} = rp$$

$\therefore$ &c. . . . . .

296. Let $z = \frac{y}{p}$, $\therefore\ y = pz$ which being substituted, the resulting equation is

$$p^3 z^3 + 2p^3 z^2 - 33p^3 z + 14p^3 = 0$$

$\therefore\ z^3 + 2z^2 - 33z + 14 = 0$, in which the coefficients are numeral.

297. Let a, b, c be the roots of the given equation Then the roots of the required equation are $\pm\sqrt{am}, \pm\sqrt{bm}, \pm\sqrt{cm}$. It has $\therefore$ six dimensions.

Assume it to be $y^6+Py^5+Qy^4+Ry^3+Sy^2+Ty+V=0$

Here $-P=+\sqrt{am}-\sqrt{am}+\sqrt{bm}-\sqrt{bm}+\sqrt{cm}-\sqrt{cm}=0$

$$P^2-2Q=am+am+bm+bm+cm+cm=2mp$$

$$\therefore Q=-mp$$

$$\left.\begin{array}{l}-R=-am\sqrt{bm}+am\sqrt{bm}-am\sqrt{cm}+am\sqrt{cm}\\ \quad -bm.\sqrt{am}+m\sqrt{ab}.\sqrt{cm}-m\sqrt{ab}.\sqrt{cm}\\ \quad -m\sqrt{ab}.\sqrt{cm}+m\sqrt{ab}.\sqrt{cm}-cm\sqrt{am}\\ \quad +bm\sqrt{am}+m\sqrt{ab}\sqrt{cm}-m\sqrt{ab}.\sqrt{cm}\\ \quad +m\sqrt{ab}.\sqrt{cm}-m\sqrt{ab}.\sqrt{cm}+cm\sqrt{am}\\ \quad -bm\sqrt{cm}+\sqrt{bm}.\sqrt{cm}-cm\sqrt{bm}\\ \qquad\qquad +cm\sqrt{bm}\end{array}\right\}=0$$

$$\left.\begin{array}{l}S=abm^2-am^2\sqrt{bc}+am^2\sqrt{bc}\\ \quad am^2\sqrt{bc}-am^2\sqrt{bc}+acm^2\\ \quad bm^2\sqrt{ac}-bm^2\sqrt{ac}-m^2\sqrt{abc^2}+m^2\sqrt{abc^2}\\ \quad +m^2\sqrt{b^2ac}-m^2\sqrt{b^2ac}+m^2\sqrt{abc^2}-m^2\sqrt{abc^2}\\ \qquad\qquad +bcm^2\end{array}\right\}=0$$

$$=m^2(ab+ac+bc)=m^2q$$

Similarly T may be found $=0$

and $V=-m^3abc=-m^3r$

$\therefore$ the equation is $y^6-mpy^4+m^2qy^2-m^3r=0$

Otherwise,

The equation is equivalent to $(y-\sqrt{ma})\times(y+\sqrt{ma})\times(y-\sqrt{mb})\times(y+\sqrt{mb})\times(y-\sqrt{mc})\times(y+\sqrt{mc})=(y^2-ma).(y^2-mb).(y^2-mc)=y^6-m.(a+b+c)y^4+m^2(ab+ac+bc)y^2-m^3abc=y^6-mpy^4+m^2qy^2-m^3r$ as before.

Otherwise.

Let $y=\pm\sqrt{mx}$ $\therefore x=\dfrac{y^2}{m}$ which being substituted in the given equation, the result

is $\frac{y^6}{m^3} - p\frac{y^4}{m^2} + q\frac{y^2}{m} - r = 0$, whence

$$y^6 - mp\,y^4 + m^2 q\,y^2 - m^3 r = 0 \text{ as before.}$$

298. If the roots be rendered all positive, the alternate signs will be negative and positive. Now the greatest negative coefficient + 1, is > greatest root.

Assume $\therefore y = 1 + 1 - x = 2 - x$ (in the equation $x^3 - x^2 + \frac{3}{2}x + 3 = 0$) and $x = 2 - y$

$\therefore y$ will be always positive.

$$\left.\begin{aligned} \therefore x^3 &= 8 - 12y^2 + 6y^2 - y^3 \\ -x^2 &= -4 + 4y - y^2 \\ +\tfrac{3}{2}x &= 3 - \tfrac{3}{2}y \end{aligned}\right\} = 10 - \frac{19}{2}y + 5y^2 - y^3$$

$$+3 = +3$$

$\therefore y^3 - 5y^2 + \frac{19}{2}y - 10 = 0$ is the equation required.

299. Let $y^3 + Py^2 + Qy + R = 0$ be the equation required.

$$\text{Then } -P = \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{a^2} + \frac{1}{c^2} + \frac{1}{b^2} + \frac{1}{c^2}$$

$$= 2\left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}\right)$$

$$= 2 \times \frac{a^2b^2 + a^2c^2 + b^2c^2}{a^2b^2c^2}$$

Now $q^2 = (ab+ac+bc)^2 = a^2b^2 + a^2c^2 + b^2c^2 + 2abc^2 + 2ab^2c + 2a^2bc$

$$= a^2b^2 + a^2c^2 + b^2c^2 + 2abc.(a+b+c)$$

$$\therefore a^2b^2 + a^2c^2 + b^2c^2 = q^2 - 2abc.(a+b+c) = q^2 - 2rp$$

$$\text{Hence } -P = 2 \times \frac{q^2 - 2rp}{r^2}$$

$$\text{Again } Q = \left(\frac{1}{a^2}+\frac{1}{b^2}\right)\times\left(\frac{1}{a^2}+\frac{1}{c^2}\right)+\left(\frac{1}{a^2}+\frac{1}{b^2}\right)\times\left(\frac{1}{b^2}+\frac{1}{c^2}\right)$$

$$+\left(\frac{1}{a^2}+\frac{1}{c^2}\right)\times\left(\frac{1}{b^2}+\frac{1}{c^2}\right)=\frac{(a^2+b^2).(a^2+c^2)}{a^2b^2c^2\times a^2}+\frac{(a^2+b^2)\times(b^2+c^2)}{a^2b^2c^2\times b^2}$$
$$+\frac{(a^2+c^2).(b^2+c^2)}{a^2b^2c^2\times c^2}$$

Let $S_2=a^2+b^2+c^2$

$$\therefore Q=\frac{(S_2-c^2).(S_2-b^2)}{r^2\times a^2}+\frac{(S_2-c^2).(S_2-a^2)}{r^2\times b^2}+\frac{(S_2-b^2).(S_2-a^2)}{r^2\times c^2}$$

But $(S_2-c^2).(S_2-b^2)=S^2-{}^2(b^2+c^2).S_2+b^2c^2=S^2-(S_2-a^2)\,S_2+\frac{r^2}{a^2}=a^2S_2+\frac{r^2}{a^2}$

$$\therefore\ \frac{(S_2-c^2).(S_2-b^2)}{r^2\times a^2}=S_2+\frac{1}{a^4}$$

Similarly $\frac{(S_2-c^2).(S_2-a^2)}{r^2\times b^2}=S_2+\frac{1}{b^4}$

&c. = &c.

$$\therefore q=3S_2+\frac{1}{a^4}+\frac{1}{b^4}+\frac{1}{c^4}=3S_2+\frac{a^4b^4+a^4c^4+b^4c^4}{a^4b^4c^4}$$

Now $(q^2-2rp)^2=(a^2b^2+a^2c^2+b^2c^2)^2=a^4b^4+a^4c^4+b^4c^4+2a^2.a^2b^2c^2+2b^2.a^2b^2c^2+2c^2.a^2b^2c^2=a^4b^4+a^4c^4+b^4c^4+2r^2S_2$

$$\therefore\ a^4b^4+a^4c^4+b^4c^4=(q^2-2rp)^2-2r^2S_2$$

$$\therefore\ Q=3S_2+\frac{(q^2-2rp)^2-2r^2S_2}{r^4}$$

Again $-R=\left(\frac{1}{a^2}+\frac{1}{b^2}\right)\times\left(\frac{1}{a^2}+\frac{1}{c^2}\right)\times\left(\frac{1}{b^2}+\frac{1}{c^2}\right)$

$$=\frac{(a^2+b^2).(a^2+c^2).(b^2+c^2)}{a^4b^4c^4}=\frac{(S_2-c^2).(S_2-b^2).(S_2-a^2)}{r^4}$$

$$=\frac{S_2^3-(a^2+b^2+c^2)S_2^2+(a^2b^2+a^2c^2+b^2c^2)S_2-a^2b^2c^2}{r^4}$$

$$=\frac{S_2^3-S_2^3+(q^2-2rp).S_2-r^2}{r^4}$$

$$=\frac{(q^2-2pr).S_2-r^2}{r^4}$$

$$\therefore\ y^3-2\times\frac{q^2-2rp}{r^2}y^2+\frac{(q^2-2rp)^2+r^2S_2.(3r^2-2)}{r^4}y-\frac{(q^2-2pr)S_2-r^2}{r^4}=0$$ is the equation required (S_2 being $=p^2-2q$)

Otherwise,

First change the roots into their squares, thence into the reciprocals of their squares, and finally, into the sums of the reciprocals of every two.

300. Diminish the roots by the quantity (e) i.e., assume $y = x - e$, $\therefore x = y + e$

$$\begin{aligned}\therefore x^3 - px^2 + qx - r = y^3 &+ 3ey^2 + 3e^2y + e^3 \\ &- py^2 - 2epy - pe^2 \\ &+ qy + qe\end{aligned}$$

Now, to take away the third term of this transformed equation, we must have $3e^2 - 2ep + q = 0$

$$\therefore e^2 - \frac{2p}{3}e + \frac{p^2}{9} = \frac{p^2}{9} - \frac{q}{3} = \frac{p^2 - 3q}{9}$$

$\therefore e = \dfrac{p \pm \sqrt{p^2 - 3q}}{3}$, which being imaginary when p^2 is $< 3q$, shews that the third term cannot then be taken away, without rendering the coefficients imaginary.

EQUATIONS OF THE THIRD DEGREE.

301. For the proof, see *Wood.*

In the equation $x^3 + qx + r = 0$ the root is

$$x = \sqrt[3]{-\frac{r}{2} + \sqrt{\frac{r^2}{4} + \frac{q^3}{27}}} + \sqrt[3]{-\frac{r}{2} - \sqrt{\frac{r^2}{4} + \frac{q^3}{27}}}$$

which may be numerically expressed, when $\sqrt{\frac{r^2}{4} + \frac{q^3}{27}}$ is real, or when q is positive, or even when q is negative, provided $\frac{r^2}{4}$ is $=$ or $> \frac{q^3}{27}$. When $\sqrt{\frac{r^2}{4} + \frac{q^3}{27}}$ is of the form $Q . \sqrt{-1}$, or when q is negative, and $\frac{q^3}{27} > \frac{r^2}{4}$, we must have recourse to the "*Arithmetic of Sines,*" to obtain the numerical value. This is the "irreducible case" of Cardan's rule, which may be solved as follows:

Since, by supposition, q is negative, we have

$$x = \sqrt[3]{-\frac{r}{2} + \sqrt{\frac{r^2}{4} - \frac{q^3}{27}}} + \sqrt[3]{-\frac{r}{2} - \sqrt{\frac{r^2}{4} - \frac{q^3}{27}}}$$

Now $-\frac{r}{2} + \sqrt{\frac{r^2}{4} - \frac{q^3}{27}} = -\frac{r}{2} . \left(1 - \sqrt{1 - \frac{4}{27}\frac{q^3}{r^2}}\right)$

Since $\frac{q^3}{27}$ is $> \frac{r^2}{4}$, or $\frac{4}{27} . \frac{q^3}{r^2} > 1$, put $\frac{4}{27} . \frac{q^3}{r^2} = \frac{1}{\cos.^2\theta}$ (a quantity necessarily > 1, since the cos. of any $\angle > 0$ is < 1)

$\therefore -\frac{r}{2} + \sqrt{\frac{r^2}{4} - \frac{q^3}{27}} = -\frac{r}{2}\left(1 - \sqrt{1 - \frac{1}{\cos.^2\theta}}\right) = -\frac{r}{2} \times \left(\frac{\cos. \theta - \sqrt{-1} \sin. \theta}{\cos. \theta}\right) = -\sqrt{\left(\frac{q}{3}\right)^3} . (\cos. \theta - \sqrt{-1} \sin. \theta)$

since $\frac{r}{2} = \sqrt{\left(\frac{q}{3}\right)^3} . \cos \theta.$

Similarly $-\frac{r}{2}-\sqrt{\frac{r^2}{4}-\frac{q^3}{27}}=-\sqrt{\left(\frac{q}{3}\right)^3}(\cos.\theta+\sqrt{-1}\sin.\theta)$

$\therefore x=-\sqrt{\frac{q}{3}}.(\cos.\theta-\sqrt{-1}\sin.\theta)^{\frac{1}{3}}+-\sqrt{\frac{q}{3}}\times(\cos.\theta+\sqrt{-1}\sin.\theta)^{\frac{1}{3}}$

$=-\sqrt{\frac{q}{3}}.\left(\cos.\frac{\theta}{3}-\sqrt{-1}.\sin.\frac{\theta}{3}+\cos.\frac{\theta}{3}+\sqrt{-1}\times\sin.\frac{\theta}{3}\right)$, by Demoivre.

$=-2\sqrt{\frac{q}{3}}.\cos.\frac{\theta}{3}$, which, by means of reference to trigonometrical tables, will give one root of the equation. The other two may be found, either by reduction, or thus:
$\cos.\theta=-\cos.(180^\circ-\theta)=$ also $-\cos.(180^\circ+\theta)$.

$\therefore$ we may assume

$$\frac{4}{27}.\frac{q^3}{r^2}=\frac{1}{\cos.^2\theta}\text{ or }=-\frac{1}{\cos.^2(180^\circ-\theta)}\text{ or }=-\frac{1}{\cos.^2(180^\circ+\theta)},$$

whose corresponding results will evidently be,

$$x=-2\sqrt{\frac{q}{3}}.\cos.\frac{\theta}{3}$$

$$=2\sqrt{\frac{q}{3}}.\cos.\left(60^\circ-\frac{\theta}{3}\right)$$

$$=2\sqrt{\frac{q}{3}}.\cos.\left(60^\circ+\frac{\theta}{3}\right)$$

If r be negative as well as q, $x=2\sqrt{\frac{q}{3}}.\cos.\frac{\theta}{3}$

$$=-2\sqrt{\frac{q}{3}}.\cos.\left(60-\frac{\theta}{3}\right)$$

$$=-2\sqrt{\frac{q}{3}}.\cos.\left(60+\frac{\theta}{3}\right)$$

For another solution, (see 304.)

In the problem, $\sqrt{\frac{r^2}{4}-\frac{q^3}{27}}=315.\sqrt{-3}$ is imaginary.

$\therefore$ Cardan's rule does not here apply. We will try the above form.

$$\text{Cos. } \theta = \frac{r}{2}\sqrt{\frac{27}{q^3}} = \frac{3r}{2q} \cdot \sqrt{\frac{3}{q}} = \frac{442}{79} \cdot \sqrt{\frac{1}{79}}$$

$$= \frac{442}{79 \times 8.8881944} = .6294794 \text{ nearly.}$$

$$\therefore \theta = 50^\circ\ 59'\ 11''\ 47''' \text{ nearly.}$$

$$\therefore \cos.\frac{\theta}{3} = \cos.(16^\circ\ 59'\ 43''\ 55''') = .9605814 \text{ nearly.}$$

$$\therefore x = 2\sqrt{\frac{q}{3}}\cos.\frac{\theta}{3} = 2\sqrt{79} \times (.9605814) = 17.0686852579$$
nearly.

The root is accurately $= 17$, as we find by trial. If more decimal places had been taken, the above result would have been more exact. The other roots may be found more easily, since we already know θ. They may also be found after reduction.

302. By assuming $x = y + 1$, we take away the second term, and get $y^3 - 7y + 6 = 0$.

Hence $\sqrt{\frac{r^2}{4} - \frac{q^3}{27}} = \sqrt{9 - \frac{343}{27}} = \frac{10}{3}\sqrt{-\frac{1}{3}}$.

$\therefore$ Cardan's rule does not apply.

But $\cos.\theta = \frac{3r}{2q}\sqrt{\frac{3}{q}} = \frac{3\times 6}{2\times 7}\sqrt{\frac{3}{7}} = .8416819$ (see 301.)

$$\therefore \theta = 32^\circ\ 19'\ 54''\ 38'''$$

$$\therefore \frac{\theta}{3} = 10^\circ\ 46'\ 38''\ 12'''$$

$$\therefore \cos.\frac{\theta}{3} = .9847632 \text{ nearly.}$$

$$\therefore y = -2\sqrt{\frac{q}{3}}.\cos.\frac{\theta}{3} = -(3.05505) \times (.9847632) \text{ nearly.}$$

$= -3.0085008$ nearly. By trial we find (-3) to be the exact root. After reduction, by the solution of quadratic equation,

it will be seen that 1 and 2 are the other roots of the equation $y^3 - 7y + 6 = 0$.

$\therefore$ the corresponding roots of the given equation are $-3+1$, $1+1$, $2+1$, or -2, 2, 3, respectively, two of them being of the form $-a$, $+a$.

Whenever an equation has roots of the form $-a$, $+a$, as in the present instance, the last coefficient has a divisor of the form a^2, and the last but one, of the form (a). It may $\therefore$ abridge labour in such cases to substitute (a), as found by inspection, for (x), the result shewing whether it be a root or not. Thus $12 = 2^2 \times 3$ and $4 = 2 \times 2$. $\therefore$ we may try 2. It verifies the equation, and $\therefore$ is a root. (-2) also succeeds. Generally, if there be n roots of the form $\pm a$, the last (n) coefficients beginning with the last, are divisible by $a^n, a^{n-1}, \ldots. a^2, a$ respectively; for the last coefficient is the product of all the roots with their signs changed, the last but $=$ sum of the products of every $(m-1)$ roots with their signs changed, (m being the whole number of roots), &c. &c. Hence, if, by inspection, we find that the last (n) coefficients of an equation, are divisible by the successive powers of the same quantity, it is reasonable to conclude that there are (n) roots equal to it, or only differing in sign, and trial may be made accordingly. The general form of such equations is $\therefore$

$$x^m + A_1 x^{m-1} + \ldots\ldots B_1 . a . x^{n-1} + B_2 . a^2 . x^{n-2} + \ldots\ldots B_{n-1} . a^{n-1} . x + B_n . a^n = 0.$$

If there be also roots of the form $\pm b$, they may be found in the same manner.

Ex. $x^6 - 3x^5 - 11x^4 + 39x^3 + 10x^2 - 108x + 72 = 0$

$$\left.\begin{aligned} \text{Here, } 72 &= 2^3 \times 3^2 \\ 108 &= 2^2 \times 3 \times 9 \\ 10 &= 2 \times 5 \end{aligned}\right\} \text{Hence, it appears there are three}$$

roots of the form ± 2, and two roots of the form ± 3. By trial, $2, 2, -2$, and $3, -3$, are found to be the roots.

303. $\left.\begin{aligned} \text{Here } q &= -9 \\ r &= 28 \end{aligned}\right\} \quad \therefore \sqrt{\frac{r^2}{4} + \frac{q^3}{27}} = 13$

$\therefore \sqrt[3]{-14 \pm 13} = -1$ or -3

$\therefore x = \sqrt[3]{-14+13} + \sqrt[3]{-14-13}$ (see 300) $= -1-3 = -4$

Let a and b be the other two roots.

Then $(-a).(-b).4 = 28$, and $-a-b+4=0$

$\therefore ab = 7$, and $a+b=4$

$$\therefore a^2 + 2ab + b^2 = 16$$
$$\text{and } 4\,ab = 28$$

$$\therefore a^2 - 2ab + b^2 = -12$$

$$\therefore a - b = \pm\sqrt{-12} = \pm 2\sqrt{-3}$$
$$\text{and } a + b = 4$$

$$\therefore a = \frac{4 \pm 2\sqrt{-3}}{2} = 2 \pm \sqrt{-3}$$

$$\text{and } b = 2 \mp \sqrt{-3}$$

$\therefore$ the three roots of the equation are $-4, 2+\sqrt{-3}, 2-\sqrt{-3}$.

N.B. a and b might have been found by reducing the equation to a quadratic, &c.

304. $x^3 - px + q = 0$.

Let the roots a, b, c, be real.

Then $x^2 - (a+b)x + ab = 0$

or $x^2 + cx + \dfrac{q}{c} = 0$, since $a+b+c=0$, and $abc = q$

Hence $x = -\dfrac{c}{2} \pm \sqrt{\dfrac{c^2}{4} - \dfrac{q}{c}}$ the two roots a, b, which are not real, unless $\dfrac{c^2}{4}$ be $> \dfrac{q}{c}$ or $c^3 > 4q$

Let $c^3 = 4q + m$

But $c^3 - pc + q = 0$ (c being a root of the equation.)

$$\therefore c^3 = pc - q = 4q + m$$

$$\therefore p = \frac{5q+m}{c} = \frac{5q+m}{(4q+m)^{\frac{1}{3}}}$$

$$\therefore \frac{p^3}{27} = \frac{(5q+m)^3}{27(4q+m)}$$

$$\therefore \frac{p^3}{27} - \frac{q^2}{4} = \frac{(125q^3 + 75mq^2 + 15m^2q + m^3)4 - (4q^2 + m)27q^2}{108\,(4q + m)}$$

$$= \frac{892q^3 + 273mq^2 + 60m^2q + 4m^3}{108\,.\,(4q + m)} = \text{a positive quantity.}$$

$$\therefore \frac{p^3}{27} > \frac{q^2}{4}$$

$\therefore$ When the roots of the cubic are real, $\frac{q^2}{4} < \frac{p^3}{27}$

$$\text{or } \frac{27q^2}{4p^3} \text{ is } < 1 \text{ or } \frac{3q}{2p}\sqrt{\frac{3}{p}} \text{ is } < 1$$

Now, since the cosine of an angle is necessarily less than the radius of the circle, or the cos. θ to radius (1) necessarily less than (1), put cos. $\theta = \pm \frac{3q}{2p}\sqrt{\frac{3}{p}}$

But cos. $\theta = 4\cos.^3\frac{\theta}{3} - 3\cos.\frac{\theta}{3} = \pm\frac{3q}{2p}\sqrt{\frac{3}{p}} = \pm\frac{q}{2}\times\left(\frac{3}{p}\right)^{\frac{3}{2}}$

$\therefore 8\left(\frac{p}{3}\right)^{\frac{3}{2}}\cos.^3\frac{\theta}{3} - 6\,.\left(\frac{p}{3}\right)^{\frac{3}{2}}\cos.\frac{\theta}{3} + q = 0$ (if the negative value be taken.)

or $\left(2\sqrt{\frac{p}{3}}\cos.\frac{\theta}{3}\right)^3 - p\,.\left(2\sqrt{\frac{p}{3}}\cos.\frac{\theta}{3}\right) + q = 0$, an equation having the same coefficients as $x^3 - px + q$. $\therefore$ its roots are the same, or $x = 2\sqrt{\frac{p}{3}}\cos.\frac{\theta}{3}$

Let us find *à priori* the assumption for cos. θ.

$$\left.\begin{array}{l} 4\cos.^3\frac{\theta}{3} - 3\cos.\frac{\theta}{3} - \cos.\theta = 0 \\ \text{Also } x^3 - px + q = 0 \end{array}\right\}\text{To render them identical,}$$

Put $x = y\,.\cos.\frac{\theta}{3}$, and substitute.

Then $y^3\cos.^3\frac{\theta}{3} - py\cos.\frac{\theta}{3} + q = 0$

$$\therefore 4\cos.^3\frac{\theta}{3} - \frac{4p}{y^2}\cos.\frac{\theta}{3} + \frac{4q}{y^3} = 0$$

Let $\frac{4p}{y^2} = 3$, $\therefore \cos.\theta = -\frac{4q}{y^3}$

But $y^2 = \frac{4p}{3}$

$$\therefore y = \pm 2\sqrt{\frac{p}{3}}$$

$$\therefore \frac{1}{y^3} = \pm \frac{1}{8}\sqrt{\frac{27}{p^3}} = \pm \frac{3}{8p}\sqrt{\frac{3}{p}}$$

$$\therefore \cos.\theta = \mp \frac{12q}{8p}.\sqrt{\frac{3}{p}} = \mp \frac{3q}{4p}\sqrt{\frac{3}{p}}$$

Also $x = y.\cos.\frac{\theta}{3} = \pm 2\sqrt{\frac{p}{3}}.\cos.\frac{\theta}{3}.$

For another method of proof, see 301.

EQUATIONS HAVING EQUAL ROOTS.

305. If an equation have (n) equal roots, the equation of limits has $(n-1)$ of them, and the two equations have a common measure of the form $(x-a)^{n-1}$, (a being one of the equal roots, see *Wood*), which being found in the usual way, will give us the (n) equal roots.

Equal roots, however, may be found from inspection of the factors of the coefficients. For, if there be (n) equal roots (a), the last (n) coefficients beginning with the last will be divisible by a^n, a^{n-1} ...... a respectively. See 302.

To apply these two methods, we have, in the problem,

(1.) An equation of limits $3x^2 - 10x + 8 = 0$, the greatest common measure of which, and the given equation, is $(x-2)$.

$\therefore$ two of the roots are 2, 2. This third is $= \dfrac{4}{2 \times 2} = 1$.

2. We have $\left.\begin{array}{r} -4 = (-1) \times 2^2 \\ \text{and } 8 = 4 \times 2 \end{array}\right\}$ Hence we may conclude that $\pm$ 2 is one of the equal roots. By trial we find that $+$ 2 is that root.

306. Here, the Equation of Limits is $3x^2 + \dfrac{20}{7}x = 0$

$$\therefore x = 0 \text{ or } -\frac{20}{21}$$

$$\therefore 0 \text{ or } -\frac{20}{21} \text{ is one of the equal roots.}$$

By trial we find that $-\dfrac{20}{21}$ is that root.

Hence the third root $= -\dfrac{10}{7} + \dfrac{40}{21} = \dfrac{10}{21}$.

Otherwise,

$$\left.\begin{aligned}\frac{4000}{9261} &= \frac{10}{21} \times \overline{\frac{20}{21}}\Big|^2 \\ 0 &= 0 \times \frac{20}{21}\end{aligned}\right\} \quad \therefore \pm \frac{20}{21} \text{ is one of the equal roots, \&c.} \quad \text{(See 305.)}$$

307. Here, the Equation of Limits is $3x^2 + 16x + 20 = 0$. The greatest common measure of which and the given equation $= x + 2$. $\therefore$ the two equal roots are -2 and -2. Hence the third root $= -8 + 4 = -4$.

Otherwise,

$$\left.\begin{aligned}16 &= 4^2 \text{ or } = 4 \times 2^2 \\ 20 &= 5\times 4 \text{ or } = 10 \times 2\end{aligned}\right\} \therefore \pm 4 \text{ or } \pm 2 \text{ is one of the equal}$$

roots. By trial we find -2 is that root.

308. The greatest common measure of the equation, and its equation of limits, is $x - 2$. $\therefore$ 2 is one of the equal roots. Hence the other root $= 1 - 4 = -3$.

Otherwise.

$$\left.\begin{aligned}12 &= 3 \times 2^2 \\ 8 &= 4 \times 2\end{aligned}\right\} \text{Hence, it appears } \pm 2 \text{ is one of the equal roots,}$$

$+ 2$ verifies the equation, and is $\therefore$ the root.

309. Here, the greatest common measure between the given equation, and its equation of limits, $3x^2 - 10x + 8 = 0$, is $x - 4$, $\therefore$ the equal roots are each $= 4$.

Hence the other root $= 5 - 8 = -3$.

Otherwise.

$$\left.\begin{aligned}48 &= 3 \times 4^2 \\ 8 &= 2 \times 4\end{aligned}\right\} \text{Hence } \pm 4 \text{ is one of the equal roots, \&c.}$$

310. For the method of finding equal roots, see *Wood* on *Depression of Equations*; also Problem 305.

The equation of limits to the given equation is $5x^4 - 52x^3 + 201x^2 - 342x + 216 = 0$, the greatest common measure of which is $x^3 - 8x^2 + 21x - 18 = 0$.

This common measure evidently does not contain three equal roots; for then 18 would be a perfect cube. If two of its roots be equal, we can find them by taking the greatest common measure of it, and its equation of limits. This common measure is $x - 3$. $\therefore$ two of the roots of $x^3 - 8x^2 + 21x - 18 = 0$ are 3. Hence the other is $= 8 - 6 = 2$.

$\therefore$ the greatest common measure of the given equation and its limit, is $(x - 3)^2 . (x - 2)$. Hence, three of the roots of that equation are 3, and the other two, 2.

If the greatest common measure of the given equation and its limit had all its roots unequal, they must be found by some other process, and the given equation would have had two of each sort. Generally, if an equation have A roots of the form $= a$, B roots $= b$, &c., then the greatest common measure will be of the form $(x - a)^{A-1} (x - b)^{B-1}$ &c. $= o$, the greatest common measure of which and its limit must be found, and so on until the last greatest common measure shall be of the form $(x-a) . (x-b) . (x-c) \ldots\ldots = o$. This being solved, we shall have the means of discovering all the sets of equal roots.

The second process (305) is shewn by the present problem to be much more convenient than the common one.

$$\text{Thus}\ \left.\begin{aligned} 108 &= 3^3 \times 2^2 \\ 216 &= 3^2 \times 2 \times 12 \\ 171 &= 3 \ \times 57 \end{aligned}\right\} \therefore \text{ we see at once the probability}$$

of there being three roots of the form ± 3 and two of the form ± 2. By trial 3, 3, 3, 2, 2, are found to succeed.

311. The equation of limits to this equation is $4x^3 - \frac{1}{2} = o$ $\therefore$ $x = \frac{1}{2}$ is one of its roots. This root is found by trial, to be also a root of the given equation. Therefore the equations have

a common measure $(x - \frac{1}{2})$. Hence two of the roots of the original equation are each $= \frac{1}{2}$. Dividing it by $(x - \frac{1}{2})^2 = x^2 - x + \frac{1}{4}$ the quotient is $x^2 + x + \frac{3}{4} = o$ whose roots are $- \frac{1}{2} + \sqrt{-\frac{1}{2}}$ and $- \frac{1}{2} - \sqrt{-\frac{1}{2}}$ the other two roots of the given equation.

Otherwise,

$$\left.\begin{array}{l} \dfrac{3}{16} = \dfrac{3}{4} \times \left(\dfrac{1}{2}\right)^2 \\ -\dfrac{1}{2} = (-1) \cdot \dfrac{1}{2} \end{array}\right\} \text{Hence there are two roots of the form}$$

$\pm \frac{1}{2}$. By trial, $\frac{1}{2}$ succeeds, and $- \frac{1}{2}$ does not. $\therefore$ &c.

EQUATIONS

HAVING PAIRS OF ROOTS OF THE FORM $(\pm a)$.

312. If an equation have pairs of roots of the form $\pm a$, $\pm b$, &c., it will be of the form $Q \times (x^2-a^2).(x^2-b^2)$, &c. $= 0$, (A). Change the signs of the roots, by changing the signs of the 2nd, 4th, &c. terms, and the resulting equation will evidently be of the form $Q' \times (x^2-a^2).(x^2-b^2)$, &c. $= 0$, (B).

Thus A and B have a common measure of the form $(x^2-a^2).\times(x^2-b^2)$, &c. $= 0$, whose roots a, $-a$, b, $-b$, &c., will be roots of the given equation.

Pairs of roots of the form $\pm a$, may also be found from the divisions of the last two coefficients. Thus, if the last coefficient be divisible by (a^2), and the last but one, at the same time, divisible by (a), we may conclude that two of the roots are either equal, or only differ in sign, and trial may be made accordingly. (See 302.)

If an equation of an even number of dimensions have all its roots in pairs of the form $(\pm a)$, it may be reduced to half the number of dimensions, by putting $y = x^2$. Thus in the equation whose roots are a, $-a$, b, $-b$, c, $-c$, &c., to $2n$ terms, we have $(x^2-a^2).(x^2-b^2).(x^2-c^2)$ ...... to n terms $= 0$; put $y=x^2$. Then $(y-a^2).(y-b^2).(y-c^2)$ ...... n terms $= 0$, an equation whose roots are a^2, b^2, c^2, &c. to (n) terms. Hence we have the roots of the given equation $= x = \pm\sqrt{y} = \pm a, \pm b$, &c.

Equations of this kind have only the *even* powers of the unknown quantity x, and are hence known to be of that kind.

In the problem, $4x^3 - 32x^2 - x + 8 = 0$. Change the signs of the roots; then, $4x^3 + 32x^2 - x - 8 = 0$.

The greatest common measure of the equations is $4x^2 - 1 = 0$; or $x^2 - \frac{1}{4} = 0$.

$\therefore + \frac{1}{2}$ and $- \frac{1}{2}$ are two of the roots sought.

The third $= 8$ which might at once have been found, and the others by reduction. The mode of divisors will evidently lead to the same result.

313. Here, the greatest common measure of the given equation, and of that whose roots are those of the given equation with their signs changed, is x^2-9.

$\therefore x = \pm 3$ gives two of the roots.

Let a and b be the other two.

Then $a + b = -3$

$ab \times 3 \times (-3) = -18$

$\therefore ab = 2$

Hence $a^2 + 2ab + b^2 = 9$

and $4ab = 8$

$\therefore a^2 - 2ab + b^2 = 1$

$\therefore a - b = \pm 1$

But $a + b = -3$

$\therefore a = -1$ or -2

$b = -2$ or -1

$\therefore$ the roots required are $3, -3, -1, -2$.

Otherwise.

$\left.\begin{array}{l} 18 = 2 \times 3^2 \\ 27 = 9 \times 3 \end{array}\right\}$ $\therefore$ there may be two roots of the form ± 3, which on trial we find to be the case.

314. Since one root is of the form $\sqrt{a}$, there is another of the form $-\sqrt{a}$, (surd roots enter equations by pairs). Hence the given equation, and $x^3 + 4x^2 - 3x - 12 = 0$, have a common measure of the form $x^2-(\sqrt{a})^2 = x^2 - a$, which is found by the usual method to be x^2-3. $\therefore$ two of the roots are $\sqrt{3}$ and $-\sqrt{3}$. Hence, the third root $= 4 - \sqrt{3} + \sqrt{3} = 4$.

315. $$\left.\begin{aligned} x^4+x^3+11x^2+9x+18=0 \\ x^4-x^3+11x^2-9x+18=0 \end{aligned}\right\}$$ The greatest common measure of which equations, is found to be $x^2+9=0$

$\therefore\ x=\pm 3\sqrt{-1}$, which gives two roots of the equation.

Let a and b be the other two roots

Then $a+b=-1+3\sqrt{-1}-3\sqrt{-1}=-1$.

$$\text{and } ab=\frac{18}{3\sqrt{-1}\times(-3).\sqrt{-1}}=\frac{18}{9}=2$$

$$\therefore\ a^2+2ab+b^2=1$$
$$\text{and } 4ab \quad =8$$

$$\therefore\ a-b=\pm\sqrt{-7}$$
$$\text{But } a+b=-1$$

$$\therefore\ a=\frac{-1\pm\sqrt{-7}}{2}$$

and $b=\dfrac{-1\mp\sqrt{-7}}{2}$ the other two roots of the equation.

N. B. In the problem, $-11x^2$ should have been printed $+11x^2$.

Otherwise.

$$\left.\begin{aligned} 18=2\times 3^2=-2\times(3\sqrt{-1})^2 \\ 9=3\times 3=-3\sqrt{-1}\times(3\sqrt{-1}) \end{aligned}\right\}$$ Hence the roots of the form $\pm a$, appear to be $\pm 3\sqrt{-1}$. By trial, $3\sqrt{-1}$ is found to be a root. $\therefore\ -3\sqrt{-1}$ is also a root, &c.

EQUATIONS,

WHOSE ROOTS ARE IN ARITHMETICAL PROGRESSION.

316. Let $a+b, a+2b, a+3b, \ldots\ldots a+nb$ be the roots.

Then $(a+b)+(a+2b)+\ldots\ldots(a+nb)=p=na+\frac{n.\overline{n+1}}{2}b$

$$\left.\begin{array}{l} \text{and } a^2+2ab+b^2 \\ \quad +a^2+4ab+4b^2 \\ \quad +a^2+6ab+9b^2 \\ \quad \cdots\cdots\cdots\cdots \\ \quad +a^2+2nab+n^2b^2 \end{array}\right\} = p^2-2q.$$

But $2ab+4ab+\ldots\ldots 2nab = 2ab\,(1+2+3+\ldots\ldots n)$

$$= 2ab\,.\,n\,.\frac{\overline{n+1}}{2} = n\,.\,\overline{n+1}\;ab$$

and $b^2+4b^2+9b^2+\ldots\ldots n^2b^2 = b^2\,(1+4+9+\ldots\ldots n^2)$

Put $1+4+9+\ldots\ldots n^2 = S$

Then $\Delta\; S = (n+1)^2 = \overline{n+1}\times\overline{n+1} = n.\overline{n+1}+n+1$

$$\therefore\; S = \frac{\overline{n-1}\,.\,n\,.\,\overline{n+1}}{3} + \frac{n.\overline{n+1}}{2}$$

$$= \frac{n.\overline{n+1}\times\overline{2n-2+3}}{6} = \frac{n.\overline{n+1}.\overline{2n+1}}{6}$$

Hence $p^2-2q = na^2+n\,.\,\overline{n+1}\;ab+\frac{n.\overline{n+1}\,.\,\overline{2n+1}}{6}b^2$

Also $\frac{p^2}{n} = na^2+n\,.\,\overline{n+1}\;ab+\frac{n.(n+1)^2}{4}b^2$

$$\therefore\; \frac{n-1}{n}p^2-2q = b^2\times\frac{n.\overline{n+1}}{12}\times(2\,.\,\overline{2n+1}-3\,.\,\overline{n+1})$$

$$= b^2\times\frac{n.\overline{n+1}}{12}\times(n-1) = \frac{n.(n^2-1)}{12}b^2$$

$$\therefore\ b^2 = \frac{12}{n^2} \times \frac{(n-1)p^2 - 2nq}{n^2-1}$$

$$\therefore\ b = \frac{2}{n} \times \sqrt{\frac{3.(n-1)p^2 - 6nq}{n^2-1}}$$

$$\therefore\ a = \frac{p}{n} - \frac{n+1}{2}.\ b = \frac{p}{n} - \frac{n+1}{n}.\ \sqrt{\frac{3.(n-1)p^2 - 6nq}{n^2-1}}$$

317. See 316. Where $n = 3$

$$p = 6$$
$$q = -4$$

$$\therefore b = \frac{2}{3} \times \sqrt{\frac{3 \times (3-1)\,6^2 + 6 \times 3 \times 4}{9-1}} = \frac{2}{3} \times \sqrt{36} = \pm 4$$

Let b be positive, or the series ascending.

$$\therefore\ a = \frac{6}{3} - \frac{4}{2} \times 4 = -6$$

$$\left.\begin{aligned} \therefore\ a + b &= -6 + 4 = -2 \\ a + 2b &= -6 + 8 = +2 \\ a + 3b &= -6 + 12 = 6 \end{aligned}\right\}$$ which are the three roots required.

Otherwise,

Let $a - b,\ a,\ a + b$ be the roots.

Then $3a = p = 6$

$$\therefore\ a = 2$$

and $(a^2 - b^2)\,a = -24$

or $4 + 12 = b^2$

$\therefore\ b^2 = \pm 4$ as before; whence we have the roots.

318. See 316. Where $n = 3$

$$p = 9$$
$$q = 23$$

$$\therefore\ b = \frac{2}{3} \sqrt{\frac{3 \times (3-1)\,81 - 6 \times 3 \times 23}{8}} = \frac{2}{3} \sqrt{\frac{36}{4}} = \frac{2}{3}$$

$$\times \left(\pm \frac{6}{2}\right) = \pm 2$$

Hence $a = 3 - 2 \times 2 = -1$

$$\left.\begin{aligned} \therefore a + b &= 2 - 1 = 1 \\ a + 2b &= 3 \\ a + 3b &= 5 \end{aligned}\right\}\text{which are the roots required.}$$

We find upon trial that these values do not satisfy the equation, unless the last term be -15 instead of -16. Hence it appears there is a mistake in the enunciation.

Otherwise.

Let $a - b$, a, $a + b$ be the roots.

Then $3a = p = 9$

$\therefore a = 3$

and $(a^2 - b^2) \times 3 = 15$

$\therefore 27 - 15 = 3b^2$

$\therefore b^2 = 4$

and $b = \pm 2$ as before.

319. See 316. Where $n = 4$

$p = 8$

$q = 14$

$$\therefore b = \frac{2}{4} \times \sqrt{\frac{3 \times 3 \times 64 - 6 \times 4 \times 14}{15}} = \pm 2$$

$$\therefore a = \frac{8}{4} - \frac{5}{2} \times 2 = 2 - 5 = -3$$

$$\left.\begin{aligned} \therefore a + b &= -3 + 2 = -1 \\ a + 2b &= -3 + 4 = 1 \\ a + 3b &= -3 + 6 = 3 \\ a + 4b &= -3 + 8 = 5 \end{aligned}\right\}\text{which are the roots required.}$$

Otherwise.

Let $a - 3b$, $a - b$, $a + b$, $a + 3b$ be the roots.

Then $4a = 8$

$\therefore a = 2$

Also $(a^2-9b^2)(a^2-b^2) = 16 - 40b^2 + 9b^4 = -15$

$$\therefore\ b^4 - \frac{40}{9}b^2 = -31$$

$$\therefore b^2 - \frac{20}{9} = \pm\sqrt{\frac{400-279}{81}} = \pm\frac{11}{9}$$

$$\therefore\ b^2 = \frac{20\pm 11}{9} = \frac{31}{9} \text{ or } 1$$

$$\therefore b = \pm\sqrt{\frac{31}{9}} \text{ or } \pm 1$$

If $b = 1$ we shall obtain the roots in the same order as before.

EQUATIONS,

WHOSE ROOTS ARE IN GEOMETRICAL PROGRESSION.

320. Let $\frac{a}{m}$ a, am be the roots.

Then $\frac{a}{m} \times a \times am = a^3 = r$

$\therefore a = r^{\frac{1}{3}}$

Also $\frac{a}{m} + a + am = a\left(1 + m + \frac{1}{m}\right) = p$

$m^2 + m + 1 = \frac{p}{a} \times m = \frac{p}{r^{\frac{1}{3}}} m$

$\therefore m^2 + \frac{r^{\frac{1}{3}} - p}{r^{\frac{1}{3}}} m = -1$

$\therefore m^2 + \frac{r^{\frac{1}{3}} - p}{r^{\frac{1}{3}}} m + \frac{r^{\frac{2}{3}} - 2pr^{\frac{1}{3}} + p^2}{4r^{\frac{2}{3}}} = \frac{p^2 - 2pr^{\frac{1}{3}} - 3r^{\frac{2}{3}}}{4r^{\frac{2}{3}}}$

$\therefore m = \frac{p - r^{\frac{1}{3}} \pm \sqrt{p^2 - 2pr^{\frac{1}{3}} - 3r^{\frac{2}{3}}}}{2r^{\frac{1}{3}}}$

But $a = r^{\frac{1}{3}}$

$\therefore$ the roots $\frac{a}{m}$, a, am are found.

N. B. When the equation is of an odd number $(2n+1)$ of dimensions, the middle root is equal to the $(2n+1)^{th}$ root of the last coefficient with its sign changed. For if $\frac{a}{r^n}$, $\frac{a}{r^{n-1}}$, ... $\frac{a}{r}$, a, ar, ... ar^{n-1}, ar^n be the roots and L the last coefficient we have, $\frac{a}{r^n} \times ar^n \times \frac{a}{r^{n-1}} \times ar^{n-1} \times$, &c. $= -$ L

$\therefore a^{m+1} = -L$

and $a = (-L)^{\frac{1}{m+1}} = -(L)^{\frac{1}{m+1}}$

321. See 320. Where $r = 27$
$p = 13$

$\therefore a = 3$

$$m = \frac{13-3+\sqrt{64}}{6} = \frac{18}{6} = 3$$

$\therefore$ the roots required are $\frac{a}{r}$, a, ar or 1, 3, 9.

Otherwise.

Find the middle root (a) by taking the cube root of 27. Then depress the equation to a quadratic, &c.

322. See 320. Where $r = 8$
$p = 7$

$\therefore a = 2$

$$m = \frac{7-2+\sqrt{49-28-12}}{4} = \frac{5+3}{4} = 2$$

$\therefore$ the required roots are $\frac{2}{2}$, 2, 2×2, or 1, 2, 4.

323. See 320. Where $r = 64$
$p = 14$

$\therefore a = 4$

$$m = \frac{14-4+\sqrt{196-112-48}}{8} = 2$$

$\therefore$ the required roots are 2, 4, 8.

EQUATIONS,

WHOSE ROOTS ARE IN HARMONICAL PROGRESSION.

Transform the equation into one, whose roots are the reciprocals of the roots of the given equation, and are, therefore, in arithmetical progression. These latter roots may be found as in (316), which being reciprocated, will give the roots required.

The general equation, however, may be thus solved independently.

First, let it be of an odd number of degrees, *viz.* $2m+1$, and the roots will be of the form $\frac{1}{a-md}$, $\frac{1}{a-\overline{m-1}d}$ .... $\frac{1}{a}$ ...... $\frac{1}{a+\overline{m-1}d}$, $\frac{1}{a+md}$, the middle term being $\frac{1}{a}$.

Let p, q, r, &c., be the first, second, &c. coefficients, R, Q, P, &c., the last, last but one, &c., coefficients.

Then $\frac{Q}{R}=(a-md)+(a-\overline{m-1}d)\ ......\ a\ ......\ (a+\overline{m-1}d)$ $+(a+md)=(2m+1)a$

$\therefore a=\frac{Q}{R}\times\frac{1}{2m+1}$. $\therefore$ the middle root is $\frac{2\overline{m+1}\times R}{Q}$

Again, if w, n, r, &c., represent the roots of any equation, whose coefficients are p, q, &c., we have $\frac{1}{w^2}+\frac{1}{n^2}+\frac{1}{r^2}+\ldots$ $=\left(\frac{1}{w}+\frac{1}{n}+\frac{1}{r}+..\right)^2-2\times$(sum of the products of the reciprocals of every two of the roots)$=\left(\frac{Q}{R}\right)^2-2\times\frac{P}{R}=\frac{Q^2-2PR}{R^2}$.

$$\left.\begin{array}{l}\text{Hence } a^2 - 2amd + m^2d^2 + a^2 + 2amd + m^2d^2 \\ \quad a^2 - 2a.\overline{m-1}d + (m-1)^2 d^2 + a^2 + 2a\times \\ \overline{m-1}d + \overline{m-1}|^2 d^2 \\ \quad \&c. \quad + \quad \&c. \\ \quad a^2 \end{array}\right\} = \frac{Q^2-2PR}{R^2}$$

Or $(2m+1)\times a^2 + 2d^2 \times \left\{m^2 + (m-1)^2 + \ldots. 2^2 + 1^2\right\} = \dfrac{Q^2-2PR}{R^2}$

Put $1 + 2^2 + \ldots. m^2 = S$

Then $\Delta\ S = (m+1)^2 = \overline{m+1}.m + m + 1$

$$\therefore\ S = \frac{\overline{m-1}.m.\overline{m+1}}{3} + \frac{m.\overline{m+1}}{2} = \frac{(2m-2+3)\times m\times(m+1)}{6}$$

$$= \frac{m.\overline{m+1}\,.\,.\overline{2m+1}}{6}$$

Hence $d^2 \times \dfrac{m.\overline{m+1}\,.\,\overline{2m+1}}{3} = \dfrac{Q^2-2PR}{R^2} - (2m+1)$
$\times \dfrac{Q^2}{R^2} \times \dfrac{1}{(2m+1)^2}.$

After the proper reductions we find

$$d = \frac{1}{(2m+1)R} \times \sqrt{\frac{6}{m.(m+1)} \times (mQ^2 - \overline{2m+1}.PR)},$$

whence, as we have already found (a), we can easily obtain all the roots.

Secondly, let the number of roots be even, *viz.* $2m$, and therefore of the form $\dfrac{1}{a-(2m-1)d}$, $\dfrac{1}{a-(2m-3)d}$ ...... $\dfrac{1}{a-d}$, $\dfrac{1}{a+d}$ ...... $\dfrac{1}{a+(2m-3)d}$, $\dfrac{1}{a+\overline{2m-1}d}$

Then $\dfrac{Q}{R}$ = sum of reciprocals of the roots $= 2m \times a$

$$\therefore\ a = \frac{Q}{2mR}$$

Again, as before, the sum of the squares of the reciprocals

$$\begin{aligned} = \frac{Q^2-2PR}{R^2} &= a^2 - 2a.(2m-1)d + (2m-1)^2 d^2 \\ &+ a^2 + 2a.(2m-1)d + (2m-1)^2 d^2 \\ &+ a^2 - 2a.(2m-3)d + (2m-3)^2 d^2 \\ &+ a^2 + 2a.(2m-3)d + (2m-3)^2 d^2 \\ &+ \quad \&c. \qquad \&c. \qquad \&c. \end{aligned}$$

$$= 2m\,a^2 + 2\,(d^2 + 3^2 d^2 + \ldots.. \overline{2m-3}|^2 d^2 + \overline{2m-1}|^2 d^2)$$

Put $1+3^2+5^2+\ldots\ldots\overline{2m-3}|^2+\overline{2m-1}|^2=S$

Then $\Delta\ S=(2m+1)^2=(2m+1)\times(2m-1)+2\times(2m+1)$

$$\therefore S=\frac{(2m-3)\times(2m-1)\times(2m+1)}{3\times 2}+\frac{2\times(2m-1)\times(2m+1)}{2\times 2}+C$$

$$=\frac{(4m^2-1)(2m-3+3)}{6}=\frac{(4m^2-1)\times m}{3}.$$

Hence $d^2=\left(\dfrac{Q^2-2PR}{2R^2}-2m\,a^2\right)\times\dfrac{3}{m\times(4m^2-1)}$

$$=\frac{1}{m^2R^2}\times\frac{3}{2\times(4m^2-1)}\times(\overline{m-1}\,.\,Q^2-2m\,PR)$$

$$\therefore d=\frac{1}{mR}\times\sqrt{\frac{3}{2\,.\,(4m^2-1)}\times(\overline{m-1}\,.\,Q^2-2m\,PR)}$$

Also $a=\dfrac{Q}{2m\,R}$. Hence the roots in this case may easily be found, and the general equation, whose roots are in harmonical progression, may thus be completely solved.

324. Since the reciprocals of the roots are in arithmetical progression, let the roots be $\dfrac{1}{a-d}$, $\dfrac{1}{a}$, $\dfrac{1}{a+d}$.

Then $\dfrac{q}{r}=a-d+a+a+d=3a$

$$\therefore a=\frac{q}{3r}$$

Also $r=\dfrac{1}{a-d}\times\dfrac{1}{a}\times\dfrac{1}{a+d}=\dfrac{1}{a}\times\dfrac{1}{a^2-d^2}$

$$\therefore a^2-d^2=\frac{1}{ar}$$

and $d^2=a^2-\dfrac{1}{ar}=\dfrac{q^2}{9r^2}-\dfrac{3}{q}=\dfrac{q^3-27r^2}{9qr^2}$

$\therefore d=\pm\dfrac{1}{3r}\sqrt{\dfrac{q^3-27r^2}{q}}$ according as the series is descending or ascending.

Hence $a\pm d=\dfrac{q}{3r}\pm\dfrac{1}{3r}\sqrt{\dfrac{q^3-27r^2}{q}}=\dfrac{q^{\frac{3}{2}}\pm\sqrt{q^3-27r^2}}{3rq^{\frac{1}{2}}}$

$$\therefore\ \frac{1}{a-d}=\frac{3rq^{\frac{1}{2}}}{q^{\frac{3}{2}}-\sqrt{q^3-27r^2}},\ \frac{1}{a}=\frac{3r}{q}$$

and $\frac{1}{a+d}=\frac{3rq^{\frac{1}{2}}}{q^{\frac{3}{2}}+\sqrt{q^3-27r^2}}$, which are the three roots required.

These roots may also be expressed in terms of p and r, or of p and q.

325. Hence $8x^3-6x^2-3x+1=0$

$$\therefore\ x^3-\frac{3}{4}x^2-\frac{3}{8}x+\frac{1}{8}=0$$

$$\therefore\ a=\frac{q}{3r}=\frac{3}{8}\times\frac{8}{3}=1$$

$$\text{and }\frac{1}{a-d}=\frac{3\times\frac{1}{8}\times\sqrt{\frac{3}{8}}}{\frac{3}{8}^{\frac{3}{2}}-\sqrt{(\frac{3}{8})^3-\frac{3^3}{8^2}}}=-\frac{1}{2}$$

and similarly $\frac{1}{a+d}=\frac{1}{4}$

326. See 324. Where $\left.\begin{array}{l}q=36\\ r=36\end{array}\right\}$

$$\therefore\ a=\frac{q}{3r}=\frac{1}{3}\quad\therefore\ \frac{1}{a}=3$$

$$\text{and }\frac{1}{a-d}=\frac{3\times36^{\frac{3}{2}}}{36^{\frac{3}{2}}-36\sqrt{36-27}}=\frac{3\times36^{\frac{1}{2}}}{36^{\frac{1}{2}}-3}=6$$

$$\text{and }\frac{1}{a+d}=\frac{3\times36^{\frac{3}{2}}}{36^{\frac{3}{2}}+36\sqrt{36-27}}=\frac{3\times6}{6+3}=2$$

$\therefore$ the roots required are 6, 3 and 2.

327. Here $ay^3-by^2-cy+1=0$

Let $y=\frac{1}{x}$

Then $\frac{a}{x^3}-\frac{b}{x^2}-\frac{c}{x}+1=0$

$\therefore\ x^3-cx^2-bx+a=0$ whose roots are in arithmetical progression, and $\therefore$ of the form $m-d$, m, $m+d$.

Now $c = m - d + m + m + d = 3m$

$$\therefore m = \frac{c}{3}$$

Again $c^2 + 2b = \overline{m-d}|^2 + m^2 + \overline{m+d}|^2$

$$= 3m^2 + 2d^2 = \frac{c^2}{3} + 2d^2$$

$$\therefore 2d^2 = \frac{2}{3}c^2 + 2b$$

$$\therefore d = \sqrt{\frac{c^2 + 3b}{3}}$$

Hence the values of x, are $\frac{c}{3} - \sqrt{\frac{c^2 + 3b}{3}}$, $\frac{c}{3}$ and $\frac{c}{3} + \sqrt{\frac{c^2 + 3b}{3}}$, or $\frac{c - \sqrt{3c^2 + 9b}}{3}$, $\frac{c}{3}$, and $\frac{c + \sqrt{3c^2 + 9b}}{3}$

$\therefore$ the values of y are $\frac{3}{c - \sqrt{3c^2 + 9b}}$, $\frac{3}{c}$, and $\frac{3}{c + \sqrt{3c^2 + 9b}}$

EQUATIONS,

HAVING ROOTS OF THE FORM $a, \frac{1}{a}, b, \frac{1}{b}$, &c.

(1) If all the roots are of the above form (the number of dimensions being ∴ even), the coefficients equally distant from either end of the equation, are equal, and of the same sign. (2) If all but one be of that form, (the number of dimensions being ∴ odd) and this one be $\pm$ unity, the coefficients equally distant from either end will be equal, but not necessarily of the same sign. (3) If there be more than one root not of the reciprocal form, there is no such law among the coefficients.

To solve the equation in the first case, resolve it into quadratic factors of the form, $x^2 - mx + 1$, &c. (See *Wood* on recurring Equations.) In the second case, divide the equation by $(x \pm 1)$, according as the last term is negative or positive, and the resulting equation will be of the first form, and may therefore be solved in the manner of the first case. In the third case, find the reciprocal equation. It will have as many roots of the reciprocal form as the original equation. They have ∴ a common measure of the first reciprocal form, which being found according to the common method, and solved by the first case, will produce the roots required.

The two first cases may also be solved in the following manner:

First, let the number of roots be even, and those roots be of the form $a, \frac{1}{a}, b, \frac{1}{b}$, &c.

Let $M =$ the middle coefficient, p, q, &c., being those taken in order from either end of the equation; in short, let the equation be of the form

$$x^{2m} + px^{2m-1} + qx^{2m-2} + \ldots\ldots Mx^m + \ldots. qx^2 + px + 1 = 0$$

Divide by x^m and we have

$$x^m + px^{m-1} + qx^{m-2} + \ldots. M + \ldots. q \times \frac{1}{x^{m-2}} + p \times \frac{1}{x^{m-1}} + \frac{1}{x^m} = 0$$

$$\therefore x^m + \frac{1}{x^m} + p . (x^{m-1} + \frac{1}{x^{m-1}}) + \ldots\ldots Q\left(x^2 + \frac{1}{x^2}\right) + R \times \left(x + \frac{1}{x}\right) + M = 0.$$

Let now $x + \dfrac{1}{x} = 2 \cos. \theta$

Then $x^m + \dfrac{1}{x^m} = 2 \cos. (m\theta)$

$$x^{m-1} + \frac{1}{x^m} = 2 \cos. (\overline{m-1} . \theta)$$

&c. = &c.

$\therefore$ by substitution, we have

$2 \cos. (m\ \theta) + 2p. \cos. (\overline{m-1}\ \theta) + 2q. \cos. (\overline{m-2}\ \theta) + \ldots\ldots 2Q. \times \cos. 2\theta + 2 R \cos. \theta + M = 0$, an equation which may be reduced to the form $\cos. \theta^m + R' \cos. \theta^{m-1} + \ldots + Q' \cos. \theta^2 + R' \times \cos. \theta + M' = 0$, or to an equation of half the number of dimensions of the given equation. Having found the values of $\cos. \theta$, we have those of $x + \dfrac{1}{x}$, and hence, by the solution of quadratics, we arrive at the values of x, as was required.

Again, let the number be odd, or of the form $2m+1$, $2m$ of the roots being of the reciprocal form, and the other ± 1, or let the equation be of the form

$$x^{2m+1} + px^{2m} + qx^{2m-1} + \ldots\ldots qx^2 + px \pm 1 = 0.$$

Divide by $x \pm 1$, having collected the terms equally distant from both ends into pairs.

Then $x^{2m} \mp (1-p)\ x^{2m-1} \pm (1-p+q)\ x^{2m-2} \pm \ldots. \pm (1-p+q)\ x^2 \mp (1-p)\ x + 1 = 0$, which being of the same form as that of the first case, may be solved in the same manner. This equation may, however, be solved independently, by dividing by $x^{\frac{2m+1}{2}}$, and after ards making assumptions similar to those above: or thus, let $x^{2m+1} + px^{2m} + qx^{2m-1} + \ldots. qx^2 + px + 1 = 0$, let $y^2 = x$.

Then $y^{2\times\overline{2m+1}} + py^{2\times 2m} + \ldots\ldots qy^4 + py^2 + 1 = 0$

$\therefore \left(y^{2m+1} + \frac{1}{y^{2m+1}}\right) + p.\left(y^{2m-1} + \frac{1}{y^{2m-1}}\right) + q\,.\left(y^{2m-3} + \frac{1}{y^{2m-3}}\right) + \ldots\ \&c. \ + Q\,.\left(y^3 + \frac{1}{y^3}\right) + R\,.\left(y + \frac{1}{y}\right) = 0,$

the number of terms being m, and the powers of y, viz. $2m+1$, $2m-1$, $2m-3$, &c., being odd.

Now $\frac{z^{2m+1} + w^{2m+1}}{z+w} = z^{2m} - z^{2m-1}w + z^{2m-2}w^2 - \ldots\ldots + zw^{2m-1} + w^{2m}$.

Hence, dividing each term of the equation in y by $y + \frac{1}{y}$ we have

$$\left.\begin{array}{l} y^{2m} - y^{2m-2} + y^{2m-4} - \ldots\ldots + \frac{1}{y^{2m-4}} - \frac{1}{y^{2m-2}} + \frac{1}{y^{2m}} \\ \quad py^{2m-2} - py^{2m-4} + \ldots\ldots - \frac{p}{y^{2m-4}} + \frac{p}{y^{2m-2}} \\ \qquad + qy^{2m-4} - \ldots\ldots \quad \frac{q}{y^{2m-4}} \\ \qquad\quad \&c. \qquad\qquad \&c. \\ \qquad\qquad + R. \end{array}\right\} = 0$$

$\therefore \left(y^{2m} + \frac{1}{y^{2m}}\right) - (1-p)\left(y^{2m-2} + \frac{1}{y^{2m-2}}\right) + (1-p+q)\left(y^{2m-4} + \frac{1}{y^{2m-4}}\right) - (1-p+q-r)\left(y^{2m-6} + \frac{1}{y^{2m-6}}\right) + \ldots\ldots \pm (1-p+q-\ldots\ldots \pm Q)\left(y^2 + \frac{1}{y^2}\right) \mp (1-p+q-\ldots\ldots \mp R) = 0,$

or $\left(x^m + \frac{1}{x^m}\right) - (1-p)\left(x^{m-1} + \frac{1}{x^{m-1}}\right) + \ldots\ldots (1-p+q-\ldots\ldots \mp Q) \times \left(x + \frac{1}{x}\right) \mp (1-p+q-\ldots\ldots \mp R) = 0$, which may be reduced to an equation of m dimensions, by putting $x + \frac{1}{x} = z$, and substituting, or by putting it $= 2\cos.\theta$, &c. as before.

328. Assume $x^3 - \frac{7}{2}x^2 + \frac{7}{2}x - 1 = \overline{x-1} \times (x^2 - mx + 1) = x^3 - \overline{m+1}\,x^2 + \overline{m+1}\,x - 1$

$$\therefore\ m + 1 = \frac{7}{2}$$

$$\therefore\ m = \frac{7}{2} - 1 = \frac{5}{2}$$

$$\therefore\ x^2 - \frac{5}{2}x = -1$$

$$\therefore\ x^2 - \frac{5}{2}x + \frac{25}{16} = \frac{25-16}{16} = \frac{9}{16}$$

$$\therefore\ x = \frac{5\pm3}{4} = 2 \text{ or } \frac{1}{2} \text{ the two roots required.}$$

Otherwise.

Divide by $x^{\frac{3}{2}}$

$$\therefore\ x^{\frac{3}{2}} - \frac{1}{x^{\frac{3}{2}}} - \frac{7}{2} \cdot \left(x^{\frac{1}{2}} - \frac{1}{x^{\frac{1}{2}}}\right) = 0$$

$$\text{or } x + 1 + \frac{1}{x} = \frac{7}{2}$$

$$\therefore\ x^2 - \frac{5}{2}x = -1 \text{ as before.}$$

329. Here, dividing by x^2, we get

$$x^2 + \frac{1}{x^2} + 2 + \frac{5}{2} \cdot \left(x + \frac{1}{x}\right) = 0$$

Put $x + \frac{1}{x} = 2\cos.\theta$

Then $x^2 + \frac{1}{x^2} = 2\cos. 2\theta = 4\cos.^2\theta - 2$

$$\therefore\ 4\cos.^2\theta + 5\cos.\theta = 0$$

$$\therefore\ \cos.\theta = 0, \text{ or } \cos.\theta = -\frac{5}{4}$$

$$\therefore\ x + \frac{1}{x} = 0 \text{ or } -\frac{5}{2}$$

$$\therefore\ x^2 = -1$$

$$\therefore x = \pm\sqrt{-1} = \sqrt{-1} \text{ or } -\sqrt{-1} = \sqrt{-1} \text{ or } \frac{1}{\sqrt{-1}}$$

$$\text{and } x^2 - \frac{5}{2}x = -1$$

$$\therefore\ x = \frac{5\pm 3}{4} = 2 \text{ or } \frac{1}{2}$$

$\therefore$ the roots required are, $\sqrt{-1}$, $\dfrac{1}{\sqrt{-1}}$, 2 and $\dfrac{1}{2}$.

330. $x^4 + px^2 = -1$

$$\therefore\ x^4 + px^2 + \frac{p^2}{4} = \frac{p^2-4}{4}$$

$$\therefore\ x^2 = \frac{-p \pm \sqrt{p^2-4}}{2}$$

$$\therefore\ x = \pm\sqrt{\frac{-p\pm\sqrt{p^2-4}}{2}}$$ Hence the roots are,

$$+\sqrt{\frac{-p+\sqrt{p^2-4}}{2}},\ +\sqrt{\frac{-p-\sqrt{p^2-4}}{2}},\ +\sqrt{\frac{-p+\sqrt{p^2-4}}{2}},$$

and $-\sqrt{\dfrac{-p-\sqrt{p^2-4}}{2}}$.

Now

$$+\sqrt{\frac{-p-\sqrt{p^2-4}}{2}} = \left(\frac{-p-\sqrt{p^2-4}}{2}\right)^{\frac{1}{2}} \times \frac{\left(\frac{-p+\sqrt{p^2-4}}{2}\right)^{\frac{1}{2}}}{\left(\frac{-p+\sqrt{p^2-4}}{2}\right)^{\frac{1}{2}}}$$

$$= \frac{\left(\frac{p^2-p^2+4}{4}\right)^{\frac{1}{2}}}{\left(\frac{-p+\sqrt{p^2-4}}{2}\right)^{\frac{1}{2}}} = \frac{1}{\left(\frac{-p+\sqrt{p^2-4}}{2}\right)^{\frac{1}{2}}}$$

Similarly $-\sqrt{\dfrac{-p-\sqrt{p^2-4}}{2}} = \dfrac{1}{-\left(\frac{-p+\sqrt{p^2-4}}{2}\right)^{\frac{1}{2}}}$

$$\therefore\ \sqrt{\frac{-p+\sqrt{p^2-4}}{2}},\ -\sqrt{\frac{-p+\sqrt{p^2-4}}{2}},$$

or $\pm\dfrac{\sqrt{2-p}+\sqrt{-2-p}}{2}$ (by taking the root) and their reciprocals are the roots required.

331. Divide by x^2, then

$$x^2 + \frac{1}{x^2} + \left(x + \frac{1}{x}\right) + q = 0 \text{ ,}$$

Now $\left(x + \frac{1}{x}\right)^2 = x^2 + \frac{1}{x^2} + 2$

$$\therefore \left(x + \frac{1}{x}\right)^2 + \left(x + \frac{1}{x}\right) = 2 - q$$

$$\therefore x + \frac{1}{x} = \frac{1 \pm \sqrt{9 - 4q}}{2}$$

$$\therefore x^2 - \frac{1 \pm \sqrt{9-4q}}{2} x \pm - 1$$

$$\therefore x = \frac{1 \pm \sqrt{9-4q} \pm \sqrt{2\sqrt{9-4q} - (4q+6)}}{4}$$

whence we can represent the roots in their reciprocal form, as in the preceding problem.

MISCELLANEOUS EQUATIONS.

332. $x^2 - px = q$

$$\therefore\ x = \frac{p \pm \sqrt{p^2+4q}}{2} = \frac{p}{2}.\left(1 \pm \sqrt{1+\frac{4q}{p^2}}\right)$$

Put $\tan.^2 z = \dfrac{4q}{p^2}$

Then $\pm \sqrt{1+\dfrac{4q}{p^2}} = \sec.\ z$

$$\therefore\ x = \frac{p}{2}.(1 \pm \sec.\ z) = \frac{p}{2}.\ \tan.\ z\left(\cos.\ z \pm \frac{1}{\sin.\ z}\right)$$

$$= \frac{p}{2}\tan.\ z.\frac{\cos.\ z \pm 1}{\sin.\ z} = \frac{p}{2}\,\frac{2\cos.^2\frac{z}{2} \pm 1 - 1}{2\sin.\frac{z}{2} \times \cos.\frac{z}{2}} \times \tan.\ z$$

$$= \frac{p}{2}\tan.\ z.\ \cot.\frac{z}{2},\ \text{or}\ \frac{p}{2}.\ \frac{\cos.^2\frac{z}{2} - 1}{\sin.\frac{z}{2}\cos.\frac{z}{2}}\tan.\ z.$$

But $\cos.^2\dfrac{z}{2} - 1 = -\sin.^2\dfrac{z}{2}$

$$\therefore\ x = \frac{p}{2}\tan.\ z.\ \cot.\frac{z}{2},\ \text{or}\ -\frac{p}{2}.\ \tan.\ z.\ \tan.\frac{z}{2}$$

333. Let a, $2a$ and b be the roots.

Then $13 = 3a + b$

Also $p^2 - 2q = a^2 + 4a^2 + b^2 = 169 - 100 = 69$

$\therefore\ b^2 = 169 - 78\,a + 9\,a^2 = 69 - 5a^2$

$\therefore\ 14\,a^2 - 78\,a = -100$

$$\therefore a^2 - \frac{39}{7}a = \frac{-50}{7}$$

$$\therefore a^2 - \frac{39}{7}a + \frac{1521}{14^2} = \frac{1521}{14^2} - \frac{1400}{14^2} = \frac{121}{14^2}$$

$$\therefore a = \frac{39 \pm 11}{14} = 2$$

$$\therefore 2a = 4$$

and $b = 7$.

334. $$a^x + \frac{1}{a^x} = b$$

$$\therefore a^{2x} - b\,a^x = -1$$

$$\therefore a^{2x} - b\,a^x + \frac{b^2}{4} = \frac{b^2 - 4}{4}$$

$$\therefore a^x = \frac{b \pm \sqrt{b^2 - 4}}{2}$$

$$\therefore x \log. a = \log. (b \pm \sqrt{b^2 - 4}) - \log. 2$$

$$\therefore x = \frac{\log. (b \pm \sqrt{b^2 - 4}) - \log. 2}{\log. a}$$

335. Let a be the third root.
Then $3a$ = the second root.
and $6a$ = first root.
Now $10 = 6a + 3a + a = 10a$

$$\left.\begin{aligned} \therefore a &= 1 \\ \therefore 3a &= 3 \\ \text{and } 6a &= 6 \end{aligned}\right\} \text{the roots required.}$$

336. Since the equations have a common root, they have a common measure of the form $x \pm a$, which, being found by the common method, gives $x^2 - 1 = 0$.

$\therefore x = \pm 1$, $+1$ is found on trial to be the common root. Let a and b, a', b' be the other roots.

Then $\left.\begin{aligned}2 &= a + b\\ 9 &= ab\end{aligned}\right\}$ $\quad$ $\left.\begin{aligned}4 &= a' + b'\\ 7 &= a'\,b'\end{aligned}\right\}$

But $a^2 + 2ab + b^2 = 4$ and $a'^2 + 2a'b' + b'^2 = 16$

and $4ab \qquad = 36$ $\quad$ and $4a'b' \qquad = 28$

$\therefore\ a^2 - 2ab + \qquad = -32$ and $a'^2 - 2a'b' + b'^2 = -12$

$\therefore\ a - b = \pm 4\sqrt{-2}$ and $a' - b' = \pm 2\sqrt{-3}$

But $a + b = 2$ $\qquad$ $a' + b' = 4$

$\therefore\ a = 1 \pm 2\sqrt{-2}$ $\qquad$ $\therefore\ a = 2 \pm \sqrt{-3}$

$b = 1 \mp 2\sqrt{-2}$ $\qquad$ $b = 2 \mp \sqrt{-3}$

Hence we have all the roots.

337. $a^{mx} + a^{mx-1} = b$

$$a^{mx} + \frac{a^{mx}}{a} = b$$

$$\therefore\ a^{mx} = \frac{ab}{a+1}$$

$$\therefore\ mx \log. a = \log. (ab) - \log. (a+1)$$

$$\therefore\ x = \frac{\log. a + \log. b - \log. (a+1)}{m \times \log. a}$$

338. $x^4 - 4x^3 . 2^{\frac{1}{4}} + 6x^2 . 2^{\frac{2}{4}} - 4x . 2^{\frac{3}{4}} + 2^{\frac{4}{4}} = 0$, which we see is the fourth power of the binomial $x - 2^{\frac{1}{4}}$

$$\therefore\ (x-2^{\frac{1}{4}})^4 = (x-2^{\frac{1}{4}}) \times (x-2^{\frac{1}{4}}) \times (x-2^{\frac{1}{4}}) \times (x-2^{\frac{1}{4}}) = 0$$

$\therefore$ the four roots are equal, each being $2^{\frac{1}{4}}$.

Again, let $x^3 + Px^2 + Qx + R = 0$ be the equation sought.

Now $P = -\frac{1}{2}a - \sqrt{-\frac{3}{4}a^2} - \frac{1}{2}a + \sqrt{-\frac{3}{4}a^2} + a$
$= 0$

$P^2 - 2Q = \frac{a^2}{4} + a\sqrt{-\frac{3}{4}a^2} - \frac{3}{4}a^2 + \frac{a^2}{4} - a\sqrt{-\frac{3}{4}a^2}$
$- \frac{3}{4}a^2 + a^2 = 0$

$\therefore Q = 0.$

$$R = \left(-\frac{1}{2}a - \sqrt{-\frac{3}{4}a^2}\right)\left(-\frac{1}{2}a + \sqrt{-\frac{3}{4}a^2}\right) \times a$$

$$= \left(\frac{1}{4}a^2 + \frac{3}{4}a^2\right) \times a^2 = a^3$$

$\therefore x^3 + a^3 = 0$ is the equation required.

339. Since $1 - \sqrt{5}$ is a root, $1 + \sqrt{5}$ is also a root, and the equation is divisible by $(x - 1 + \sqrt{5}) \times (x - 1 - \sqrt{5})$ $= (x-1)^2 - 5 = x^2 - 2x - 4$. The quotient arising from the division is

$$x^2 + 3x + 2 = 0$$

$$\therefore x^2 + 3x + \frac{9}{4} = \frac{9-8}{4} = \frac{1}{4}$$

$$\therefore x = \frac{-3 \pm 1}{2} = -1 \text{ or } -2$$

340. Since surd roots of the form $a \pm \sqrt{b}$, enter equations by pairs, $3 - \sqrt{2}$ is also a root.

Let a be the third root.

Then $11 = 3 - \sqrt{2} + 3 + \sqrt{2} + a$

$\therefore a = 5$

$\therefore$ the roots are $3 + \sqrt{2}$, $3 - \sqrt{2}$ and 5.

341. Let $x^n + px^{n-1} + qx^{n-2} + \ldots\ldots q'x + p' = 0$ be the equation to be transformed, and $y^m + Py^{m-1} + Qy^{m-2} + \ldots\ldots Q'y + P' = 0$ the transformed equation. Also, let $a, b, c, \ldots\ldots b', a'$, represent the roots of the former equation, A, B, C, ... B', A' of the latter.

Then, since there are $n.\dfrac{n-1}{2}$ combinations in n things taken two and two together, $n.\dfrac{n-1}{2}$ is the number of roots in the transformed equation.

$$\therefore\ m = n \, . \, \frac{n-1}{2}$$

$$\text{Again, } -2\,\mathrm{P} = \begin{cases} (a-b)^2 + (a-c)^2 + (a-d)^2 + \ldots\ldots (a-a')^2 \\ (b-a)^2 + (b-c)^2 + (b-d)^2 + \ldots\ldots (b-a')^2 \\ (c-a)^2 + (c-b)^2 + (c-d)^2 + \ldots\ldots (c-a')^2 \\ \&c. \qquad \&c. \qquad \&c. \qquad \ldots\ldots \quad \&c. \\ (a'-a)^2 + (a'-b)^2 + (a'-c)^2 + \ldots\ldots (a'-b')^2 \end{cases}$$

$$\text{But } (a-b)^2 + (a-c)^2 + (a-d)^2 + \ldots\ldots (a-a')^2 = \begin{cases} a^2 - 2ab + b^2 \\ a^2 - 2ac + c^2 \\ \&c. \quad \&c. \quad \&c. \\ a^2 - 2aa' + a'^2 \end{cases}$$

$$= (n-1)\,.\,a^2 - 2a(-p-a) + \mathrm{S}_2 - a^2$$

$$= na^2 + 2ap + \mathrm{S}_2$$

Similarly $(b-a)^2 + (b-c)^2 + \ldots\ldots = n\,.\,b^2 + 2bp + \mathrm{S}_2$

&c. $=$ &c. to n terms.

Whence $\mathrm{S}_2 =$ sum of the squares $= a^2 + b^2 + \ldots\ldots$

$$\therefore\ -2\,\mathrm{P} = n\,.\,\mathrm{S}_2 + 2p\,.\,(-p) + n\,.\,\mathrm{S}_2$$

$$= 2n\,.\,\mathrm{S}_2 - 2p^2$$

$$\therefore\ \mathrm{P} = p^2 - n\,\mathrm{S}_2 = p^2 - n\,.\,p^2 + 2nq = 2nq - \overline{n-1}\,.\,p^2$$

$$\text{Again } 2\,.\,(\mathrm{P}^2 - 2\mathrm{Q}) = \begin{cases} (a-b)^4 + (a-c)^4 + \ldots\ldots (a-a')^4 \\ (b-a)^4 + (b-c)^4 + \ldots\ldots (b-a')^4 \\ \&c. \qquad \&c. \qquad \&c. \\ (a'-a)^4 + (a'-b)^4 + \ldots\ldots (a'-b')^4 \end{cases}$$

Let $\mathrm{S}_3 = a^3 + b^3 + \ldots\ldots$

$\mathrm{S}_4 = a^4 + b^4 + \ldots\ldots$

&c. $=$ &c.

$$\left.\begin{matrix} (a-b)^4 \\ +\ (a-c)^4 \\ +\ (a-d)^4 \\ +\ \&c. \end{matrix}\right\} = \begin{cases} a^4 - 4a^3 b + 6a^2b^2 - 4ab^3 + b^4 \\ a^4 - 4a^3 c + 6a^2c^2 - 4ac^3 + c^4 \\ \&c. \qquad \&c. \qquad \&c. \\ a^4 - 4a^3 a' + 6a^2a'^2 - 4aa'^3 + a'^4 \end{cases}$$

$$= (n-1)\,.\,a^4 - 4a^3\,.\,(-p-a) + 6a^2(\mathrm{S}_2 - a^2)$$

$$-\ 4a\,.\,(\mathrm{S}_3 - a^3) + \mathrm{S}_4 - a^4 = na^4 + 4a^3p + 6a^2\mathrm{S}_2 - 4a\mathrm{S}_3 + \mathrm{S}_4$$

Similarly $(b-a)^4 + (b-c)^4 + \ldots\ldots = nb^4 + 4b^3p + 6b^2\mathrm{S}_2 - 4b\,.\,\mathrm{S}_3 + \mathrm{S}_4$

&c. $=$ &c. to n terms.

$$\therefore\ 2\,(\mathrm{P}^2 - 2\mathrm{Q}) = 2n\,.\,\mathrm{S}_4 + 4p\,.\,\mathrm{S}_3 + 6\mathrm{S}_2^2 + 4p\mathrm{S}_3 + n\mathrm{S}_4$$

$$\therefore\ \mathrm{P}^2 - 2\mathrm{Q} = n\,\mathrm{S}_4 + 4p\,.\,\mathrm{S}_3 + 3\mathrm{S}_2^2.$$

$$\text{Now } S_4 = -pS_3 - qS_2 - rS_1 - 4s$$
$$S_3 = -pS_2 - qS_1 - 3r$$
$$S_2 = -pS_1 - 2q$$
$$S_1 = -p$$
$$\therefore n.S_4 = np^4 - 4np^2q + 4npr + 2nq^2 - 4ns$$
$$4p.S_3 = -4p^4 + 12p^2q - 12pr$$
$$3S_2^2 = 3p^4 - 12p^2q + 12q^2$$

$\therefore P^2 - 2Q = (n-1)p^4 - 4nqp^2 + 4(n-3).pr + 2.(n+6)q^2 - 4ns$, or $= 4n^2q^2 - 4n.(n-1)qp^2 + (n-1)^2p^4 - 2Q$, since $P = 2nq - (n-1)p^2$

$\therefore + 2Q = (n-1).(n-2).p^4 - 4n.(n-2)qp^2 - 4(n-3)\times pr + 2(n-2).(2n+3)q^2 + 4ns$

$\therefore Q = \dfrac{(n-1).(n-2)}{2}.p^4 - 2n.(n-2)qp^2 - 2.(n-3)\times pr + 2.(n-2).(2n+3)q^2 + 4ns.$

By proceeding in a similar manner we may find all the coefficients in succession. Thus, having given $(w-1)$ of the first coefficients P, Q, &c., let it be required to find the w^{th} coefficient X.

If $\Sigma_0, \Sigma_1, \Sigma_2$ ..... represent the sums of the successive powers of the roots of the transformed equation, it is known that Σ_1 involves the first coefficient P only.

Σ_2 involves the first and second P and Q only.

Hence Σ_w involves the first w coefficients, $(w-1)$ of which being given, let us find the w^{th}.

$$2 \times \Sigma_w = \begin{cases} (a-b)^{2w} + (a-c)^{2w} + \ldots\ldots (a-a')^{2w} \\ (b-a)^{2w} + (b-c)^{2w} + \ldots\ldots (b-a')^{2w} \\ \quad \&c. \qquad \&c. \qquad \&c. \\ (a'-a)^{2w} + (a'-b)^{2w} + \ldots\ldots (a'-b')^{2w} \end{cases}$$

$$\begin{aligned} &\text{Now } (a-b)^{2w} \\ &+ (a-c)^{2w} \\ &+ \ldots\ldots = \end{aligned} \begin{cases} a^{2w} - 2w.a^{2w-1}b + 2w\dfrac{2w-1}{2}a^{2w-2}b^2 - \ldots\ldots + b^{2w} \\ a^{2w} - 2w.a^{2w-1}c + 2w\dfrac{2w-1}{2}a^{2w-2}c^2 - \ldots\ldots + c^{2w} \\ \&c. \qquad \text{to } w-1 \text{ terms} \end{cases}$$

$$= (w-1)a^{2w} - 2w.a^{2w-1}(S_1 - a) + 2w\frac{2w-1}{2}a^{2w-2}.(S_2 - a)$$
$$+ \ldots\ldots S_{2w} - a^{2w}$$
$$= wa^{2w} - 2wa^{2w-1}S_1 + 2w\frac{2w-1}{2}a^{2w-2}S_2 - \ldots\ldots + S_2$$

Let $2w = C_1\ 2w\,\frac{2w-1}{2} = C_2$, &c. $=$ &c.

Then $(a-b)^{2w}+(a-c)^{2w}+\ldots = wa^{2w} - C_1\,a^{2w-1}\,S_1 + C_2\,a^{2w-2}\times S_2 - \ldots + S_{2w}$

Similarly $(b-a)^{2w}+(b-c)^{2w}+\ldots = wb^{2w} - C_1\,b^{2w-1}\,S_1 + C_2\,b^{2w-2}\times S_2 - \ldots\, S_{2w}$

&c. $=$ &c. to w terms.

$\therefore 2\times\Sigma_w = 2w\,S_{2w} - 2C_1\,S_1\,S_{2w-1} + 2C_2\,S_2\,S_{2w-2} - \ldots \pm C_w\,S_w\,S_w$, $C_w\,S_w\,S_w$ or $C_w\,S_w^2$, being the middle term (the terms equally distant from either end, being the same, are collected.)

$\therefore \Sigma_w = S_0\,S_{2w} - C_1\,S_1\,S_{2w-1} + C_2\,S_2\,S_{2w-2} - \ldots \pm \frac{C_w\,S_w^2}{2}$

where $+$ or $-$ is used according as w is even or odd, and $C_1 = 2w$, &c. $C_w = 2w\,.\,\frac{2w-1}{2}\,.\,\frac{2w-2}{3}\ldots\frac{w+1}{w}$.

Let us take for an example the general cubic, $x^3 + px^2 + qx + r = 0$, and suppose P, Q given, required R.

Then $\Sigma_3 = S_0\,S_6 - 6S_1\,S_5 + \frac{6\times5}{2}\,.\,S_2\,S_4 - \frac{6\times5\times4}{2\times3\times2}\times S_3^2$

$= 3S_6 - 6S_1\,S_5 + 15S_2S_4 - 10\,S_3^2$

Now $\Sigma_3 = -P\Sigma_2 - Q\Sigma_1 - 3R = -P^3 + PQ - 3R$

$\therefore\ = \frac{PQ-P^3}{3} - S_6 + 2S_1\,S_5 - 5S_2\,S_4 + \frac{10}{3}\,S_3^2$

$= \frac{PQ-P^3}{3} - S_6 - 2p\,S_5 - 5p^2\,S_4 + 10q\,S_4 + \frac{10}{3}\,S_3^2$

By finding the values of $S_6\,S_5$, &c., in terms of the coefficients p, q, r, &c. R may be expressed in terms of these coefficients, and of P and Q.

For the use of this transformation, see *Garnier's Analyse Algébrique*. Chap. II.

342. For the investigation, see *Wood* or *Maclaurin*.

The equation completed is $x^6 \pm 0\times x^5 + 3x^4 \pm 0\times x^3 - 4x^2 \pm 0\times x - 12 = 0$

In the series $\frac{6}{1}, \frac{5}{2}, \frac{4}{3}, \frac{3}{4}, \frac{2}{5}, \frac{1}{6}$, if each term be di-

vided by that which immediately precedes it, and the results $\frac{5}{12}, \frac{8}{15}, \frac{9}{16}, \frac{8}{15}, \frac{5}{12}$, be placed over the terms according to the rule, we obtain

$$x^6 \pm 0^{\frac{5}{12}} \times x^5 + 3^{\frac{8}{15}} x^4 \pm 0^{\frac{9}{16}} \times x^3 - 4^{\frac{8}{15}} x^2 \pm 0^{\frac{5}{12}} \times x - 12 = 0$$

and $+ \quad - \quad + \quad + \quad + \quad - \quad +$

are signs corresponding to the terms under which they are placed, since $0 \times \frac{5}{12} < 3$, $9 \times \frac{8}{15} > 0 \times 0$, $0 \times \frac{9}{16} > -12$

$$16 \times \frac{8}{15} > 0 \times 0, \text{ and } 0 \times \frac{5}{12} < (-4) \times (-12).$$

There are four changes of signs, and $\therefore$ four impossible roots. The roots are (as we find by the solution of a cubic) $\pm \sqrt{-2}$, $\pm \sqrt{-3}$ and $\pm \sqrt{2}$

343. Since impossible roots enter equations by pairs, an equation of an odd number of dimensions must evidently have, at least, one possible or real root.

Again, the pairs of impossible roots being necessarily of the form $a + b\sqrt{-1}$, $a - b\sqrt{-1}$, and the last term $=$ the product of all the roots with their signs changed; if this last be negative, or have a sign different from that of the first term, such sign cannot have been introduced by the impossible roots, since $(-a-b\sqrt{-1}) \times (-a+b\sqrt{-1}) = a^2 - (b\sqrt{-1})^2 = a^2 + b^2$ a positive quantity. $\therefore$ there must be at least one negative real root. If the number of dimensions be even, and the last term negative, there must be at least two real roots, one negative and the other positive; for there must be one negative real root, and as impossible roots enter by pairs, and a positive quantity multiplied by a negative one produces a negative result, there must be at least another real root, *viz.*, a positive one.

344. Let $\sqrt[n]{1} = x$, x being the general representative of the roots of unity.

Then $x^n = 1 = \cos. 2p\pi + \sqrt{-1} \sin. 2p\pi$, where p is any whole number whatever.

$\therefore x = \cos. \frac{2p}{n}\pi + \sqrt{-1} \sin. \frac{2p}{n}\pi$, by Demoivre's Theorem.

Let $\cos. \frac{\pi}{n} + \sqrt{-1} \sin. \frac{\pi}{n} = r$, then $0, 1, \ldots\ldots n-1$ being successively substituted for p, the corresponding values of x, will be, $\cos. \frac{0}{n}\pi + \sqrt{-1} \sin. \frac{0}{n}\pi = r^0 = 1$

$$\cos. \frac{2}{n}\pi + \sqrt{-1} \sin. \frac{2}{n}\pi = r^2$$

$$\&c. \quad = \quad \&c.$$

$$\cos. \frac{2 . (n-1)}{n}\pi + \sqrt{-1} \sin. \frac{2(n-1)}{n}\pi = r^{2(n-1)}$$

Now $r^{2n} = \cos. \frac{2n}{n}\pi + \sqrt{-1} \sin. \frac{2n}{n}\pi = \cos. 2\pi + \sqrt{-1} \times \sin. 2\pi = 1 = r^0$

$$r^{2(n+1)} = \cos. \frac{2(n+1)}{n}\pi + \sqrt{-1} \sin. \frac{2(n+1)}{n}\pi =$$
$$\cos. (2\pi + \frac{2}{n}\pi) + \sqrt{-1} \sin. (2\pi + \frac{2}{n}\pi) = r^2$$

Similarly $r^{2(n+2)} = r^4$, &c. = &c.

$\therefore$ there are but n different values of x

Again, $\cos. \frac{2(n-1)}{n}\pi = \cos. (2\pi - \frac{2}{n}\pi) = \cos. \frac{2}{n}\pi$

But $\sin. \frac{2(n-1)}{n}\pi = \sin. (2\pi - \frac{2}{n}\pi) = - \sin. \frac{2}{n}\pi$

$\therefore r^{2(n-1)} = \cos. \frac{2}{n}\pi - \sqrt{-1} \sin. \frac{2}{n}\pi$ which differs from r^2 only in sign. Similarly it may be shewn that r^{2n-4} differs from r^4 only in the sign of its second term, and so on, in pairs throughout (n being an odd number). Hence, it appears, there are

$\frac{n-1}{2}$ pairs of roots of the form $A \pm \sqrt{-1}.B$ when n is odd number. We shall $\therefore$ have $\frac{n-1}{2}$ quadratic factors of the equation $x^n - 1$ of the form $(x - A - \sqrt{-1}\,B).(x-A+\sqrt{-1}\,B) = (x - A)^2 + B^2 = x^2 - 2\,Ax + (A^2 + B^2)$, where A, and B are the cosine and sign of the same angle respectively, $\therefore$ since $A^2 + B^2 = 1$ these factors will be of the form $x^2 - 2\,A\,x + 1$

Hence, substituting for A, $\cos.\frac{2}{n}\pi$, $\cos.\frac{4}{n}\pi$ ...... $\cos.\frac{2.(n-1)}{n}\pi$ successively, we shall have $x^n - 1 = (x - 1).(x^2 - 2\cos.\frac{2}{n}\pi.x + 1) \times (x^2 - 2\cos.\frac{4}{n}\pi.x + 1) \times \ldots\ldots (x^2 - 2\cos.\frac{2(n-1)}{n}\pi.x + 1)$

The roots have been found to be $r^0, r^2, r^4, r^6, \ldots. r^{2(n-1)}$, r being $= \cos.\frac{\pi}{n} + \sqrt{-1}\sin.\frac{\pi}{n}$

345. $x^m + 1 = 0$, let $y = -x$, then, since m is odd $-y^m + 1 = 0$, or $y^m - 1 = 0$, whose roots, by the preceding problem, are r^0, r^2, r^4. r^6, ... r^{2m-2}, r being $= \cos.\frac{\pi}{m} + \sqrt{-1}\sin.\frac{\pi}{m}$

$\therefore$ the roots of $x^m + 1$, or the corresponding values of x will be those of $\frac{y}{-1}$, or of $\frac{y}{r^m}$, and $\therefore \frac{r^0}{-1} (= -1), \frac{r^2}{r^m}, \frac{r^4}{r^m} \ldots\ldots \frac{r^{2m-2}}{r^m}$ or r^m, $\frac{1}{r^{m-2}}$, $\frac{1}{r^{m-4}}$, ... r^{m-2} are the roots of $x^m + 1$, according to the enunciation of the problem.

LIMITS TO ROOTS.

346. Diminish the roots by such a quantity that every term shall be positive; then since all the roots are negative that quantity is evidently greater than the greatest positive root. Again, change the signs of the roots, then the quantity, (found as above) which is $>$ greatest positive root of the transformed equation, will, when its sign is changed, be less than the least negative root.

Thus, let $x = y + e$

Then $\left.\begin{array}{r} y^3 + 3y^2e + 3e^2y + e^3 \\ - 4y^2 - 8ey - 4e^2 \\ - y - e \\ + 20 \end{array}\right\} = 0$ in which, if all the terms be positive we must have $3e > 4$, $3e^2 > 8e + 1$, and $e^3 + 20 > 4e^2 + e$; 3 when substituted for (e) answers these conditions. $\therefore$ 3 is $>$ greatest root.

Again, change the signs of the roots, then $x^3 + 4x^2 - x - 20 = 0$

Then, putting $x = y + e$

$\left.\begin{array}{r} y^3 + 3y^2e + 3e^2y + e^3 \\ + 4y^2 + 8ey + 4e^2 \\ - y - e \\ - 20 \end{array}\right\} = 0$ in which, if all the terms be positive, we must have $3e^2 + 8e > 1$, and $e^3 + 4e^2 > e + 20$.

2 2 is the least whole number that answers these conditions, and is $\therefore$ a limit $>$ greatest root of the equation in y, and $\therefore$ -2 is a limit less than the least root of the given equation.

347. Diminish the roots by the number 3, or assume $y = x - 3$ and $\therefore x = y + 3$

$$\left.\begin{array}{r} \text{Then } y^3 + 9y^2 + 27y + 27 \\ - 7y^2 - 42y - 63 \\ + 7y + 21 \\ + 10 \end{array}\right\} = y^3 + 2y^2 - 8y - 5 = 0$$

which equation has two negative roots; and only one of the roots of the given equation is negative; $\therefore$ one of the positive roots is greater than 3, and the other less.

348. The limiting equation is $3x^2 - q = 0$, whose roots are $x = \sqrt{\frac{q}{3}}, -\sqrt{\frac{q}{3}}$

$\therefore$ if a, b, c, be the roots of the given equation taken in order, $\sqrt{\frac{q}{3}}$ lies between a and b.

Again, since $x^3 - qx + r = 0$, diminish the roots by $\sqrt{q}$, and we have

$$\left.\begin{array}{r} y^3 + 3\sqrt{q}\, y^2 + 3qy + q^{\frac{3}{2}} \\ - qy \quad - q^{\frac{3}{2}} \\ + r \end{array}\right\} = 0$$

$= y^3 + 3\sqrt{q}\, y^2 + 2qy + r$ every term of which being positive $\sqrt{q}$ is greater than the greatest root. $\therefore a$ lies between $\sqrt{q}$ and $\sqrt{\frac{q}{3}}$ and a^2 lies between q and $\frac{q}{3}$

Otherwise,

As before, $\sqrt{q}$ is greater than (a) the greatest root.

Also $\sqrt{\dfrac{q}{n \cdot \frac{n-1}{2}}}$ is $<$ than the greatest root, (*Wood*) or $\sqrt{\frac{q}{3}}$ is $< a$.

$\therefore a$ lies between $\sqrt{q}$ and $\sqrt{\frac{q}{3}}$.

349. This is evident, from the consideration, that there must be one result at least $= 0$ in passing from $+$ to $-$, or from $-$ to $+$. If x, indeed, represent the abscissa of a curve, the results arising from the substitution of different numbers for x in the equation will be represented by the corresponding ordinates, which increase to their maximum, after which they decrease to 0, then become negative, still decrease to a minimum, return to 0, &c. &c.

350. Take the most unfavourable case in supposing all the coefficients from M negative, and equal to P, and find such a value of x, that this value, and all greater than it, when substituted in the form $x^n - P(x^{n-m} + \ldots x^2 + x + 1)$ shall give positive results, and $\therefore$ *à fortiori*, in the given equation, whose negative part is less, and positive part greater, than those in the above form; that is, find x such, that

$$x^n \text{ shall always be} > P\frac{(x^{n-m+1}-1)}{x-1} > \frac{Px^{n-m+1}}{x-1} - \frac{P}{x-1}$$

$$\text{Now, } \frac{Px^{n-m+1}}{x-1} \text{ is always} > \frac{Px^{n-m+1}}{x-1} - \frac{P}{x-1}$$

$$\text{Assume} \therefore x^n = \text{or} > \frac{Px^{n-m+1}}{x-1}$$

$$\text{or } x^{m-1} \times (x-1) > P$$

or $\dfrac{x^{m-1}}{(x-1)^{m-1}} \times (x-1)^m > P$, which is the case when $(x-1)^m = \text{or} > P$, and $\therefore$ when $x - 1 = \text{or} > P^{\frac{1}{m}}$, or when $x = \text{or} > 1 + P^{\frac{1}{m}}$.

$\therefore x^n - P.(x^{n-m} + \ldots\ldots x^2 + x + 1)$ is always positive when $x = \text{or} > 1 + P^{\frac{1}{m}}$, and $\therefore$ *à fortiori*.

$x^n + px^{n-1} + \ldots\ldots - Mx^{n-m} - \ldots\ldots = 0$ is always positive when $x = \text{or} > 1 + P^{\frac{1}{m}}$.

$\therefore 1 + P^{\frac{1}{m}}$ is greater than the greatest root, &c.

APPROXIMATION.

351. With regard to the accuracy of the operation, see *Wood.*

Approximate values of x and y are 2 and $\frac{1}{2}$ respectively.

$$\left.\begin{array}{l}\text{Let } x = 2 + v \\ \quad y = .5 + z\end{array}\right\}$$

$$\text{Then } \left.\begin{array}{l} x^2 + xy = 4 + 4v + v^2 \\ \qquad\qquad 1 + .5v + zv + 2z\end{array}\right\} = 5$$

$$\text{and } \left.\begin{array}{l} 2xy - y^2 = 2 + v + 2zv + 4z \\ \qquad - .25 \qquad\qquad - z + z^2\end{array}\right\} = 2$$

Or neglecting terms of two dimensions from their comparative smallness.

$$\left.\begin{array}{l} 5 + (4.5)\,v + 2z = 5 \\ 1.75 + v + 3z = 2\end{array}\right\} \therefore \left.\begin{array}{l} 4.5v + 2z = 0 \\ v + 3z = .25\end{array}\right\}$$

$$\therefore z = -\frac{4\frac{1}{2}}{2}v = -\frac{9}{4}v = \frac{-v + \frac{1}{4}}{3} = \frac{-4v + 1}{12}$$

$$\therefore 27v = 4v - 1$$

$$\therefore 23v = -1 \quad \therefore v = -\frac{1}{23} = -.043478261.$$

$$\text{and } z = \frac{.043478261 + .25}{3} = .097826087$$

$$\left.\begin{array}{l} \therefore x = 1.956521739 \text{ more nearly} \\ \text{and } y = .597826087 \text{ more nearly}\end{array}\right\}$$

By repeating the operation with these new values, new ones again will be found more accurate than the former, and so on, without limit.

x and y may, however, be found by the solution of a quadratic, as follows: let $x = vy$.

Then $\left.\begin{aligned} v^2y^2 + vy^2 &= 5 \\ 2vy^2 - y^2 &= 2 \end{aligned}\right\}$ $\therefore \dfrac{v^2+v}{2v-1} = \dfrac{5}{2}$, whence v may be found by the solution of a quadratic, from which x and y may also be found.

352. For the accuracy of the operation, see *Wood.*

By trial, 4.1 is nearly a root of the equation.

Let $\therefore x = 4.1 + v$

$$\left.\begin{aligned} \therefore x^3 &= 68.921 + 50.43v \text{ nearly} \\ x^2 &= 16.81 + 8.2v \text{ nearly} \\ x &= 4.1 + v \end{aligned}\right\} = 90$$

$\therefore 59.46v = 90 - 89.881 = .169$

$$\therefore v = \frac{.169}{59.46} = .00284$$

$\therefore x = 4.10284$ nearly, which being used as a new value of x, and the operation repeated, the root may be more nearly found, &c.

353. For the accuracy, &c., see *Wood.*

By trial, 3.8 is nearly a root.

Let $\therefore x = 3.8 + v$

$$\therefore x^3 - x - 50 = \left.\begin{aligned} &54.872 + 44.42v \\ &-(3.8) - v \\ &-50 \end{aligned}\right\} = 0 \text{ nearly}$$

$\therefore 1.072 + 43.42v = 0$

$\therefore v = -.0246 \ldots\ldots$

$\therefore x = 3.8 - .0246 \ldots\ldots = 3.7754 \ldots\ldots$ nearly, and similarly the operation may be repeated.

354. $x^3 + 2x - 30 = 0$, of which 2.9 is found by trial to be an approximate root.

Let $\therefore x = 2.9 + v$

$$\text{Then } x^3 + 2x - 30 = \left\{\begin{aligned} &24.389 + 25.23v \\ &5.8 + 2v \\ &-30 \end{aligned}\right\} \text{ nearly.}$$

$\therefore .189 + 27.23v = 0$ nearly.

$\therefore v = -.00694$ nearly.

$\therefore x = 2.89306$ nearly.

Again, let $x = 2.893 + v'$, and operate as before, &c. &c.

355. $x^3 - 2x - 5 = 0$.

By trial, $x = 2.1$ nearly.

Let $x = 2.1 + v$

$$\left.\begin{array}{l}\text{Then } x^3 \\ -2x \\ -5\end{array}\right\} = \left.\begin{array}{l}9.261 + (4.41) \times 3v \\ -(4.2) - 2v \\ -5\end{array}\right\} \text{nearly} = 11.23v + .061 = 0$$

$$\therefore v = -\frac{.061}{11.23} = -.00543 \text{ nearly.}$$

$\therefore x = 2.09457$ nearly. With this new value of x repeat the operation, and it will be found that x more nearly $= 2.0945515$.....

For the required explanation, see the *Introduction to Barlow's Mathematical Tables*, page 31, where the defects of the various methods of Approximation are clearly pointed out. This valuable book should be in the hands of every mathematician.

SYMMETRICAL FUNCTIONS OF THE ROOTS OF EQUATIONS.

356. Let the general equation be $x^n + p_1 x^{n-1} + p_2 x^{n-2} + \dots\dots + p_m x^{n-m} + \dots\dots p_n = 0$, where p_1, p_2, &c., are positive, negative, or zero, its roots being a, b, c .....

Assume $S_1 = a + b + c + \dots\dots = -p_1$

$S_2 = a^2 + b^2 + c^2 + \dots\dots$ &c. &c.

Also let ${}^mS_a$ = sum of products formed by the combinations (m and m together) of a, b, c ..... into which a enters, ${}^mS_b$ = ... into which (b) enters.

&c. = &c.

Then ${}^mS_a = a \times$(sum of products, &c. ($m-1$, and $m-1$) ..... into which a does not enter.)

$= a.(\pm p_{m-1}$ − sum of the products into which a does enter,) according as m is even or odd.

$\therefore\ {}^mS_a = a\,(\mp p_{m-1} - {}^{m-1}S_a)$

Similarly ${}^{m-1}S_a = a\,(\pm p_{m-2} - {}^{m-2}S_a)$

${}^{m-2}S_a = a.(\mp p_{m-3} - {}^{m-3}S_a)$

&c. = &c.

$\therefore\ {}^mS_a = \pm\,(ap_{m-1} + a^2 p_{m-2} + a^3 p_{m-3} + \text{\&c.} \dots)$

Similarly ${}^mS_b = \pm\,(bp_{m-1} + b^2 p_{m-2} + b^3 p_{m-3} + \text{\&c.})$

&c. = &c.

$\therefore\ {}^mS_a + {}^mS_b + \dots = \pm\,(S_1 p_{m-1} + S_2 p_{m-2} + S_3 p_{m-3} + \dots\dots)$

But this sum evidently $= \mp\, m p_m = \mp\, S_0 p_m$, according as m is odd or even. Divide ∴ by ± 1 and transpose.

and $S_0 p_m + S_1 p_{m-1} + S_2 p_{m-2} + \dots\dots S_{m-1} p_1 + S_m = 0$.

Now in the example $x^n - 1 = 0, p_1, p_2, p_3, p_4 \dots\dots p^{n-1}$ are each equal to zero.

$\therefore$ $S_1, S_2, S_3 \ldots\ldots S_{n-1}$ involving in each term some one of these zeros as a multiplier, must be $= 0$.

But $S_n = - S_0 p_n = (-n) \times (-1) = n$. Again, $S_{n+1} = - S_n - S_0 p_n = -n + n = 0$. Similarly $S_{n+2}, S_{n+3}, \ldots\ldots S_{2n-1}$ each $= 0$. But $S_{2n} = - S_n \times p_{n+1} - S_0 p_n = 0 + n = n$ (since $p_{n+1} = 0$). Hence the truth of the problem is evident.

Otherwise

Sine $x^n - 1 = 0$

$\therefore$ $x^n = 1 = \cos. 2p\pi + \sqrt{-1} \sin. 2p\pi$, p being any integer.

$$\therefore\ x = \cos. \frac{2p}{n}\pi + \sqrt{-1} \sin. \frac{2p}{n}\pi$$

And $x^m = \cos. \dfrac{2pm}{n} \pi + \sqrt{-1} \sin. \dfrac{2pm}{n} \pi$ a form exhibiting the m^{th} powers of the roots of $x^n - 1$, p being supposed 0, 1, 2 .... $n-1$ successively.

Let $\dfrac{m}{n} = w$ a whole number.

Then $x^m = \cos. (2pw\pi) + \sqrt{-1} \sin. 2pw\pi = 1$ whatever is the value of p. Hence, when $\dfrac{m}{p}$ is a whole number, $S_m = 1 + 1 + \ldots n$ terms $= n$. The other condition may be similarly proved.

357. Let the general equation be $x^n + p\, x^{n-1} + p_2\, x^{n-2} + \ldots p_n = 0$, then the equation of limits is $nx^{n-1} + (n-1)\, p_1 \times x^{n-2} + (n-2).\, p_2\, x^{n-3} + \ldots$ and it is known, that if a, b, c, &c., be the roots of the equation,

$$\frac{nx^{n-1} + (n-1)\, p_1\, x^{n-2} + \ldots}{x^n + p_1\, x^{n-1} + \ldots p_n} = \frac{1}{x-a} + \frac{1}{x-b} + \ldots$$

Let $x = 1$ and divide each of the terms.

$$\text{Then } \frac{n + (n-1).\, p_1 + (n-2)\, p_2 + \ldots}{1 + p_1 + p_2 + \ldots p_n} = \begin{cases} 1 + a + a^2 + a^3 + \ldots \\ 1 + b + b^2 + b^3 + \ldots \\ \&c. \qquad \&c. \end{cases}$$

$$= n + A + B + C + \ldots \text{ to infinity}$$

$$\therefore A + B + C + \ldots = \frac{n+(n-1)p_1+(n-2)p_2+\ldots p_{n-1}}{1+p_1+p_2+\ldots p_n} - n$$

$$= -\frac{p_1+2p_2+3p_3+\ldots(n-1)\,p_{n-1}+np_n}{1+p_1+p_2+p_3+\ldots p_n}$$

N. B. By the aid of this form we may sum a great variety of infinite series.

358. Let S be the value required.

Then $S = \frac{1}{a}(b+c+\ldots) + \frac{1}{b}(a+c+\ldots)+\frac{1}{c}(a+b+\ldots)+$ &c.

$$= \frac{1}{a}(p-a) + \frac{1}{b}(p-b) + \frac{1}{c}(p-c) + \ldots$$

$$= p.\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\ldots\right)-n$$

$$= p.\frac{Q}{R} - n \text{ the value required.}$$

359. Let S = the sum of the functions.

Then $S = a^2\left(\frac{1}{b} + \frac{1}{c} + \ldots\right) + b^2\left(\frac{1}{a}+\frac{1}{c}+\ldots\right)+ c^2 \times \left(\frac{1}{a} + \frac{1}{b} + \ldots\right) +$ &c.

$$= a^2\left(\frac{Q}{R} - \frac{1}{a}\right)+b^2\left(\frac{Q}{R}-\frac{1}{b}\right)+c^2\left(\frac{Q}{R}-\frac{1}{c}\right)+ \&c.$$

$$= (a^2 + b^2 + c^2 + \ldots)\frac{Q}{R} - (a + b + c + \ldots)$$

$\therefore S = (p^2 - 2q)\frac{Q}{R} - p$ as was to be demonstrated.

MISCELLANEOUS ALGEBRA.

360. The equation is $(x-\sqrt{-2}).(x+\sqrt{-2}).(x-3).(x-4)$
$=(x^2+2).(x^2-7x+12)=x^4-7x^3+14x^2-14x+24$
$\therefore\ x^4-7x^3+14x^2-14x+24=0$ is the equation required.

361. The equation is $(x-1-\sqrt{-2}).(x-1+\sqrt{-2})\times$
$(x-2-\sqrt{-2}).(x-2+\sqrt{-2})$
$=(\overline{x-1}|^2+2)\times(\overline{x-2}|^2+2)=(x^2-2x+3)(x^2-4x+6)$
$=x^4-6x^3+17x^2-24x+18=0$

362. Since $1+\sqrt{a^3}$ is a root, and $\sqrt{a^3}$ is supposed a surd, $1-\sqrt{a^3}$ is also a root. Also $-\sqrt{-b}$ is supposed imaginary or b is positive, $\therefore +\sqrt{-b}$ is a root.
Hence $(x-1-\sqrt{a^3}).(x-1+\sqrt{a^3}).(x+\sqrt{-b}).(x-\sqrt{-b})=$
$(\overline{x-1}|^2-a^3)\times(x+b)=(x^2-2x+1-a^3).(x+b)=$
$x^4-2x^3+(1-a^3+b)x^2-2bx+b.(1-a^3)=0$

363. The equation is $(x-2-\sqrt{-3})(x-2+\sqrt{-3}).\times$
$(x-1).(x+5)=(\overline{x-2}|^2+3)(x^2+4x-5)=x^4-14x^2+$
$48x-35=0$

364. Let it be continued to n factors.
Assume $a.(a+r)(a+2r).(a+3r)\dots(a+(n-1)r)=a^n+P_1\times$
$a^{n-1}+P_2a^{n-2}+P_3a^{n-3}+\dots$

Then it is known that $P = r + 2r + 3r + \ldots (n-1)r = \frac{n.\overline{n-1}}{2}r$

Also $P^2 - 2Q = r^2 + 4r^2 + \ldots (n-1)^2 r^2 = r^2 (1+4+\ldots (n-1)^2)$

Now, let $1+4+\ldots (n-1)^2 = S$

Then $\Delta S = n^2 = (n-1+1)\, n = n.(n-1) + n$

$$\therefore\ S = \frac{\overline{n-2}\,.\,\overline{n-1}\,.\,n}{3} + \frac{(n-1).\,n}{2} \text{ there being no correction.}$$

$$\therefore\ S = \frac{n.(n-1)}{6}\,.(2n-4+3) = \frac{n.(n-1).(2n-1)}{2\times 3}$$

$$\therefore\ 2\,Q = \frac{n^2.\overline{n-1}|^2}{4}\,r^2 - \frac{n.(n-1).(2n-1)}{2\times 3}\,r^2$$

$$= \frac{(6n^2 - 14n + 4)}{2.3.4}\,r^2$$

$$\therefore Q = \frac{3n^2 - 7n + 2}{2.3.4}\,r^2$$

By the same process the remaining coefficients may be found, and thence we shall have an expression, such as was required.

The last coefficient $P_n = r \times 2r \times$, &c. $\times\ 0 = 0$

365. Log. $\frac{A^2 \sqrt{B^2 - C^2}}{C \sqrt[5]{D^3\, E\, F}} = \log. (A^2 \sqrt{B^2 - C^2} - \log. (C \sqrt[5]{D^3\, E\, F})$

But $\log. A^2 \sqrt{B^2 - C^2} = 2 \log. A + \log. (B^2 - C^2)^{\frac{1}{2}} = 2 \log. A + \frac{1}{2} \log. (B^2 - C^2)$, and $\log. (C\, D^{\frac{3}{5}}\, E^{\frac{1}{5}}\, F^{\frac{1}{5}}) = \log. C + \frac{3}{5} \log. D + \frac{1}{5} \log. E + \frac{1}{5} \log. F.$

$$\therefore \log. \frac{A^2 \sqrt{B^2 - C^2}}{C \sqrt[5]{D^3\, E\, F}} = 2 \log. A + \frac{1}{2} \log. (B^2 - C^2) - \log. C - \frac{1}{5}(3 \log. D + \log. E + \log. F),$$

whence the log. of the quantity will be found, by means of which as a reference in the tables, the quantity itself may be found.

366. **Besides the m real roots, there can only be pairs of imaginary roots of the form $a \pm b\sqrt{-1}$, where, a is positive negative or zero, (b is necessarily unequal to zero). Now, the differences arising from combining imaginary roots of the form $a \pm b\sqrt{-1}$ or of $\pm b\sqrt{-1}, \pm b'\sqrt{-1}$, &c., are of the form $\pm (b + b')\sqrt{-1}$, whose square is negative. All other differences in which two imaginary roots, or in which an imaginary and a real root are involved, are of the form $a \pm b\sqrt{-1}$, the square of which is imaginary.**

Again, since there are m real roots, taking them two and two we form $m. \frac{m-1}{2}$ differences, the squares of which are positive.

$\therefore$ with regard to sign, $m. \frac{m-1}{2}$ of the roots of the transformed equation are positive, and the rest either negative or imaginary.

Hence, the last term being $=$ product of all the roots with their signs changed, can only be affected in sign by the $m. \frac{m-1}{2}$ real roots. It is $\therefore$ positive or negative, according as $m. \frac{m-1}{2}$ is even or odd.

367. This can be effected by the Method of Divisors, since there must be corresponding rational factors equal to these roots in the last term; unless any of the roots are reciprocals of others. In this case, the method of finding reciprocal roots may be applied.

The doctrine of limits will also shew the value of rational roots.

368. For the proof of the rule see *Wood*, Method of Divisors.

To apply this rule,

Let $x = 1, 0, -1, -2$, &c., be substituted successively, and the corresponding results will be 12, 112, 390, &c., which having a great number of divisors, it will be convenient to diminish the roots, and thereby the coefficients. Diminish the roots by e, i. e., assume $y = x - e$, and in the resulting equation it will be seen, $e = 2$ will be a convenient assumption; for then the equation will become

$y^4 - 9y^3 + 10y^2 + 8y = 0$ of which one root $= 0$

$\therefore x = y + 2 = 2 \quad \therefore 2$ is a root of the given equation.

But $y^3 - 9y^2 + 10y + 8 = 0$

For y substitute $1, 0, -1, -2$, &c. successively, and the results will be $10, 8, -12$, whose respective divisors are $1, 2, 5$; $1, 2, 4$ and $1, 2, 3, 4, 6$. $\therefore$ the sums and differences of these divisors and of $1, 0, 1$ are $-4, -1, 0, 2, 3, 6$; $-4, -2, -1, 1, 2, 4$; and $-5, -3, -2, -1, 0, 2, 3, 4, 5, 7$ respectively, in which the progressions are, $6, 4, 2$; $6, 2, -2$; $3, 1, -1$; $3, -1, -5$; $2, 1, 0$; &c. &c. $\therefore$ the divisors to be tried are $y^2 - 2y + 4$, $y^2 - 4y + 2$, $y^2 - 2y + 1$, &c., whence we have two roots of the equation in y, and thence $x = y + 2$ is known, &c.

369. Since $1764 = 6^2 \times 7^2$

$$\text{Log.}\,(1767) = \log.\,(1764 + 3) = \log.\,1764.\left(1 + \frac{3}{1764}\right)$$

$$= \log.\,(1764) + \log.\left(1 + \frac{3}{1764}\right)$$

$$= \log.\,(6^2 \times 7^2) + \frac{3}{1764} - \frac{3^2}{2(1764)^2} + \ldots.$$

$$= 2\,(\log.\,6 + \log.\,7) + \frac{3}{1764} - \frac{9}{6223392} \text{ nearly.}$$

$$= 2.\,(\log.\,6 + \log.\,7) + .001699 \ldots. \text{ nearly.}$$

370. Let the equation be divided by $x - a$, till the remainder, R, contain no power of x, and let Q be the quotient.

Then $x^n - px^{n-1} + \ldots.. \pm W = Q \times (x - a) + R = 0$

Now, let $x = a$, then $R = -\,Q \times (a - a) = 0$

$\therefore$ the expression $x^n - \ldots\ldots \pm W$ is divisible by $x-a$, if a be a root of the equation $x^n - px^{n-1} + \ldots\ldots \pm W = 0$.

Otherwise.

Since a is a root of the equation, we have

$$\left.\begin{array}{l} x^n - px^{n-1} + \ldots\ldots \pm W = 0 \\ a^n - pa^{n-1} + \ldots\ldots \pm W = 0 \end{array}\right\}$$

$\therefore x^n - a^n - p\,.\,(x^{n-1} - a^{n-1}) + \ldots\ldots \pm V\,(x-a) = 0$, an expression which is divisible by $x-a$; for put $x - a = y$

Then $x^m - a^m = (y+a)^m - a^m = y^m + m\,a\,y^{m-1} + \ldots\ldots\, m a^{m-1} y$

$$\therefore \frac{x^m - a^m}{x-a} = \frac{x^m - a^m}{y} = y^{m-1} + m\,a\,y^{m-2} + \ldots\ldots m\,a^{m-1}$$

$\therefore x^m - a^m$ is divisible by $x-a$, and $\therefore$ the above expression is.

Hence, &c. &c.

371. The hyp. log. $e = 1$ (e being the hyperbolic base) $\therefore b = b$. hyp. log. e. $=$ hyp. log. e^b.

$$\therefore \text{hyp. log.} \frac{\sqrt{a^2+x^2}+a}{\sqrt{a^2+x^2}-a} = \text{hyp. log. } e^b$$

$$\therefore \frac{\sqrt{a^2+x^2}+a}{\sqrt{a^2+x^2}-a} = e^b \therefore \sqrt{a^2+x^2} + a = e^b\sqrt{a^2+x^2} - ae^b \times$$

$$\therefore \sqrt{a^2+x^2} = \frac{a\,.\,(1+e^b)}{e^b-1}$$

$$\therefore a^2 + x^2 = a^2\,.\left(\frac{e^b+1}{e^b-1}\right)^2$$

$$\therefore x^2 = \frac{4a^2e^b}{(e^b-1)^2}$$

$$\therefore x = \pm\frac{2ae^{\frac{b}{2}}}{e^b-1}$$

372. Let a and $a+3$, b and c be the roots of the equation.

Then $2a + 3 + b + c = 45$

$$\therefore 2a + b + c = 42$$

Also $a^2 + 3a + ab + ac + ab + 3b + ac + 3c + bc = -40$

or $a^2 + 2ab + 2ac + bc + 3a + 3b + 3c = -40$

and $p^2 + 2q = 2025 + 80 = a^2 + a^2 + 6a + 9 + b^2 + c^2 = 2105$

$$\left.\begin{aligned} \text{or } 2a^2 + 6a + b^2 + c^2 &= 2096 \\ 2a + b + c &= 42 \\ \text{and } a^2 + 2a(b+c) + 3.(a+b+c) &= -40 \end{aligned}\right\}$$ from which we can find a, b, and c.

By substitution $a^2 + 2a.(42-2a) + 3.(42-a) = -40$

$\therefore\ a^2 + 84a - 4a^2 + 126 - 3a = -40$

$\therefore\ 3a^2 - 81a = 166 \quad \therefore\ a^2 - 27a = \dfrac{166}{3}$, whence a can be found, and similarly of the rest.

THEORY OF NUMBERS.

NOTATION.

373. Let the multiple $m.(a-1)$ of $a-1$ consist of n digits, $d_0, d_1 \ldots\ldots d_{n-1}$.

Then $d_0+d_1 a+d_2 a^2+\ldots\ldots d_{n-1}a^{n-1}=m.(a-1)$

But $d_1 a=d_1.(a+1-1)=d_1+d_1.(a-1)$

$d_2 a^2=d_2.(\overline{a-1}+1)^2=d_2+2d_2(a-1)+d_2.(a-1)^2$

&c. = &c.

$\therefore m.(a-1)=d_0+d_1+\ldots\ldots Q\times(x-a)$ $(Q=d_1+2d_2+d_2\times(a-1)+\ldots\ldots)$

$\therefore m=\dfrac{d_0+d_1+\ldots\ldots}{a-1}+Q$, and m is a whole number.

$\therefore d_0+d_1+\ldots\ldots d_{n-1}$ is either $=a-1$ or a multiple of it.

374. Let the multiple of 11 be $(d_0+d_1 10+d_2 10^2+\ldots\ldots d_{n-1}10^{n-1})\times 11=N$, d_0, d_1, &c., being the digits of the multiplier.

Then $N=d_0+(d_0+d_1)10+(d_1+d_2)10^2+(d_2+d_3)10^3+\ldots\ldots(d_{n-3}+d_{n-2})10^{n-2}+(d_{n-2}+d_{n-1})10^{n-1}+d_{n-1}10^n$

But the number represented by these digits inverted (M) $=d_{n-1}+(d_{n-2}+d_{n-1})10+(d_{n-3}+d_{n-2})10^2+\ldots\ldots(d_0+d_1)10^{n-1}+d_0 10^n=d_{n-1}\times(1+10)+d_{n-2}(10+100)+d_{n-3}.(100+1000)+\ldots\ldots d_1.(10^{n-2}+10^{n-1})+d_0(10^{n-1}+10^n)$

But $10^{n-2}+10^{n-1}=10^{n-2}\times(10+1)=11\times10^{n-2}$

and $10^{n-1}+10^n=10^{n-1}\times(10+1)=11\times10^{n-1}$

$\therefore M=(d_{n-1}+d_{n-2}10+d_{n-3}10^2+\ldots\ldots d_1 10^{n-2}+d_0\times10^{n-1})11$

$\therefore \frac{N}{11}$ and $\frac{M}{11}$ have the same digits, the order of them only being different.

N. B. It can evidently make no difference in the value of (M), whether any of the compound digits of (N) be less or not less than 10. For suppose $d_2 + d_3 = 10 + a$, then we have $(d_2+d_3)\times 10^3 + (d_3 + d_4)\,10^4 = a.10^3 + (d_3 + d_4 + 1)\,10^4$ and the corresponding part of (M) $= a.10^{n-3} + (d_3 + d_4 + 1)\,10^{n-4} = (a + 10).\ 10^{n-3} + (d_3 + d_4)\,10^{n-4} = (d_2 + d_3)\,10^{n-3} + (d_3 + d_4)\,10^{n-4}$ the same as before. The same may be proved of all compounds $d_0 + d_1$, $d_1 + d_2$, &c. $\therefore$ &c.

375. Let the number (N) $= d_0 + d_1.\ 10 + d_2.\ 10^2 + \ldots d_{p-1}.\ 10^{p-1}$

$$
\begin{aligned}
\text{Then } N = d_0 &+ d_1.\ 11 + d_2.\ 11^2 + d_3.\ 11^3 + \&c.\\
&- d_1 \quad - 2d_2.\ 11 - 3d_3.\ 11^2\\
&+ d_2 \quad + 3d_3.\ 11\\
&\qquad\quad - d_3
\end{aligned}
$$

$$= 11m - 11n + 11 \times \left\{d_1 + d_2.(11-2) + d_3.(11^2 - 3.11 + 3) + \ldots\right\}$$

$$
\begin{aligned}
= 11.(m - n + d_1 + 9d_2 &+ 90d_3 + 909d_4 + 9090d_5 + \&c.)\\
&+ d_3 \qquad\qquad + d_5
\end{aligned}
$$

$$= 11\left\{m - n + 11n + e + 9\,(d_2 + 10d_3 + 101d_4 + \ldots)\right\}$$

$$= 11\left\{m + n + e + 9.\,(n + d_2 + 10d_3 + \ldots)\right\}$$

$\therefore \frac{N}{11\times 9} = \frac{m+n+e}{9} + n + d_2 + 10d_3 + \ldots$ whence the proof required is manifest.

376. Let $N = d_{n-1}\,10^{n-1} + d_{n-2}\,10^{n-2} + \ldots d_2\,10^2 + d_1\,10 + d_0$

$N' = d_{n-1} + d_{n-2}\,10 + \ldots d_2\,10^{n-3} + d_1\,10^{n-2} + d_0\,10^{n-1}$

$\therefore N - N' = d_{n-1}.\,(10^{n-1} - 1) + 10\,d_{n-2}.\,(10^{n-3} - 1) + \ldots 10\,d_1 \times (1 - 10^{n-3}) + d_0.\,(1 - 10^{n-1})$ which is divisible by $10 - 1$ or 9, since every term is divisible by $10 - 1$.

377. Let $\frac{N}{D}$ be the fraction, its corresponding terminating decimal being $d_1 d_2 d_3 \ldots d_m$, $d_1 d_2 \ldots$ being the digits of the decimals.

Then $\frac{N}{D} = \frac{d_1\, d_2\, d_3\, \ldots\, d_m}{10m}$ which, when reduced to its lowest terms, must have powers of 2 and 5 only in the denominator, because 10 is divisible only by 2 or 5, and $\therefore 10^m$ is divisible only by a quantity of the form $2^p \times 5^q$, p and q being integers.

$\therefore$ D is of the form $2^p \times 5^q$.

378. Every square number may be thus represented, $(d_0 + d_1\, 10 + d_2\, 10^2 + \ldots\, d_{n-1}\, 10^{n-1})^2 = d_0^2 + 2d_0 . d_1\, 10 + \ldots$ from which two first terms will evidently be found the two first digits.

Thus, let $d_0 = 1, 2, 3, 4, 5, 6, 7, 8, 9$ successively, and let the corresponding values of d_1 be 1, 2, 3, &c. the results shewing how many substitutions need be made, to ascertain the truth of the problem. It is found that when $d_0 = 2$, and $d_1 = 1$, or when $d_0 = 0$ we have the two first digits $= 4$, or $= 0$, and in no other case have they the same value. See also *Barlow's* Theory of Numbers, p. 84.

379. $N = a_0 + a_1 . 10 + a_2 . 10^2 + a_3 . 10^3 + \ldots a_n . 10^n$

$$\left.\begin{aligned} &= a_0 + a_1\, 7 + a_2\, 7^2 + \text{\&c.} \\ &\quad + a_1 . 3 + a_2 . 6 \times 7 + \text{\&c.} \\ &\quad\quad + a_2 . 3^2 + \text{\&c.} \end{aligned}\right\} = a_0 + 3a_1 + 3^2 a_2 + \ldots 3^n a_n$$

$+ Q.7\, (Q = a_1 + a_2 . (7+6) + \ldots)$

$$\therefore \frac{N}{7} = \frac{a_0 + 3a_1 + 3^2 a_2 + \ldots\, 3^n a_n}{7} + Q$$

$\therefore$ &c. &c.

380. It is evident that no weight exceeding the sum of $2^0 + 2^1 + 2^2 + 2^3 + \ldots\, 2^{n-1}$ can be weighed.

Now it is also evident that this sum may be represented by the

number 11111... n terms expressed in the binary scale, and similarly, any part of that sum, $2^0 + 2^3 + 2^6$ may be expressed by 1001001.

Hence, if any proposed weight, is to be weighed, and we have weights 2^0, 2^1, 2^2... 2^{n-1}, the sum of them taking not less than the given weight, that weight can be balanced by a certain number of them, which are found by transforming the number expressing the given weight into the binary scale. Thus required what selection must be made of the weights 2^0, 2^1, 2^2.... to balance 459 pounds.

Divide 459 by 2 successively, until the last quotient is 1; then the corresponding remainders will be the successive digits of the number in the binary scale, beginning with units, and the last quotient will be the last digit. (See *Barlow*).

These digits are 1, 1, 0, 1, 0, 0, 1, 1, 1.

$$\therefore 459 = 2^0 + 2^1 + 2^3 + 2^6 + 2^7 + 2^8$$

381. Divide 1317 by 2 successively, and the remainders 1, 0, 1, 0, 0, 1, 0, 0, 1, 0, 1 are the digits in the binary scale, (see the preceding problem) of the number 1317, beginning with units.

$\therefore$ that number is 10100100101, and the weights to be selected are 2^0, 2^2, 2^5, 2^8, 2^{10}.

N. B. Problems of this kind may be much diversified. Thus, suppose I wish to pay P £, but have coins of the value of p^0, p^1, p^2,... pounds only; how am I to make the payment without exchanging any of my coins?

Transform the number P into its equivalent expressed in that system whose radix is p, and the digits will exhibit how many coins, and of what kind, are to be taken.

Thus, let $P = 320$

$p = 5$

Dividing 320 successively by 5, we get 0, 4, 2, 2 for corresponding remainders.

$\therefore$ 2240 is 320 expressed in the system whose radix is 5.

$$\therefore 320 = 4 \times 5 + 2 \times 5^2 + 2 \times 5^3$$

$$\therefore\ 4 \text{ of the coins each} = 5£ \quad 2 = 25£ \quad 2 = 125£ \Big\} \text{are to be taken.}$$

Verbum sat.

382. Let v be the radix of the system; then since there are $(n-1).(n-2)\ldots 2\times 1$ permutations in $(n-1)$ things b, c, d, &c., taken $n-1$ together there will be $(n-1).(n-2)\ldots 2\times 1$ numbers in which a is the first digit; similarly it may be shewn that there are $(n-1).(n-2)\ldots 2\times 1$ in which b is the first digit and so on. In the same manner it will appear that the numbers $a, b, c, d\ldots$ are each repeated $(n-1).(n-2)\ldots 2\times 1$ times as 2[d], 3[d], &c. digits.

Hence the sum of the numbers (A) whose digits are $a, b, c\ldots n$ terms, taken in every order,

$$= (n-1).(n-2)\ldots 2\times 1 \times \overline{a+b+c+d\ldots} \times (1+v+v^2+\ldots n \text{ terms})$$

And similarly the sum of the numbers (P) whose digits are $p, q, r\ldots n$ terms, taken in every order,

$$= (n-1).(n-2)\ldots 2\times 1 \times (p+q+r+\ldots)\times(1+v+v^2+\ldots n \text{ terms.})$$

$$\therefore A : P :: a+b+c+\ldots : p+q+r+\ldots$$

Otherwise.

Since by the question, the quantities a, b, c, &c... are to be permuted n and n together, there can be no reason why one of them should be repeated more times than another in each place of the digits, or that any one place of the digits in the sum, should involve $a, b, c\ldots$ differently from another. Also, since the number of $p, q, r\ldots$ is the same as that of $a, b, c\ldots$, these must be similarly involved in the sum of the numbers formed from them. Let $\therefore$ a be the first digit m times, then b is first digit m times, and so of the rest. Also $a, b, c\ldots$ are second digits m times, &c. &c. $\therefore$ A $= m\times(a+b+c+\ldots)\times(1+v+v^2+\ldots)$

Hence also $B = m.(p+q+r+\ldots)(1+v+v^2+\ldots)$

$$\therefore A : B :: a+b+c+\ldots : p+q+r+\ldots$$

383. Any number in common notation may be represented by the form $d_0 + d_1 . 10 + d_2 . 10^2 + \ldots . d_{n-1} . 10^{n-1}$ (n being the number of digits). These digits may evidently be any of the numbers 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, without deranging the formula; if, however, any of them be 10 or $>$ than 10, the ten can be transferred to the next digit, and the excess less than ten, will remain as a new digit. As, indeed, in the common system, d_0, d_1... serve to exhibit the number of units, 10, 10^2, 10^3... it is evident they ought to be less than ten, when the formula is reduced to its most simple terms. There cannot $\therefore$ be more than ten different digits, 0, 1, 2, 3, 4...9, and there may be as many as ten used in the representation of any number.

POLYGONAL NUMBERS.

384. Triangular numbers are the successive sums of arithmetical series, whose first term is unity and common difference is also unity.

Hence the general triangular number may be expressed by

$1 + 2 + 3 + \dots. n = (1 + n). \frac{n}{2}$

Now $(1 + n). \frac{n}{2} \times 8 + 1 = 4n^2 + 4n + 1 = (2n + 1)^2$ which is a square number, $\therefore$, &c. ...

385. Hexagonal numbers are the successive sums of arithmetical series, whose first term is unity and common difference $= 4$.

$\therefore$ the n^{th} term in the series of hexagonal numbers $= 1 + 5 + 9 + \dots. 1 + 4\,(n-1) = (2 + 4.(n-1).)\,\frac{n}{2} = (4\,n-2). \frac{n}{2} = n.\,(2n-1)$

The $(2n-1)^{th}$ term in the series of triagonal numbers

$= 1+2+3+\dots. 2n-1 = (1 + 2n - 1). \frac{2n-1}{2} = \frac{2n}{2} \times (2n - 1)$

$= n.\,(2n-1)$

$\therefore$ the n^{th} term, &c....&c.

PRIME NUMBERS.

386. If (m) be integral, every coefficient in the expansion of $(a + b)^m$ is integral (*Barlow*, p. 17). The general coefficient is of the form $m. \frac{m-1}{2}. \frac{m-2}{3}....$ Hence when m is prime, $\frac{m-1}{2}. \frac{m-2}{3}....$ must be integral; i.e., every coefficient except the first and last, is divisible by m, and $\therefore$ the sum of the intermediate terms is divisible by m.

Now $x^m = (x-1+1)^m = (x-1)^m + 1 + m. Q_1$, Q_1 being the sum of all the terms except the first and last, divided by m.

Similarly $(x-1)^m = (x-2)^m + 1 + m. Q_2$

$$(x-2)^m = (x-3)^m + 1 + m. Q_3$$

$$\&c. = \&c.$$

$$(x-\overline{x-1})^m = (x-x)^m + 1 + m\, Q_x$$

$$\therefore x^m = 1 + 1 + ... x \text{ terms} + m. (Q_1 + Q_2 + ... Q_x)$$

$$= x + m. (Q_1 + Q_2 + ... Q_x)$$

$$\therefore \frac{x^m}{m} = \frac{x}{m} + Q_1 + Q_2 + ... Q_x$$

$\therefore \frac{x^m}{m}$ leaves the same remainder as $\frac{x}{m}$

Similarly if m be any prime > 2, $\frac{x^m}{2m}$, $\frac{x}{2m}$ leave the same remainders; also if m be any prime $> 2 \times 3$, $\frac{x^m}{2 \times 3m}$, $\frac{x}{2 \times 3m}$ leave the same remainders, &c. &c.

Again $\frac{n^p}{p} = \frac{n^{p-1}}{p} \times n = (w + \frac{1}{p})\, n = nw + \frac{n}{p}$

But $\frac{a^p}{p}$ leaves the same remainder as $\frac{n^p}{p}$

$\therefore p^{th}$ remainder $=$ first remainder.

Again $\frac{n^{p+1}}{p} = \frac{n^{p-1}}{p} \times n^2 = \left(w + \frac{1}{p}\right)n^2 = n^2 w + \frac{n^2}{p}$

$\therefore \frac{n^{p+1}}{p}$ and $\frac{n^2}{p}$ leave the same remainder.

$\therefore \frac{a^{p+1}}{p}$ and $\frac{a^2}{p}$ leave the same remainder.

Generally $\frac{n^{p+q}}{p} = \frac{n^{p-1}}{p} \times n^{q+1} = \left(w + \frac{1}{p}\right) \times n^{q+1} = wn^{q+1} + \frac{n^{q+1}}{p}$.

$\therefore$ the $(p+q)^{th}$ remainder $=$ the $(q+1)^{th}$ remainder, q being any number whatever.

$\therefore$ after the $(p-1)^{th}$ remainder, they recur.

387. $a^n = (a - 1 + 1)^n = (a-1)^n + 1 + n.Q_1$ where Q is integral, since every coefficient of $(a+b)^n$ expanded is integral, (see *Barlow*, p. 17), and n is prime.

Similarly $(a-1)^n = (a-2)^n + 1 + n.Q_2$

&c. = &c.

$$\left\{a - (a-1)\right\}^n = (a-a)^n + 1 + n\,Q_a$$

$$\therefore a^n = a + n.(Q_1 + Q_2 + Q_3 + \dots Q_a)$$

$$\therefore \frac{a^n - a}{n} = Q_1 + Q_2 + Q_3 + \dots Q_a = P$$

$\therefore a.\frac{(a^{n-1} - 1}{n}$ is divisible by n; but a is not divisible by n;

$\therefore \frac{a^{n-1} - 1}{n}$ is an integer (A).

Similarly $\frac{b^{n-1} - 1}{n}$ is an integer (B)

$\therefore \frac{a^{n-1} - 1}{n} - \frac{b^{n-1} - 1}{n} = \frac{a^{n-1} - b^{n-1}}{n}$ is an integer. Q. E. D.

388. Let $a = mp + n_1$ (n) being less than (a).

$\therefore \frac{a}{p}, \frac{a^2}{p}, \frac{a^3}{p}$, &c. leave the same remainders as n, n^2, n^3, &c. since every term except the last of the expansions is divisible by p.

Now $\frac{n^p - n}{p}$ is an integer, and since n is less than, and $\therefore$ prime to p, $\frac{n^{p-1} - 1}{p}$ is integral (see preceding problem.)

Put $\therefore \frac{n^{p-1} - 1}{p} = w$

$\therefore \frac{n^{p-1}}{p} = w + \frac{1}{p}$ or we obtain a remainder $= 1$ when we take the $(p-1)^{th}$ term.

389. Let $\frac{n}{d}, \frac{n'}{d'}$, be the two fractions.

Then $\frac{n}{d} + \frac{n'}{d'} = \frac{32}{45} = \frac{nd' + n'd}{dd'}$.

Assume $dd' = 45 = 5 \times 9$, and $\therefore nd' + n'd = 32$

But since d and d' are prime to each other, $d = 5$, and $d' = 9$

$\therefore 9n + 5n' = 32$

$\therefore n + n' + \frac{4n}{5} = 6 + \frac{2}{5}$

Put $\frac{4n}{5} - \frac{2}{5} = w$ (a whole number)

$\therefore 4n - 2 = 5w$

$\therefore n = w + \frac{2 + w}{4}$

Put $\frac{2 + w}{4} = v$ (a whole number)

$\therefore w = 4v - 2$

Hence $n = 4v - 2 + \frac{2 + 4v - 2}{4} = 5v - 2$

and $n' = \frac{32}{5} - \frac{9n}{5} = \frac{32 - 45v + 18}{5} = 10 - 9v$ in which values v may $= 0, \pm 1, \pm 2, \pm 3, \ldots..$

Let $v = 0$

Then $n = -2$ and $n' = 10$

Let $v = 1$

Then $n = 3$ and $n' = 1$

&c. &c. &c.

CONTINUED FRACTIONS AND INDETERMINATE PROBLEMS.

390. $\sqrt{2} = 1 + \sqrt{2} - 1 = 1 + \dfrac{1}{\sqrt{2}+1}$

$\sqrt{2} + 1 = 2 + \sqrt{2} - 1 = 2 + \dfrac{1}{\sqrt{2}+1}$

$\sqrt{2} + 1 = 2 + \sqrt{2} - 1 = 2 + \dfrac{1}{\sqrt{2}+1}$

&c. = &c. = &c.

$$\therefore \sqrt{2} = 1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \ldots}}}} = 1 +$$

Now $\dfrac{1}{2+\frac{1}{2}} = \dfrac{2}{5}$

and $\dfrac{1}{\frac{2}{5}+2} = \dfrac{5}{12}$

$\dfrac{1}{\frac{5}{12}+2} = \dfrac{12}{29}$

$\dfrac{1}{\frac{12}{29}+2} = \dfrac{29}{70}$

&c. = &c.

$\therefore \sqrt{2} = 1 + \dfrac{2}{5}$ nearly, $= 1 + \dfrac{5}{12}$ more nearly, $= 1 + \dfrac{12}{29}$ more nearly, &c. &c.

391. $$\sqrt{3} = 1 + \sqrt{3} - 1 = 1 + \cfrac{1}{\cfrac{\sqrt{3}+1}{2}}$$

$$\frac{\sqrt{3}+1}{2} = 1 + \frac{\sqrt{3}-1}{2} = 1 + \frac{1}{\sqrt{3}+1}$$

$$\sqrt{3}+1 = 2 + \sqrt{3} - 1 = 2 + \cfrac{1}{\cfrac{\sqrt{3}+1}{2}}$$

$$\frac{\sqrt{3}+1}{2} = 1 + \frac{\sqrt{3}-2}{2} = 1 + \frac{1}{\sqrt{3}+1}$$

$$\sqrt{3}+1 = 2 + \sqrt{3} - 1 = 2 + \cfrac{1}{\cfrac{\sqrt{3}+1}{2}}$$

&c. = &c. = &c.

$$\therefore \sqrt{3} = 1 + \cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{2+1 \text{ \&c.}}}}}}}$$

Hence the fractions converging to $\sqrt{3}$ are

$$\frac{1}{1},\quad 1+\frac{1}{1+\frac{1}{2}},\quad 1+\cfrac{1}{1+\cfrac{1}{2+1}},\quad 1+\cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{1+\frac{1}{2}}}} \quad \text{\&c. \&c.}$$

or $1,\ 2,\ \frac{5}{3},\ \frac{7}{4},\ \frac{16}{9}$, &c. .....

392. $X^2 - Y^2 = 24$

$$\begin{aligned} \therefore (X - Y) \times (X + Y) &= 1 \times 24 \\ &\text{or } 2 \times 12 \\ &\text{or } 3 \times 8 \\ &\text{or } 4 \times 6 \\ &\text{or } 6 \times 4 \\ &\text{or } 8 \times 3 \\ &\text{or } 12 \times 2 \\ &\text{or } 24 \times 1 \end{aligned}$$

$$\therefore \left.\begin{array}{l}X+Y=24\\X-Y=1\end{array}\right\}\text{or}\left.\begin{array}{l}X+Y=12\\X-Y=2\end{array}\right\}\text{or}\left.\begin{array}{l}X+Y=8\\X-Y=3\end{array}\right\}\text{or}\left.\begin{array}{l}X+Y=6\\X-Y=4\end{array}\right\}$$

$$\text{or}\left.\begin{array}{l}X+Y=4\\X-Y=6\end{array}\right\}\text{or}\left.\begin{array}{l}X+Y=3\\X-Y=8\end{array}\right\}\text{or}\left.\begin{array}{l}X+Y=2\\X-Y=12\end{array}\right\}\text{or}\left.\begin{array}{l}X+Y=1\\X-Y=24\end{array}\right\}$$

$\therefore$ the values of x are $\frac{25}{2}$, 7, $\frac{11}{2}$, 5, 5, $\frac{11}{2}$, 7, and the corresponding values of y are $\frac{+23}{2}$, 5, $\frac{5}{2}$, 1, -1, $\frac{-5}{2}$, -5.

For the complete solution, see *Barlow*, page 106.

393. $mx + ny = p$

$$\left.\begin{array}{l}\therefore\ x = \dfrac{p}{m} - \dfrac{ny}{m}\\[2ex] \text{and } y = \dfrac{p}{n} - \dfrac{mx}{n}\end{array}\right\}$$ in which, if m be prime to n, the least integral values of y and x, which will render x and y integral, are m and n respectively; since p is divisible both by m and n. If m and n be not prime to each other, these are not the least values of m and n.

Let $p = qm = rn$, $\therefore m = \frac{p}{q}$, $n = \frac{p}{r}$

$\therefore \frac{p}{q}x + \frac{p}{r}y = p$

$\therefore rx + qy = rq$

$$\left.\begin{array}{l}\therefore\ x = q - \dfrac{q}{r}y\\[2ex] y = r - \dfrac{r}{q}x\end{array}\right\}$$ in which the least integral values of y and x are the nearest integers, which, when substituted for y and x in $\frac{q}{r}\cdot y$, $\frac{r}{q}\cdot x$, render them integral.

MISCELLANIES.

394. Let x be the number required.

Then $x = 2q + 1 = 3q' + 2 = 5q'' + 3$ (q, q', q'' being the quotients corresponding to the divisors, 2, 3 5, respectively).

Now, these quotients being integral, we must seek integer values of them from the two equations,

$$\left.\begin{aligned} 2q + 1 &= 3q' + 2 \\ 5q'' + 3 &= 3q' + 2 \end{aligned}\right\}$$

$$2q = 3q' + 1$$

$$5q'' = 3q' - 1$$

$$\therefore q = q' + \frac{q' + 1}{2}$$

$$\text{and } q'' = \frac{3q' - 1}{5}$$

Put $\dfrac{q' + 1}{2} = w$ an integer

$$\therefore q' = 2w - 1$$

$$q'' = \frac{6w - 4}{5} = w + \frac{w - 4}{5}$$

Let $\dfrac{w - 4}{5} = v \quad \therefore w = 5v + 4$, where v may $= 0, 1, 2,$ &c.

The corresponding values of w are 4, 9, 14, 19, &c.
and of q'.... 7, 17, 27, 37, &c.
and of x.... 23, 53, 83, 113, &c.

395. Let x be the representative of the roots of unity.

Then $x^3 = 1 = \cos. 2m\pi + \sqrt{-1} \sin. 2m\pi$, m being any integer.

$\therefore x = \cos.\frac{2m\pi}{3} + \sqrt{-1}\sin.\frac{2m\pi}{3}$, by Demoivre's Theorem

$\therefore x^n = \cos.\frac{2mn\pi}{3} + \sqrt{-1}\sin.\frac{2mn\pi}{3}$ (n being integral)

$= (\cos. 2mn\pi + \sqrt{-1}\sin. 2mn\pi)^{\frac{1}{3}} = (1)^{\frac{1}{3}} = x$, since $2mn$ is an even number.

Again, let n be fractional and $= \frac{p}{q}$

$\therefore x^{\frac{p}{q}} = (\cos. 2mp\pi + \sqrt{-1}\sin. 2mp\pi)^{\frac{1}{3q}}$

$= (\cos. 2mpq\pi + \sqrt{-1}\sin. 2mpq\pi)^{\frac{1}{3q}}$ (since $2mpq$ is even,)

$= (\cos. 2mp\pi + \sqrt{-1}\sin. 2mp\pi)^{\frac{1}{3}}$ (by Demoivre.)

$= (1)^{\frac{1}{3}} = x =$ cube root of unity.

Generally, it may be shewn in the same way, that any power of any root of unity is itself a root of unity.

396. Let x be one part, then $2 - x$ is the other, and $x - 2 + x$ or $2x - 2 =$ difference of the parts.

But $2 - x + x^2 - (x + (2 - x))^2 = 2 - x + x^2 - (x + 4 - 4x + x^2) = 2 - x + x^2 - x - 4 + 4x - x^2 = 2x - 2 =$ difference of the parts, Q. E. D.

397. Since S, R, Q, P, &c. = sum of products of the roots with their signs changed, taken n and n, $n - 1$ and $n - 1$, $n - 2$ and $n - 2$, &c., together respectively, and the numbers of combinations of n things taken n and n, $n - 1$ and $n - 1$, &c, together are $1, n, \frac{n.n-1}{1.2}, \frac{n.(n-1).(n-2)}{1.2.3}$, &c. respectively,

$\therefore$ S consists of one term

R consists of n terms

Q consists of $\frac{n.n-1}{1.2}$

P consists of $\frac{n.n-1.n-2}{1.2.3}$

&c. &c.

Again, since in $(n-1)$ roots (all except (a)) taken, $n - 1$ and

$n-1$, $n-2$ and $n-2$, $n-3$ and $n-3$, &c., there may be formed 1, $n-1$, $\frac{(n-1).(n-2)}{1.2}$, &c., products respectively.

$\therefore$ R contains only 1 term $=(r)$ which is not divisible by (a)
Q.... contains $(n-1)$ terms $=(q)$....
P.... contains $\frac{(n-1).(n-2)}{1.2}$ terms $=(p)$....
&c. &c.

Hence S contains one term $=(s.a)$ divisible by a,
R contains $n-1$ terms $=(r'.\times a)$...
Q contains $\frac{n.(n-1)}{1.2}-(n-1)=\frac{(n-1).(n-2)}{1.2}$ terms $=(q'\times a)$....
&c. &c.

Hence $S = sa$
$R=r+ar'$, $Q=q+aq'$, &c. $=$ &c. &c.

Now $\frac{S}{a}=s=-$ (product of all the roots except (a) with their signs changed) $=-r$ evidently.

$$\therefore \frac{S}{a}+R=\frac{S}{a}+r+r'\times a=-r+r+r'\times a=r'a$$

$$\therefore \frac{R'}{a}=r'$$

Again, $\frac{R'}{a}+Q=r'+q+aq'$

But, since $ar'=-a\times(-r')=-a\times$ (sum of the products of $(n-1)$ roots with their signs changed, taken $(n-2)$ together $=-a\times q$

$$\therefore r'=-q$$

Similarly $q'=-p$
&c. $=$ &c.

$$\therefore \frac{R'}{a}+Q=aq'=Q'$$

$$\therefore \frac{Q'}{a}=q'$$

$$\therefore \frac{Q'}{a}+P=\text{an integer}=q'+p+ap'=ap'$$

$$\therefore \frac{P'}{a} = p'$$

&c. = &c.

$$\therefore \frac{S}{a}, R', Q', P', \&c., \text{ are integers.}$$

N. B. This elegant property of the coefficients which may also be proved by substituting a for x, &c. &c., may be applied to find the roots of equations, provided those roots be integral.

Thus $x^5 - 8x^4 + 27x^3 - 52x^2 + 56x - 24 = 0$

$$\text{Since } R' = \frac{S}{a} + R$$

$$Q' = \frac{R'}{a} + Q = \frac{S}{a^2} + \frac{R}{a} + Q$$

$$P' = \frac{Q'}{a} + P = \frac{S}{a^3} + \frac{R}{a^2} + \frac{Q}{a} + P$$

&c. = &c.

$\therefore$ if by trial we find such a value of a as shall make $\frac{S}{a^m} + \frac{R}{a^{m-1}} + \frac{Q}{a^{m-2}} + \ldots$ to (m) terms, or $\frac{S + aR + a^2Q + \ldots}{a^m}$ an integer, that value will *in all probability* be a root, and substitution may be made accordingly. Three or four terms will be sufficient for this trial.

In the above equation $S = -24$

$R = 56$

$Q = -24$

$$\therefore \frac{-24 + 3 \times 56 - 9 \times 24}{3^3} = \frac{-324}{27} = -12$$

$\therefore$ we may conclude 3 to be a root.

Other applications may also be made, which we leave to the ingenuity of the reader.

398. Let x be the number required.

Then $x = 3q + 1 = 5q' + 3$, q and q' being the quotients corresponding to the divisors 3 and 5 respectively.

Hence $3q = 5q' + 2$

$$\therefore q = 2q' + 1 - \frac{q' + 1}{3}$$

Put $\frac{q' + 1}{3} = w$ an integer.

$\therefore q' = 3w - 1$

and $q = \frac{15w - 5 + 2}{3} = 5w - 1$

Let $w = 0, 1, 2,$ &c. then $q = -1, 4, 9,$ &c.

$x = -2, 13, 28,$ &c.

$\therefore$ the least value of x is 13

399. Let $a = b + x \quad \therefore a - b = x$

Then $a^n = b^n + n.b^{n-1}.x + \ldots. x^n$

$\therefore \frac{a^n - b^n}{x} = nb^{n-1} + n.\frac{n-1}{2}.b^{n-2}x + \ldots. x^{n-1}$

$\therefore a^n - b^n$ is divisible by x or $a - b$

But $a^n + b^n = 2b^n + n.b^{n-1}x + \ldots. x^n$ which is not divisible by x, or $a - b$, unless $2b^{n-1}$ is, i.e. unless $\frac{2b^{n-1}}{a-b}$ or $\frac{b}{a-b}$ be an integer.

Again, let $a = x - b$ or $x = a + b$

Then $a^n = x^n - n.x^{n-1}b + \ldots. \pm b^n$ according as n is even or odd.

$\therefore a^n - b^n = x^n - n.x^{n-1}b + \ldots. \mp n.xb^{n-1} + 0$ or $-2b^n$

$\therefore a^n - b^n$ is divisible by x or $a + b$, if n be even, or if n be odd, provided $\frac{2b^n}{a+b}$ be an integer, but in all other cases, $a^n - b^n$ is not divisible by $a + b$.

400. Let $2x + 1$ represent the odd number, x being any integer whatever.

Then $(2x + 1)^2 + 3 = 4x^2 + 4x + 4$ which is evidently divisible by 4.

401. $(2n)^3 = 8n^3$ which is even.

$(2n+1)^3 = 8n^3 + 12n^2 + 6n + 1 = 2.(4n^3 + 6n^2 + 3n)+1$ which is odd.

$\therefore$ the root of every odd cube is also odd.

Let $\therefore$ $2x + 1$ be the root of the given cube.

Then the middle term of the series $(2x + 1)^2 = 4x^2 + 4x + 1$

and the common difference $= 1$

$\therefore$ the last term $= 4x^2 + 5x + 1$

and the first term $= 4x^2 + 3x + 1$

$$\therefore \text{ the sum of the series } = (8x^2 + 8x + 2)\,\frac{2x+1}{2}$$

$$= (4x^2 + 4x + 1)\,.\,(2x + 1)$$

$$= 8x^3 + 12x^2 + 6x + 1$$

$$= (2x + 1)^3 \quad \text{Q. E. D.}$$

402. This is nothing more than finding two squares together = a given square, the sides of those squares being rational.

Let a^2 be the given square, and x^2, y^2 those sought.

Then $a^2 = x^2 + y^2$

$\therefore a^2 - y^2 = x^2 = (a - y).(a + y)$

Put $a + y = \dfrac{px}{q}$

and $a - y = \dfrac{qx}{p}$

$$\therefore 2a = \frac{p^2 + q^2}{pq}\,x$$

$$2y = \frac{p^2 - q^2}{pq}\,x$$

$$\therefore x = \frac{2apq}{p^2 + q^2}$$

$y = \dfrac{p^2 - q^2}{p^2 + q^2} \times a$ where p and q may be assumed of any magnitude. If $a =$ two squares, put $p^2 + q^2 = a$ or any factor of it, and we have the above forms integral.

Let $a = 5 = 2^2 + 1$

Then $p^2 = 2^2$, $q^2 = 1$

$$\therefore x = \frac{4 \times 5}{5} = 4$$

$$y = \frac{1 \times 5}{5} = 1$$

Let $a = 30$

Then $x = \dfrac{2apq}{p^2+q^2} = \dfrac{60pq}{p^2+q^2}$

$y = \dfrac{p^2-q^2}{p^2+q^2} \times 30$ Let $p = 2$ and $q = 1$

$\therefore\ x = \dfrac{60 \times 2}{5} = 24$

$y = \dfrac{3 \times 30}{5} = 18$

$\therefore$ if the hypothenuse be 30, the legs will be 24 and 18. Similar results may be found *ad libitum*.

403. Log. $\mathrm{N} = \log.(\mathrm{N} \times 1) = \log.\mathrm{N} + \log. 1$.

Let e be the hyperbolic base.

Then $\cos.\theta + \sqrt{-1}\sin.\theta = e^{\theta\sqrt{-1}}$

$\therefore\ \cos. 2n\pi + \sqrt{-1}\sin. 2n\pi = e^{2n\pi\sqrt{-1}}$

$\therefore\ 1 = e^{2n\pi\sqrt{-1}}$

$\therefore$ log. $(1) = 2n\pi\sqrt{-1}$ where n may be any number from 0 to ∞; when $n = 0$, log. $1 = 0$.

$\therefore$ log. N = log. N and is real in this case only, since all the other values involve $\sqrt{-1}$.

Again, log. $(-\mathrm{N}) = \log.\mathrm{N} + \log.(-1)$

But since $\cos.\theta + \sqrt{-1}\sin.\theta = e^{\theta\sqrt{-1}}$

and $\therefore \cos.(2n+1).\pi + \sqrt{-1}\sin.(2n+1).\pi =$

$\therefore\ -1 = e^{(2n+1)\pi\sqrt{-1}}$

$\therefore$ log. $(-1) = (2n+1)\pi\sqrt{-1}$ which for every value of n is imaginary. $\therefore$ log. (-1) is always imaginary.

Hence log. $(-\mathrm{N}) = \log.\mathrm{N} + \log.(-1)$ is always imaginary.

ARITHMETIC OF SINES.

404. Let (a) be the given angle, (x) one of the parts and $\therefore$ $(a-x)$ the other part.

Then $\dfrac{\tan.(a-x)}{\tan.x} = \dfrac{n}{1}$, $\dfrac{n}{1}$ being the given ratio.

$$\therefore \frac{\tan.a - \tan.x}{1+\tan.a.\tan.x} = n\tan.x$$

$$\therefore \tan.a - \tan.x = n\tan.x + n\tan.a.\tan.^2x$$

$$\therefore \tan.^2x + \frac{n+1}{n\tan.a}\tan.x = \frac{1}{n}$$

$$\therefore \tan.x = \pm\sqrt{\frac{(n+1)^2}{4n^2\tan.^2a} + \frac{1}{n}} - \frac{n+1}{2n\tan.a}$$

$$\text{But } \frac{(n+1)^2}{4n^2\tan.^2a} + \frac{1}{n} = \frac{n^2+2n+1+4n\tan.^2a}{4n^2\tan.^2a}$$

$$= \frac{n^2-2n+1+4n.(1+\tan.^2a)}{4n^2\tan.^2a}$$

$$= \frac{(n-1)^2+4n.\sec.^2a}{4n^2\tan.^2a}$$

$\therefore \tan.x = \dfrac{\pm\sqrt{(n-1)^2+4n.\sec.^2a}-(n+1)}{2n\tan.a}$ $\therefore$ tan. x is known, and, by reference to the table we shall obtain x. Hence $a-x$ is known.

The tan. x may easily be constructed, and thence will appear the geometrical solution of the problem.

405. $$\text{Tan.}\frac{A}{2} = \frac{\sin.\frac{A}{2}}{\cos.\frac{A}{2}} = \frac{2\sin.\frac{A}{2}\cos.\frac{A}{2}}{2\cos.^2\frac{A}{2}} = \frac{\sin.A.}{1+\cos.A}$$

Since $\cos. A = \cos.^2 \frac{A}{2} - \sin.^2 \frac{A}{2} = 2 \cos.^2 \frac{A}{2} - 1$, &c.

This may be proved geometrically by taking in the circle whose radius AB, or AC = unity, the ∠ BAC = ∠ A, &c.

Thus, produce CA to D, join DB, and bisect BAC by the radius AE, and draw the tan. ET meeting AB produced in T. Also draw TN perpendicular AC.

$$\text{Then } \angle \text{ TAE} = \frac{1}{2} \angle \text{ TAC} = \angle \text{ D}$$

∴ AE is parallel to DT.

and the triangles TAE, TDN are ∴ similar.

∴ TE : AE :: TN : DN

$$\text{or } \tan. \frac{A}{2} : 1 :: \sin. A : 1 + \cos. A. \quad \therefore \text{ \&c.}$$

406. $\left.\begin{array}{l}\text{Sin. } (a+b) = \sin. a. \cos. b + \cos. a. \sin. b \\ \text{and } \sin. (a-b) = \sin. a. \cos. b - \cos. a. \sin. b\end{array}\right\}$ (See *Woodhouse* or *Creswell*), and generally $(x+y).(x-y) = x^2 - y^2$

$$\begin{aligned}\therefore \sin. (a+b). \sin. (a-b) &= \sin.^2 a. \cos.^2 b - \cos.^2 a. \sin.^2 b \\ &= \sin.^2 a. (1-\sin.^2 b) - (1-\sin.^2 a). \sin.^2 b \\ &= \sin.^2 a - \sin.^2 b.\end{aligned}$$

The geometrical proof will not be difficult.

407. $$\begin{aligned}\text{Cos. } 2a = \cos. (a+a) &= \cos. a. \cos. a - \sin. a. \sin. a \\ &= \cos.^2 a - \sin.^2 a = \cos.^2 a - (1-\cos.^2 a) \\ &= 2 \cos.^2 a - 1.\end{aligned}$$

408. $$\frac{\sin. (a-b)}{\sin. a. \sin. b} + \frac{\sin. (b-c)}{\sin. b. \sin. c} + \frac{\sin. (a-c)}{\sin. a. \sin. c}$$

$$= \frac{\sin. c. \sin. (a-b)}{\sin. a. \sin. b. \sin. c} + \frac{\sin. a. \sin. (b-c)}{\sin. a. \sin. b. \sin. c} + \frac{\sin. b. \sin. (a-c)}{\sin. a. \sin. b. \sin. c}$$

But $\sin. c. \sin. (a-b) + \sin. a. \sin. (b-c) + \sin. b. \sin. (a-c) = 0$.

$$\left.\begin{array}{l}\text{For } \sin. c. \sin. a. \cos. b - \cos. a. \sin. b. \sin. c \\ + \sin. a. \sin. b. \cos. c - \cos. b. \sin. c. \sin. a \\ + \sin. b. \sin. a. \cos. c - \cos. c. \sin. a. \sin. b\end{array}\right\} = 0$$

$\therefore \dfrac{\sin.(a-b)}{\sin. a. \sin. b} + \dfrac{\sin.(b-c)}{\sin. b. \sin. c} + \dfrac{\sin.(a-c)}{\sin. a. \sin. c} = 0$, if the common denominator $\sin. a. \sin. b. \sin. c.$ be finite; i. e. if a, b, c be each > 0, and $< \pi$, $> \pi$ and $< 2\pi$, &c. &c.

409. Let x be the arc required.

Then $\cos. x = \tan. x = \dfrac{\sin. x}{\cos. x}$ to radius $= 1$.

$$\therefore \cos.^2 x = \sin x$$

$$\therefore 1 - \sin.^2 x = \sin. x$$

$$\therefore \sin.^2 x + \sin. x = 1$$

and $\sin.^2 x + \sin. x + \dfrac{1}{4} = 1 + \dfrac{1}{4} = \dfrac{5}{4}$

$$\therefore \sin. x = \frac{-1 \pm \sqrt{5}}{2}$$

$\therefore x$ is that arc whose sine $= \dfrac{-1 + \sqrt{5}}{2}$, or that whose sine $= \dfrac{-1 - \sqrt{5}}{2}$, the radius being unity. Reduce these surd values to decimals, and refer to the tables (*Hutton's*), where are to be found the rules necessary for finding the arcs corresponding to the two sines.

410. Let a be the given $\angle$, x one of the parts required, and $\therefore a - x$ the other part.

Then $\dfrac{\sin.(a-x)}{\sin. x} = \dfrac{n}{1}$, $\dfrac{n}{1}$ being the given ratio.

$$\therefore \sin. a. \cos. x - \cos. a. \sin. x = n \sin. x$$

$$\therefore \sin. a. \frac{\cos. x}{\sin. x} = \cos. a + n$$

$$\therefore \cot. x = \frac{\cos. a}{\sin. a} + n = \cot. a + n \operatorname{cosec}. a$$

whence x is known, by reference to the tables.

$$\therefore a - x \text{ is also known.}$$

410. Let the two arcs be a and b

Then $$\frac{\cos. a + \cos. b}{\cos. a \backsim \cos. b} = \frac{2 \cos. \frac{a+b}{2}. \cos. \frac{a-b}{2}}{2 \sin. \frac{a+b}{2}. \sin. \frac{a-b}{2}} = \frac{\cot. \frac{a+b}{2}}{\tan. \frac{a-b}{2}}$$

For $\cos. (A + B) + \cos. (A - B) = 2 \cos. A. \cos. B = 2 \times \cos. \frac{A+B+(A-B)}{2} \times \sin. \frac{A+B-(A-B)}{2}$

and $\cos. (A+B) \backsim \cos. (A-B) = 2 \sin. A. \sin. B. =$ &c.

$\therefore \cos. a + \cos. b : \cos. a \backsim \cos. b :: \cot. \frac{a+b}{2} : \tan. \frac{a-b}{2}.$

411. $\text{Sin.} (60+A) = \sin. 60. \cos. A + \cos. 60. \sin. A.$
$\sin. (60-A) = \sin. 60. \cos. A - \cos. 60. \sin. A.$
$\therefore \sin. (60+A) - \sin. (60-A) = 2 \sin. A. \cos. 60 = \sin. A$
$\therefore \sin. (60+A) = \sin. (60 - A) + \sin. A.$

412. $$\frac{\sin. a + \sin. b}{\cos. a + \cos. b} = \frac{2 \sin. \frac{a+b}{2} \cos. \frac{a-b}{2}}{2 \cos. \frac{a+b}{2} \cos. \frac{a-b}{2}} = \frac{\sin. \frac{a+b}{2}}{\cos. \frac{a+b}{2}}$$
$= \tan. \frac{a+b}{2}$

For $\sin. (A+B) + \sin. (A-B) = 2 \sin. A. \cos. B$
and $\cos. (A+B) + \cos. (A-B) = 2 \cos. A. \cos. B.$
i. e. the sum of the sines of two arcs $= 2 \times$ (the sin. of half their sum) $\times$ (cos. of half their difference), and the sum of the cosines of two arcs $= 2 \times$ (cos. of half their sum) $\times$ (cos. of half their difference).

413. $\text{Tan. } 2A = \tan. (A+A) = \frac{\tan. A + \tan. A}{1 - \tan. A \tan. A} = \frac{2 \tan. A}{1 - \tan.^2 A}$

Let $\tan. 2 A = 2 \tan. A + x$

$\therefore x + 2 \tan. A = \frac{2 \tan. A}{1 - \tan.^2 A}$

$$\therefore x = \frac{2 \tan. A - 2 \tan. A + 2 \tan.^3 A}{1 - \tan.^2 A}$$

$$= \frac{2 \tan.^3 A}{1 - \tan.^2 A}$$ which is positive when tan. A is positive, and less than unity (the radius), or negative and greater than unity; i. e. when A is positive and less than 45°, or when it is negative and greater than 45°.

$\therefore$ tan. 2A is greater than 2 tan. A when A is positive and less than 45°, or when A is negative and greater than 45.

N. B. If an arc lying on one side of the diameter be called positive, the arc adjacent on the other side of the diameter is termed negative.

The limits of tan. 2A with respect to 2 tan. A, may similarly be found, for the other quadrants. Also when tan. 2A is less than 2 tan. A. This we leave to the reader.

414. Let $2 \cos. \theta = u + \frac{1}{u}$

Then, if $2 \cos. (m-1)\theta = u^{m-1} + \frac{1}{u^{m-1}}$

and $2 \cos. m\theta = u^m + \frac{1}{u^m}$ we shall have

$$2 \cos. (m+1)\theta = u^{m+1} + \frac{1}{u^{m+1}}$$

For $\cos. (A+B) + \cos. (A-B) = 2 \cos. A \cos. B.$

Put $A = m\theta$, $B = \theta$

Then $\cos. (m+1)\theta + \cos. (m-1)\theta = 2 \cos. m\theta \,.\, \cos. \theta$

$$\therefore \cos. (m+1)\theta = \left(u^m + \frac{1}{u^m}\right)\left(u + \frac{1}{u}\right) \times \frac{1}{2} - \left(u^{m-1} + \frac{1}{u^{m-1}}\right) \times \frac{1}{2}$$

$$\therefore 2 \cos. (m+1)\theta = u^{m+1} + \frac{1}{u^{m+1}} + u^{m-1} + \frac{1}{u^{m-1}} - u^{m-1} - \frac{1}{u^{m-1}} = u^{m+1} + \frac{1}{u^{m+1}}$$

$\therefore$ if the proposition be true for any two successive values of m (m being integral), it is also true for the next higher value of m.

But $2 \cos. 2\theta = 2(2^2 \cos. \theta - 1) = 4 \cos.^2 \theta - 2$

$$= u^2 + 2 + \frac{1}{u^2} - 2 = u^2 + \frac{1}{u^2}$$

and $2 \cos. \theta = u + \frac{1}{u}$ by supposition.

$\therefore\ 2 \cos. 3\theta = u^3 + \frac{1}{u^3}$

&c. $=$ &c.

$2 \cos. n\theta = u^n + \frac{1}{u^n}$ if n be integral. For, greater information on the subject, see *Woodhouse's* Trigonometry.

Otherwise.

Since $u^2 - 2u \cos. \theta = -1$, solve the equation in u.

$\therefore\ u = \cos. \theta + \sqrt{\cos.^2 \theta - 1} = \cos. \theta + \sqrt{-1} \sin. \theta$

Hence $\frac{1}{u} = \frac{1}{\cos. \theta + \sqrt{-1} \sin. \theta} = \frac{\cos. \theta - \sqrt{-1} \sin. \theta}{\cos.^2 \theta + \sin.^2 \theta}$

$= \cos. \theta - \sqrt{-1} \sin. \theta$

Hence we have $u^n = (\cos. \theta - \sqrt{-1} \sin. \theta)^n = \cos. n\theta + \sqrt{-1} \sin. n\theta$

and $\frac{1}{u^n} = (\cos. \theta - \sqrt{-1} \sin. \theta)^n = \cos. n\theta - \sqrt{-1} \sin. n\theta$

$\therefore\ u^n + \frac{1}{u^n} = 2 \cos. n\theta$

and $u^n - \frac{1}{u^n} = 2\sqrt{-1} \sin. n\theta$, where n may have any value whatever.

415. $\text{Tan.}(45+A) = \frac{\tan. 45 + \tan. A}{1 - \tan. 45 \times \tan. A} = \frac{1 + \tan. A}{1 - \tan. A}$

$\tan.(45-A) = \frac{\tan. 45 - \tan. A}{1 + \tan. 45 \times \tan. A} = \frac{1 - \tan. A}{1 + \tan. A}$

$$\therefore \text{tan.}\,(45+A) - \text{tan.}\,(45-A) = \frac{1+\text{tan.}\,A}{1-\text{tan.}\,A} - \frac{1-\text{tan.}\,A}{1+\text{tan.}\,A}$$

$$= \frac{1+2\,\text{tan.}\,A + \text{tan.}^2 A - 1 + 2\,\text{tan.}\,A - \text{tan.}^2 A}{1-\text{tan.}^2 A}$$

$$= \frac{4\,\text{tan.}\,A}{1-\text{tan.}^2 A}$$

But $2\,\text{tan.}\,2A = 2\,\text{tan.}\,(A+A) = 2 \times \dfrac{\text{tan.}\,A + \text{tan.}\,A}{1-\text{tan.}\,A.\text{tan.}\,A} = \dfrac{4\,\text{tan.}\,A}{1-\text{tan.}^2 A}$

$$\therefore \text{tan.}\,(45+A) = \text{tan.}\,(45-A) + 2\,\text{tan.}\,2\,A.$$

416. Let the arc be A

Then $\text{tan.}\,A + \text{cot.}\,A = \dfrac{\text{sin.}\,A}{\text{cos.}\,A} + \dfrac{\text{cos.}\,A}{\text{sin.}\,A}$

$$= \frac{\text{sin.}^2 A + \text{cos.}^2 A}{\text{sin.}\,A \times \text{cos.}\,A} = \frac{1}{\text{sin.}\,A\,\text{cos.}\,A}$$

$$= \frac{1}{\frac{\text{sin.}\,2A}{2}} = \frac{2}{\text{sin.}\,2A} = 2\,\text{cosec.}\,2\,A$$

since $\text{sin.}\,2A = 2\,\text{sin.}\,A\,\text{cos.}\,A$, and $\text{cosec.}\,2\,A = \dfrac{1}{\text{sin.}\,2\,A}$

417. Let A be the angle

Then $\text{tan.}\,A \backsim \text{cot.}\,A = \text{tan.}\,A \backsim \dfrac{1}{\text{tan.}\,A}$

$$= \frac{\text{tan.}^2 A \backsim 1}{\text{tan.}\,A}$$

$$= \pm \frac{1-\text{tan.}^2 A}{\text{tan.}\,A.}$$, according as tan. A is less or greater than unity (the radius).

But $\text{cot.}\,2\,A = \dfrac{1}{\text{tan.}\,2A} = \dfrac{1}{\frac{2\,\text{tan.}\,A}{1-\text{tan.}^2 A}} = \dfrac{1-\text{tan.}^2 A}{2\,\text{tan.}\,A}$

$$\therefore \text{tan.}\,A \backsim \text{cot.}\,A = \pm\, 2\,\text{cot.}\,2\,A.$$

418. By the above problem, we have
$\text{tan.}\,A = \text{cot.}\,A \pm 2\,\text{cot.}\,2\,A$, according as tan. A is less or greater than the radius, i. e. as A is $<$ or $>$ $45°$

$\therefore \tan. 50^\circ = \cot. 50^\circ - 2 \cot. (100^\circ)$

But $\cot. 100^\circ = \cot. (90 + 10) = \dfrac{1 - \tan. 90 \times \tan. 10}{\tan. 90 + \tan. 10}$

$$= \frac{-\infty \times \tan. 10}{\infty} = -\tan. 10$$

and $\cot. 50 = \tan. 40$

$\therefore \tan. 50 = \tan. 40 + 2 \tan. 10.$

419. Let tan. A, and tan. B, be the given tangents.

Then $\tan. (A \pm B) = \dfrac{\sin. (A \pm B)}{\cos. (A \pm B)}$

$$= \frac{\sin. A. \cos. B \pm \sin. B. \cos. A}{\cos. A. \cos. B \mp \sin. A. \sin. B}$$

Divide by cos. A. cos. B

$$\text{Then } \tan. (A \pm B) = \frac{\dfrac{\sin. A}{\cos. A} \pm \dfrac{\sin. B}{\cos. B}}{1 \mp \dfrac{\sin. A}{\cos. A}.\dfrac{\sin. B}{\cos. B}} = \frac{\tan. A \pm \tan. B}{1 \mp \tan. A \tan. B}$$

420. Let the radius = unity.

Then $\cos. (A+B) = \cos. A. \cos. B - \sin. A. \sin. B$

and $\cos. (A-B) = \cos. A. \cos. B + \sin. A. \sin. B$

$\therefore \cos. (A+B) + \cos. (A-B) = 2 \cos. A. \cos. B$, and to introduce radius r, we must divide each function of the arcs by (r). (see *Woodhouse.*)

$$\therefore \frac{\cos. (A+B)}{r} + \frac{\cos. (A-B)}{r} = \frac{2 \cos. A}{r} \times \frac{\cos. B}{r}$$

$$\therefore \frac{\cos. (A+B)}{2} + \frac{\cos. (A-B)}{2} = \frac{\cos. A \times \cos. B}{r}$$

421. Let A be the arc whose sine is s

Then $\sin. A = s$

$\cos. A = \sqrt{1 - s^2}$

$\text{vers}. A = 1 - \cos. A = 1 - \sqrt{1 - s^2}$

$$\text{sec. A} = \frac{1}{\text{cos. A}} = \frac{1}{\sqrt{1-s^2}}$$

$$\text{tan. A} = \frac{\text{sin. A}}{\text{cos. A}} = \frac{s}{\sqrt{1-s^2}}$$

$$\text{cot. A} = \frac{\text{cos. A}}{\text{sin. A}} = \frac{\sqrt{1-s^2}}{s}$$

422. $\text{Tan. } (A + B + C) = \text{tan. } m\pi = 0$

But $\text{tan. } (A+B+C) = \dfrac{\text{tan. A} + \text{tan. } (B+C)}{1 - \text{tan. A} \times \text{tan. } (B+C)} = 0$

$\therefore \text{tan. A} + \text{tan. } (B + C) = 0$

or $\text{tan. A} + \dfrac{\text{tan. B} + \text{tan. C}}{1 - \text{tan. B} \times \text{tan. C}} = 0$

$\therefore \text{tan. A} + \text{tan. B} + \text{tan. C} - \text{tan. A} \times \text{tan. B} \times \text{tan. C} = 0$

$\therefore \text{tan. A} + \text{tan. B} + \text{tan. C} = \text{tan. A} \times \text{tan. B} \times \text{tan. C}$

423. Let A, B, C be the three parts of the quadrant. Then $\text{tan. A} \times \text{tan. B} + \text{tan. A} \times \text{tan. C} + \text{tan. B} \times \text{tan. C} = r^2$

For $\text{tan. A} \times \text{tan. C} + \text{tan. B} \times \text{tan. C} = (\text{tan. A} + \text{tan. B})\ \text{tan. C} = (\text{tan. A} + \text{tan. B}) \times \text{cot. } (A + B)$, since $A + B + C = 90^\circ$

and $\text{cot. } (A+B) = \dfrac{1 - \text{tan. A} \times \text{tan. B}}{\text{tan. A} + \text{tan. B}}$

$\therefore \text{tan. A} \times \text{tan. B} + \text{tan. A} \times \text{tan. C} + \text{tan. B} \times \text{tan. C} = \text{tan. A} \times \text{tan. B} + 1 - \text{tan. A} \times \text{tan. B} = 1$, to radius $=$ unity.

or $\dfrac{\text{tan. A}}{r} \times \dfrac{\text{tan. B}}{r} + \dfrac{\text{tan. A}}{r} \times \dfrac{\text{tan. C}}{r} + \dfrac{\text{tan. B}}{r} \times \dfrac{\text{tan. C}}{r} = 1$

to radius $= r$

$\therefore \text{tan. A} \times \text{tan. B} + \text{tan. A} \times \text{tan. C} + \text{tan. B} \times \text{tan. C} = r^2$

424. Let $A + B + C = (2n+1)\dfrac{\pi}{2}$, where A, B, C are the parts of the odd multiple $(2n+1)$ of $\dfrac{\pi}{2}$

$$\text{Then cot. } (2n+1)\frac{\pi}{2} = \text{cot. } \left(n\pi + \frac{\pi}{2}\right)$$

$$= \frac{1 - \text{tan. } n\pi \times \text{tan.} \frac{\pi}{2}}{\text{tan. } n\pi + \text{tan. } \frac{\pi}{2}}$$

$$= \frac{1 - 0 \times \infty}{0 + \infty} = \frac{1 - \text{finite quantity}}{\infty} = 0$$

$$\therefore \text{cot. } (\overline{A+B}+C) = \frac{\text{cot. } (A+B) \times \text{cot. } C - 1}{\text{cot. } (A+B) + \text{cot. } C} = 0$$

$$\therefore \text{cot. } (A+B) \times \text{cot. } C = 1$$

$$\text{But cot. } (A+B) = \frac{\text{cot. } A \times \text{cot. } B - 1}{\text{cot. } A + \text{cot. } B}$$

$$\therefore \frac{\text{cot. } A \times \text{cot. } B - 1}{\text{cot. } A + \text{cot. } B} \times \text{cot. } C = 1$$

or $\text{cot. } A \times \text{cot. } B \times \text{cot. } C - \text{cot. } C = \text{cot. } A + \text{cot. } B$

$\therefore \text{cot. } A + \text{cot. } B + \text{cot. } C = \text{cot. } A \times \text{cot. } B \times \text{cot. } C$

425. $(\cos. a + \sqrt{-1} \sin. a) \times (\cos. b + \sqrt{-1} \sin. b)$ $= \cos. a \times \cos. b + \sqrt{-1}. (\sin. a. \cos. b + \cos. a. \sin. b) - \sin. a \times \sin. b = \cos. (a+b) + \sqrt{-1} \sin. (a+b)$

Similarly $(\cos. (a+b) + \sqrt{-1} \sin.. (a+b)) \times (\cos. c + \sqrt{-1} \sin. c) = \cos. (a+b+c) + \sqrt{-1}. \sin. (a+b+c)$, &c. = &c.

$\therefore \cos. (a+b+c+\ldots\ldots) + \sqrt{-1}. \sin. (a+b+c+\ldots\ldots) = (\cos. a + \sqrt{-1} \sin. a).(\cos. b + \sqrt{-1} \sin. b).(\cos. c + \sqrt{-1} \sin. c)$ &c.

or, dividing and multiplying by cos. a, cos. b, &c.

$\frac{\cos. (a+b+c+\ldots.)}{\cos. a. \cos. b. \cos. c.}\left\{1 + \sqrt{-1}\frac{\sin. (a+b+c+\ldots)}{\cos. (a+b+c+.)}\right\} = (1 + \sqrt{-1} \times \tan. a). (1 + \sqrt{-1} \tan. b) \times (1 + \sqrt{-1} \tan. c) \times$ &c.

$= 1 + \sqrt{-1}. A + \sqrt{-1} \times \sqrt{-1}. B + \sqrt{-1} \times \sqrt{-1} \times \sqrt{-1} C + \ldots.$ by the theory of equations.

But $\sqrt{-1}. A = \sqrt{-1} A$

$\sqrt{-1} \times \sqrt{-1} B = -B$

$$\sqrt{-1} \times \sqrt{-1} \times \sqrt{-1}\, C = -\sqrt{-1}\, C$$
$$\sqrt{-1} \times \sqrt{-1} \times \sqrt{-1} \times \sqrt{-1}\, D = D$$
$$\&c. \quad = \quad \&c.$$

$$\therefore \frac{\cos.(a+b+c+\ldots)}{\cos. a. \cos. b. \cos. c..} + \sqrt{-1} \tan.(a+b+c+\ldots.) \times$$
$$\frac{\cos.(a+b+c+\ldots.)}{\cos. a. \cos. b. \cos. c\ldots} = 1 - B + D - F + \ldots.. + \sqrt{-1} \times (A - C + E - \ldots.)$$

$\therefore$ equating the real and imaginary quantities, we have
$$\frac{\cos.(a+b+c+\ldots.)}{\cos. a \cos. b \cos. c\ldots.} = 1 - B + D - F + \ldots.$$
and $\tan.(a+b+c+\ldots) \times (1-B+D-F\ldots) = A-C+E-G+\ldots$
$$\therefore \tan.(a+b+c+\ldots) = \frac{A-C+E-G+\ldots.}{1-B+D-F+\ldots.}$$

Hence, also we have $\cos.(a+b+c+\ldots.) = \cos. a. \cos. b. \cos. c\ldots \{1-B+D-F\ldots..\}$

and $\sin.(a+b+c+\ldots)$ (= tan. × cos.) $= \cos. a. \cos. b. \cos. c.. \{A-C+E-G+\ldots\}$

Many curious, and perhaps, useful propositions may hence be established.

Required to express $\tan.(a+b+c+\ldots)$ in terms of $\tan.(2a)$, $\tan.(2b)$, &c., or, generally, in terms of $\tan.(ma)$ $\tan.(mb)$, $\tan.(mc)$, &c.

Let ${}^1S_1$ = sum of tangents of a, b, c, &c.

${}^2S_1$ = sum of tangents of $2a, 2b, 2c$, &c.

&c. = &c.

${}^1S_2$ = sum of products of every two of tangents of a, b, c, &c. &c., and generally, let ${}^mS_n$ express the sum of the products of every n of the quantities $\tan.(ma)$, $\tan.(mb)$, $\tan.(mc)$, &c.

Then by the above form we have

$$\tan.(m\phi) = \frac{m \tan.\phi - m.\frac{m-1}{2}.\frac{m-2}{3}.\tan.^3\phi + \&c.}{1 - m.\frac{m-1}{2}\tan.^2\phi + m.\frac{m-1}{2}.\frac{m-2}{3}.\frac{m-3}{4}.\tan.^4\phi - \&c.}$$

Also tan. $(ma + mb + mc + ...) = \dfrac{{}^m S_1 - {}^m S_3 + {}^m S_5 - .....}{1 - {}^m S_2 + {}^m S_4 - .....}$

Put $a + b + c + ... = \phi$

Then tan. $m\phi = \dfrac{{}^m S_1 - {}^m S_3 + {}^m S_5 - ....}{1 - {}^m S_2 + {}^m S_4 - ....}$

$$= \frac{m \tan. \phi - m. \dfrac{m-1}{2}. \dfrac{m-2}{3} \tan.^3 \phi + \&c.}{1 - m. \dfrac{m-1}{2} \tan.^2 \phi + \&c.}$$

$\therefore$ we have an equation involving tan. ϕ or tan. $(a+b+c+....)$ and tan. (ma), tan. (mb), tan. (mc), &c., and consequently by solving that equation, or finding the values of tan. ϕ, we can express tan. ϕ in terms of tan. (ma), tan. (mb), &c.

Let $m = 2$

Then tan. $2\phi = \dfrac{{}^2 S_1 - {}^2 S_3 + {}^2 S_5 - ..}{1 - {}^2 S_2 + {}^2 S_4 - ..} = \dfrac{2 \tan. \phi}{1 - \tan.^2 \phi}$

Let $\therefore \dfrac{{}^2 S_1 - {}^2 S_3 + {}^2 S_5 - ..}{1 - {}^2 S_2 + {}^2 S_4 - ..} = Q$

$\therefore Q - Q \times \tan.^2 \phi = 2 \tan. \phi$

$\therefore \tan.^2 \phi + \dfrac{2}{Q} \tan. \phi = 1$

$\therefore \tan. \phi = \dfrac{-1 \pm \sqrt{Q^2 + 1}}{Q}$

If $m = 3$, a cubic equation must be solved, &c. &c.

426. This is Demoivre's Theorem, which is deducible from one more general, *viz.* cos. $(A \pm B \pm C \pm ...) \pm \sqrt{-1} \times$ sin. $(A \pm B \pm C \pm ...) = (\cos. A \pm \sqrt{-1} \sin. A). (\cos. B \pm \sqrt{-1} \sin. B) \times (\cos. C \pm \sqrt{-1} \sin. C) \times$ &c., to prove which we proceed as follows:

Put $\cos. A + \sqrt{-1} \sin. A = a$, $\cos. A - \sqrt{-1} \sin. A = a'$

$\cos. B + \sqrt{-1} \sin. B = b$, $\cos. B - \sqrt{-1} \sin. B = b'$

&c. = &c. &c. = &c.

$a \times b = \cos. A. \cos. B - \sin. A. \sin. B + \sqrt{-1} (\sin. A. \times \cos. B + \cos. A. \sin. B)$

$= \cos.\ (A + B) + \sqrt{-1}\ \sin.\ (A + B)$

Similarly $a \times b \times c = \cos.\ (A + B + C) + \sqrt{-1}\ \sin.\ (A + B + C)$

&c. $=$ &c.

(1) and $a \times b \times c \times d \times \&c. = \cos.\ (A + B + C + D + ...) + \sqrt{-1}\ \sin.\ (A + B + C + D + ...)$

Again $a' \times b' = \cos.\ A.\ \cos.\ B - \sin.\ A.\ \sin.\ B - \sqrt{-1}\ (\sin.\ A.\ \cos.\ B + \cos.\ A.\ \sin.\ B.)$

$= \cos.\ (A + B) - \sqrt{-1}\ \sin.\ (A + B)$

Similarly $a' \times b' \times c' = \cos.\ (A + B + C) - \sqrt{-1}\ \sin.\ (A + B + C)$

&c. $=$ &c.

(2) and $a' \times b' \times c' \times d' \times \&c. = \cos.\ (A + B + C + \&c.) - \sqrt{-1}\ \sin.\ (A + B + C + \&c.)$

Again $a \times b' = \cos.\ A\ \cos.\ B + \sin.\ A\ \sin.\ B + \sqrt{-1}\ (\sin.\ A\ \cos.\ B - \cos.\ A.\ \sin.\ B)$

(3) $= \cos.\ (A - B) + \sqrt{-1}\ \sin.\ (A - B)$

or $= \cos.\ (B - A) - \sqrt{-1}\ \sin.\ (B - A)$

From (1) it appears that when all the factors are positive, the result is of the same form with any one of the factors; the angle in both of its terms being the sum of the angles in the factors, and both terms positive.

From (2) we learn, that when each of the factors is negative in the second term, the result is also negative in the second term, and the angle in each of its terms is the sum of the angles in the factors.

But from (3) we find that when one factor is positive in the second term, and the other negative, the result of these *two* factors is positive or negative in the second term, according as the first factor (arranging the angles in the order of the factors) is positive or negative in its second term; and the $\angle$ in this result $=$ the difference of the angles of the factors.

Hence then it is manifest that, generally, $\cos.\ (A \pm B \pm C \pm ....) \pm \sqrt{-1}\ \sin.\ (A \pm B \pm C \pm ....) = (\cos.\ A \pm \sqrt{-1}\ \sin.\ A)$

$\times$ (cos. B $\pm \sqrt{-1}$ sin. B) $\times$ (cos. C $\pm \sqrt{-1}$ sin. C) $\times$ &c., where the signs are taken according to the above-mentioned circumstances.

The above very general theorem is more so than it appears to be, inasmuch as it comprehends the form cos. $(A \pm B \pm$ &c. $\pm A' \pm B' \pm$ &c.$) \pm \sqrt{-1}$ sin. $(A \pm B \pm$ &c. $\pm A' \pm B'$ &c.$)$

$$= \frac{(\text{cos. A} \pm \sqrt{-1}\ \text{sin. A}) \times (\text{cos. B} \pm \sqrt{-1}\ \text{sin. B}) \times \&\text{c.}}{(\text{cos. A}' \pm \sqrt{-1}\ \text{sin. A}') \times (\text{cos. B}' \pm \sqrt{-1}\ \text{sin. B}') \times \&\text{c.}};$$ for

$$\frac{1}{\text{cos. A}' \pm \sqrt{-1}\ \text{sin. A}'} = \frac{\text{cos. A}' \mp \sqrt{-1}\ \text{sin. A}'}{\text{cos.}^2\text{A}' + \text{sin.}^2\text{A}'} = \text{cos. A}'$$

$\mp \sqrt{-1}$ sin. A′

Now, let $A = \pm B = \pm C = \pm D$ &c. to m terms

Then cos. $(mA) \pm \sqrt{-1}$ sin. $(mA) =$ (cos. A $\pm \sqrt{-1}$ sin. A) $\times$ (cos. A $\pm \sqrt{-1}$ sin. A) $\times$ &c. to m terms $=$ (cos. A $\pm \sqrt{-1}$ sin. A)^m, the signs being taken + and +, − and − only, which is the solution required when m is integral.

The case when m is any rational fractional of the form $\frac{p}{q}$, may be proved as follows:

Let $A = \frac{q}{p} \times \theta$

then $pA = q\theta$

and (cos. A $\pm \sqrt{-1}$ sin. A)$^p =$ cos. $pA \pm \sqrt{-1}$ sin. $pA =$ cos. $q\theta$ $\pm \sqrt{-1}$ sin. $q\theta =$ (cos. $\theta \pm \sqrt{-1}$ sin. θ)^q

$\therefore$ (cos. A $\pm \sqrt{-1}$ sin. A)$^{\frac{p}{q}} =$ cos. $\theta \pm \sqrt{-1}$ sin. θ

$$= \text{cos.}\ \frac{p}{q}\text{A} \pm \sqrt{-1}\ \text{sin.}\ \frac{p}{q}\text{A}$$

Let now m be irreducible, and of the form $s + r^{\frac{m}{n}}$. Then by a similar train of reasoning we shall prove the truth of this case, and also that of any other which may present itself. This we leave to the student.

Otherwise.

By expanding sin. θ and cos. θ and $e^{\pm\sqrt{-1}\theta}$ it will be seen that cos. $\theta \pm \sqrt{-1}$ sin. $\theta = e^{\pm\theta\sqrt{-1}}$ (e being the hyperbolic base.)

$\therefore$ cos. $(A \pm B \pm C \pm \ldots\ldots) \pm \sqrt{-1}\cdot$sin. $(A \pm B \pm C \pm \ldots.)$

$= e^{\pm(A \pm B \pm C \pm \ldots\ldots)\sqrt{-1}} = e^{\pm A\sqrt{-1}} \times e^{\pm B\sqrt{-1}} \times$ &c.

$=$ (cos. $A \pm \sqrt{-1}$ sin. A) $\times$ (cos. $B \pm \sqrt{-1}$ sin. B) $\times$ &c. as before.

This method proves the problem in one step, thus,

cos. $m\,A \pm \sqrt{-1}$ sin. $m\,A = e^{\pm mA\sqrt{-1}} = \left(e^{\pm A\sqrt{-1}}\right)^m$

$=$ (cos. $A \pm \sqrt{-1}$ sin. $A)^m$ whatever may be the value of m.

It is not, however, founded on such obvious principles as the preceding method.

427. The sine of an arc is the perpendicular let fall from one extremity of the arc, upon the diameter passing through the other.

By the perpendiculars let fall from each extremity of the arc upon the diameters passing through the other extremity, two right-angled triangles will be formed, having one angle at the centre common to the triangles, and their hypothenuses being radii of the circle, will be equal. $\therefore$ the sides opposite equal $\angle$ are equal; or the sines are equal.

428. Let the sin. $A = a$ be given.

Then cos. $2\,A =$ cos. $(A + A) =$ cos. A. cos. $A -$ sin. A. sin. A;

$=$ cos. $^2A -$ sin. ^2A

$= 1 -$ sin. $^2A -$ sin. ^2A

$= 1 - 2a^2$, the radius being supposed equal to unity.

429. The chord of an arc $= 2$ sin. of half that arc.

$\therefore$ chord of $120 = 2$ sin. $60 = \dfrac{\text{sin. } 60}{(\frac{1}{2})}$

$= \dfrac{\text{sin. } 60}{\text{cos. } 60} =$ tan. 60

430. The cosines will be sides opposite equal angles in similar right-angled triangles, formed by the radii r, R, cosines c, C, and sines s, S.

But the versed sines v, V, are equal to the differences between the radii and cosines.

and $r : R :: c : C$ from similar triangles

$\therefore r-c : R-C :: r : R$

or $v : V :: r : R$.

431. Since the sine lies on the same side of the diameter as the arc, with respect to that diameter, it must have the same sign as the arc.

$\therefore$ sin. $(-A) = -$ sin. A.

The cosine is identically the same for $(-A)$ as for $+A$.

$\therefore$ cos. $(-A) =$ cos. A

The secant is referred to the centre and not to the diameter, and therefore suffers no change of sign with regard to that of the arc.

$\therefore$ sec. $(-A) =$ sec. A

Otherwise,

$$\text{sin. } (-A) = \text{sin. } (0-A) = \text{sin. } 0 \times \text{cos. } A - \text{cos. } 0 \times \text{sin. } A$$
$$= -1 \times \text{sin. } A = - \text{sin. } A$$

&c. &c. &c.

Otherwise,

A variable quantity cannot change sign without passing through zero, or infinity, and *vice versâ*.

Hence, the sine passing through zero, when A changes its sign, changes its sign also.

The cosine does not pass through zero during such change, and $\therefore$ does not change its sign.

The secant never = zero; it $= \infty$ when A = 90, and afterwards changes sign, &c. &c.

432. The $\cot.(A \pm B) = \dfrac{\cos.(A \pm B)}{\sin.(A \pm B)}$

$$= \frac{\cos. A. \cos. B \mp \sin. A. \sin. B}{\sin. A. \cos. B \pm \cos. A \sin. B}$$

Divide both denominator and numerator by sin. A. sin. B.

and we get $\cot.(A \pm B) = \dfrac{\cot. A. \cot. B \mp 1}{\cot. B \pm \cot. A}$

$$\therefore \frac{\cot.(A \pm B)}{r} = \frac{\frac{\cot. A}{r} \times \frac{\cot. B}{r} \mp 1}{\frac{\cot. B}{r} \pm \frac{\cot. A}{r}} = \frac{\cot. A. \cot. B \mp r^2}{r(\cot. B \pm \cot. A)}$$

$$\therefore \cot.(A \pm B) = \frac{\cot. A. \cot. B \mp r^2}{\cot. B \pm \cot. A}$$

The rule for introducing the radius (r) is obvious.

It is, multiply every term by that power of r whose index = difference between the highest number of dimensions in any one term, and that of the term itself.

433. $\text{Tan.}^3 60 = 3 \tan. 60$

For $\tan. 60 = \dfrac{\sin. 60}{\cos. 60} = \dfrac{\sqrt{\frac{3}{4}}}{\frac{1}{2}} = \sqrt{3}$

$\therefore \tan.^3 60 = (\sqrt{3})^3 = \sqrt{27} = 3\sqrt{3} = 3 \tan. 60.$

434. Let $a + mb$ be the mean arc,

$a + (m - p)b$, $a + (m + p)b$ be the extremes

a being first term, and b the common difference

Then $\cos.(a + \overline{m - p}.b) = \cos.(\overline{a + mb} - pb)$

$$= \cos.(a + mb).\cos. pb + \sin.(a + mb).\sin. pb$$

and $\cos.(a + \overline{m + p}.b) = \cos.(a + mb).\cos. pb - \sin.(a + mb).\sin. pb$

$\therefore \cos.(a + \overline{m - p}.b) + \cos.(a + \overline{m + p}.b) = 2 \cos. pb. \cos.(a + mb)$

$\therefore \cos.(a + \overline{m - p}.b) + \cos.(a + \overline{m + p}.b) = \dfrac{2}{r}.\cos. pb. \cos.(a + mb)$

$\therefore r : 2\cos.(pb) :: \cos.(a + mb) : \cos.(a + \overline{m - p}.b) + \cos.(a + \overline{m + p}.b)$

a result which indicates an error in the enunciation of the problem.

435. $$\tan.\left(45 - \frac{z}{2}\right) = \frac{1-\tan.\frac{z}{2}}{1+\tan.\frac{z}{2}}$$

and $$\tan. z + \sec. z = \frac{\sin. z}{\cos. z} + \frac{1}{\cos. z} = \frac{\sin. z+1}{\cos. z}$$

$$= \frac{2\sin.\frac{z}{2}.\cos.\frac{z}{2} + \cos.^2\frac{z}{2} + \sin.^2\frac{z}{2}}{\cos.^2\frac{z}{2} - \sin.^2\frac{z}{2}} = \frac{\left(\cos.\frac{z}{2}+\sin.\frac{z}{2}\right)^2}{\cos.^2\frac{z}{2} - \sin.^2\frac{z}{2}}$$

$$= \frac{\cos.\frac{z}{2}+\sin.\frac{z}{2}}{\cos.\frac{z}{2} - \sin.\frac{z}{2}} = \frac{1+\tan.\frac{z}{2}}{1-\tan.\frac{z}{2}} = \frac{1}{\tan.^3 u}$$

Again, $$\tan. z - \sec. z = \frac{\sin. z - 1}{\cos. z} = \frac{2\sin.\frac{z}{2}.\cos.\frac{z}{2} - \left(\cos.^2\frac{z}{2} + \sin.^2\frac{z}{2}\right)}{\cos.^2\frac{z}{2} - \sin.^2\frac{z}{2}}$$

$$= - \frac{-\left(\cos.\frac{z}{2} - \sin.\frac{z}{2}\right)^2}{\cos.^2\frac{z}{2} - \sin.^2\frac{z}{2}} = \frac{-\left(\cos.\frac{z}{2} - \sin.\frac{z}{2}\right)}{\cos.\frac{z}{2} + \sin.\frac{z}{2}}$$

$$= - \frac{1-\tan.\frac{z}{2}}{1+\tan.\frac{z}{2}} = -\tan.^3 u.$$

$$\therefore \sqrt[3]{\tan. z + \sec. z} + \sqrt[3]{\tan. z - \sec. z} = \frac{1}{\tan. u} - \tan. u$$

$$= \frac{1-\tan.^2 u}{\tan. u}.$$

$$= 2 \times \frac{1}{\tan 2u}$$

$$= 2 \times \cot. 2u.$$

436. The terms of the equation being expanded, we have $\sin. B + \sin. A.\cos. B - \cos. A.\sin. B + \sin. 2A.\cos. B + \cos. 2A \times \sin. B = \sin. A.\cos. B + \cos. A.\sin. B + \sin. 2A.\cos. B - \cos. 2A \times \sin. B.$

$\therefore$ sin. B $-$ 2 cos. A. sin. B $+$ 2 cos. 2 A. sin. B $= 0$

$\therefore$ 2 cos. 2 A $-$ 2 cos. A $= -1$

$\therefore$ 2 cos. ^{2}A $- 1 -$ cos. A $= -\frac{1}{2}$

$\therefore$ cos. ^{2}A $- \frac{1}{2}$ cos. A $= \frac{1}{4}$

$\therefore$ cos. A $= \frac{1 \pm \sqrt{5}}{4}$, whence, and by reference to the tables, the numerical values of A may easily be obtained, one of which is 72°.

437. Tan. 30 $=$ tan. $\frac{60}{2}$

But tan. 60 $= \frac{2 \text{ tan. } 30}{1 - \text{tan.}^2 30}$ to radius $=$ unity

and tan. 60 $= \frac{\text{sin. } 60}{\text{cos. } 60} = \frac{\sqrt{\frac{3}{4}}}{\frac{1}{2}} = \sqrt{3}$

$\therefore \sqrt{3} - \sqrt{3}$ tan.^{2} 30 $=$ 2 tan. 30

$\therefore$ tan.^{2} 30 $+ \frac{2}{\sqrt{3}}$ tan. 30 $= 1$

$\therefore$ tan. 30 $= -\frac{1}{\sqrt{3}} \pm \sqrt{\frac{1}{3} + 1} = \frac{-1 \pm 2}{\sqrt{3}}$

$= \frac{1}{\sqrt{3}}$ or $- \frac{3}{\sqrt{3}}$ ($= -\sqrt{3}$) to radius $= 1$

$\therefore$ tan. 30 $= \frac{10000}{\sqrt{3}}$ or $-$ 10000 $\sqrt{3}$ to radius 10000

$= \frac{10000}{1.7320508}$ or $= -$ 10000 $\times$ (1.7320508)

$=$ 5773.50.... or $= -$ 17320.508....

438. Sin. (A $-$ B) $= \frac{r}{2} =$ sin. 30° or $=$ sin. ($\pi - 30$) or $=$ sin. ($2\pi + 30$), &c.

and generally sin. (A $-$ B) $=$ sin. ($2m\pi + 30$) or $=$ sin. ($\overline{2m+1}\,\pi - 30$) where m is any number whatever.

$\therefore A-B = 2m\pi + 30$, or $= \overline{2m+1}.\pi - 30$

Also cos. $(A+B) = \frac{r}{2} =$ cos. 60, or $=$ cos. $(2\pi - 60)$, &c.

and generally cos. $(A+B) =$ cos. $(2m\pi - 60)$

$\therefore A + B = 2n\pi - 60$

and $A - B = 2m\pi + 30$ or $\overline{2m+1}.\pi - 30$

$\left.\begin{array}{l}\therefore A = (n+m)\pi - 15, \text{ or } = (n+m)\pi + 45 \\ \text{and } B = (n-m)\pi - 45, \text{ or } = (n-m)\pi - 75\end{array}\right\}$ which have innumerable values, since n and m may be any positive integral numbers whatever.

Let $n = 0$ and $m = 0$

Then $\left.\begin{array}{l}A = -15^\circ, \text{ or } 45^\circ \\ B = -45^\circ, \text{ or } -75^\circ\end{array}\right\}$ which are particular values of A and B.

439. Let it be required to find the sine of any arc between 45° and 90°, as $45 + a$, where a is less than 45.

Then sin. $(45 + a) =$ sin. $(90 - \overline{45 - a}) =$ cos. $(45 - a)$ which is known by the table.

Again cos. $(45+a) =$ cos. $(90-45+a) =$ sin. $(45-a)$ which is also known by the table.

$\therefore$ the table exhibits the sines and cosines of every arc in the first quadrant.

Suppose now, the arc to be between 90 and 180.

Then sin. $(90 + 45 + a) =$ sin. $(180 - \overline{45 - a}) =$ sin. $(45-a) =$ which is known by the table.

and cos. $(90 + 45 + a) =$ cos. $(180 - \overline{45 - a}) = -$ cos. $(45 - a)$ which is also known by the table.

Again, let the arc be between 180 and 270.

$\left.\begin{array}{l}\text{Then, sin. } (180 + 45+a) = - \text{ sin. } (45+a) = - \text{ cos.}(45-a) \\ \text{and cos. } (180+45+a) = - \text{ cos. } (45+a) = - \text{ sin.}(45-a)\end{array}\right\}$ which are known by the table.

Again, let the arc be between 270 and 360.

$$\left.\begin{array}{l}\text{Then sin. }(270+45+a) = \sin.(360-\overline{45-a}) = -\sin.(45-a)\\ \text{and cos. }(360-\overline{45-a}) = \quad . \quad . \quad . \quad . \quad = \quad \cos.(45-a)\end{array}\right\}$$

which are known by the table.

$\therefore$ the table exhibits the sines and cosines of any arcs whatever.

440. $$\text{Tan. } 30^\circ = \frac{\sin.\ 30^\circ}{\cos.\ 30^\circ} = \frac{\frac{1}{2}}{\pm\sqrt{\frac{3}{4}}} = \frac{1}{\pm\sqrt{3}}$$

$$\text{But tan. } 30 = \tan.(2\times 15) = \frac{2\tan.\ 15}{1-\tan.^2 15} = \frac{1}{\pm\sqrt{3}}$$

$$\therefore \sqrt{3}.\ \tan.\ 15 = 1 - \tan^2 15$$

$$\therefore \tan.^2 15 \pm 2\sqrt{3}\tan.\ 15 + 3 = 1 + 3 = 4$$

$$\therefore \tan.\ 15 = \pm 2 \mp \sqrt{3}, \text{ to radius = unity.}$$

441. $$\text{Sin. }(a+b) = \sin.\ a.\ \cos.\ b + \cos.\ a.\ \sin.\ b$$
$$\sin.(a-b) = \sin.\ a.\ \cos.\ b - \cos.\ a.\ \sin.\ b$$
$$\therefore \sin.(a+b) = 2\sin.\ a.\ \cos.\ b - \sin.(a-b)$$

Let $b = 1'$ and let $\sin.\ 1' = m$, and $\cos.\ 1' = \sqrt{1-m^2} = n$

Then $\sin.(a+1') = 2\sin.\ a \times n - \sin.(a-1')$, in which formula, if for (a) we substitute $1'$, $2'$, $3'$, &c. successively, we shall determine the sines of $2'$, $3'$, &c. in terms of the given quantities m and n.

Thus $\sin.\ 2' = 2nm$

$$\sin.\ 3' = 2n\sin.\ 2' - \sin.\ 1' = 4n^2m - m$$
$$= m.(4-4m^2-1) = m.(3-4m^2)$$
$$\sin.\ 4' = 2nm.(3-4m^2) - 2nm = 4nm.(1-2m^2)$$

&c. = &c. See *Woodhouse's* Trigonometry.

442. Let a be the first term, b, the common difference and $a+mb$ the mean term.

$$\left.\begin{array}{l} a+(m-p)\ b\\ a+(m+p).\ b\end{array}\right\}\text{the equidistant extremes.}$$

Then $\sin.(a+\overline{m+p}.b) = \sin.(a+mb).\cos.\ pb + \cos.(a+mb).\times \sin.\ pb$

$\sin.(a + \overline{m - p}.b) = \sin.(a + mb).\cos.pb - \cos.(a + mb)$ $\sin.pb$

$$\therefore \frac{\sin.(a+\overline{m+p}.b)+\sin.(a+\overline{m-p}.b)}{r} = \frac{2\sin.(a+mb).}{r} \times \frac{\cos.pb}{r}$$

$\therefore r : 2\cos.pb :: \sin.(a + mb) : \sin.(a + \overline{m + p}.b) +$ $\sin.(a + \overline{m - p}.b)$ a result which shews an error in the enunciation.

To apply this proposition as required, let $r = 1$

Then $\sin.(a + \overline{m + p}.b) + \sin.(a + \overline{m - p}.b) = 2\cos.p.b \times$ $\sin.(a + mb)$

$\therefore \sin.(a + mb + pb) = 2\cos.pb \times \sin.(a + mb) -$ $\sin.(a + mb - pb)$

Hence, having given sin. pb, or cos. pb, sin. $(a + mb)$ and sin. $(a + mb - pb)$, we can always obtain sin. $(a + mb + pb)$; from which we get cos. $(a + mb + pb)$, and $\therefore$ sec. $(a+mb+pb)$, tan. $(a + mb + pb)$, cot. $(a+mb + pb)$, &c. and $\therefore$ by substituting different numbers in succession for $a + mb + pb$, we obtain the sines, cosines, &c. of every arc, which, properly arranged, will form the table.

Thus, let $a = 1', 2', 3', 4'$, &c. and $b = 1'$

$m + p$ being also $1', 2', 3'$, &c.

Then, by *Woodhouse* having found the value of sin. $1' = m$, and thence cos $1' = n$, by the form, we have

$\sin.2' = 2\cos.1' \times \sin.1' - 0 = 2mn$

$\sin.3' = 2n \times 2mn - \sin.1' = 4mn^2 - m$

&c. = &c.

Hence, we have the sines of all arcs, and $\therefore$ the cosines, tangents, &c. Different methods, however, should in particular cases be resorted to, for which see *Woodhouse's* Trigonometry.

443. Let A, B, C represent the angles of a Δ, and a, b, c their opposite sides, and let there be given A, a and $b + c$ to find b and c.

$b : c :: \sin.B : \sin.C$

$\therefore b : b+c :: \sin.B : \sin.B + \sin.C$

$\therefore$ $\sin. B + \sin. C = (b + c). \frac{\sin. B}{b} = (b + c)\frac{\sin. A}{a}$ and is $\therefore$ known.

But $\sin. B + \sin. C = 2 \sin. \frac{B + C}{2}. \cos. \frac{B - C}{2}$

and $\frac{B + C}{2}$ being $= \frac{\pi - A}{2}$, $\sin. \frac{B + C}{2} = \cos. \frac{A}{2}$

$\therefore \cos. \frac{B - C}{2} = \frac{b + c}{a}. \frac{\sin. A}{2 \cos. \frac{A}{2}} = \frac{b + c}{a} \sin. \frac{A}{2}$, and is $\therefore$ known.

$\therefore$ $\frac{B - C}{2}$ is known and $= m$.

Also $\frac{B + C}{2} \ldots\ldots\ldots = \frac{\pi - A}{2} = n$

$\therefore$ $B = n + m$ and $C = n - m$ are known

$$\left.\begin{array}{l} \therefore\ b = \dfrac{a}{\sin. A} \times \sin. B \\ \text{and } c = \dfrac{a}{\sin. A} \times \sin. C \end{array}\right\} \text{are known.}$$

444. $\cos. B = \frac{1}{2}. \frac{\sin. A}{\sin. C} = \frac{1}{2}. \frac{a}{c}$

But $\cos. B$ also $= \frac{a^2 + c^2 - b^2}{2ac}$ (*Woodhouse.*)

$$\therefore \frac{a}{2c} = \frac{a^2 + c^2 - b^2}{2ac}$$

$$\therefore a^2 = a^2 + c^2 - b^2$$

$$\therefore c^2 = b^2$$

and $c = b$, or the triangle is isosceles.

445. Variable quantities in changing sign must become zero or infinite.

Hence the sine being positive from zero to 180, where from a finite quantity it becomes zero, must afterwards change sign, or become negative; it becomes 0 again for the arc $= 360^\circ$, after which it again changes sign, &c. &c.

The cosine changes sign after the first quadrant, because it has then passed through zero. For the same reason it again changes sign after the third quadrant. The tangent changes sign after the first quadrant, because it has then passed through infinity. After the second quadrant it again changes sign, because it has then passed through zero. It becomes infinite also at the end of the third quadrant, and $\therefore$ again changes sign, &c. $\therefore$ the tangent changes its sign in every quadrant, taking them in order.

The secant, never passing through zero, and only becoming infinite at the end of the first and third quadrants, has changes of signs through those points only.

Since the secants, in the first and third quadrants, have different signs, the change having taken place in passing from the first quadrant to the second; $\therefore$ sec. A differs in sign from sec. $(180 + A)$ supposing A less than 90. If, however, A be $>$ 90 and $=$ $90 + A'$.

Then sec. $A = \text{sec.}\ (90 + A') = -\ \text{sec.}\ A'$.

sec. $(290 + A') = +\ \text{sec.}\ A'$, which have different signs, both having changed signs. A similar mode of proof will explain the other cases.

A more simple manner of proving this problem, would have been to have considered the functions of the arcs as positive or negative with regard to their position, with respect to the diameter of the circle, or its centre.

446. $\text{Cos.}\ C = \dfrac{a^2 + b^2 - c^2}{2ab}$ (*Woodhouse.*)

$\therefore 2ab.\ \cos.\ C = a^2 + b^2 - c^2$

$\therefore c^2 = a^2 - 2ab.\ \cos.\ C + b^2$

and $c = \pm \sqrt{a^2 - 2ab.\ \cos.\ C + b^2}$

and $\sin.\ A : a :: \sin.\ C : c$

$$\therefore \sin.\ A = \frac{a.\ \sin.\ C}{c} = \frac{a . \sin C}{\pm \sqrt{a^2 - 2ab \cos.\ C + b^2}}$$

447. (Fig. 52.) Let B C D be the horizontal plane M C the mountain, M being its summit.

Take any station A, and with the quadrant measure the $\angle$ M A m, which put $= \theta$, retreat to another station B, in such a direction, that the plane M B A be perpendicular to the horizon, and let the distance between A and B be measured, which put $= a$. Also measure the $\angle$ M B C $= \theta'$

Then, if M m be perpendicular to the horizon, it lies in the same plane with M A, M B, B A, &c.

and we have $\text{M}m = \text{A}m \times \tan.\theta = \text{B}m \times \tan.\theta' = a\tan.\theta' + \text{A}m.\tan.\theta'$.

$$\therefore \text{A}m = \frac{a.\tan.\theta'}{\tan.\theta - \tan.\theta'},$$

$$\therefore \text{M}m = \frac{a \times \tan.\theta \times \tan.\theta'}{\tan.\theta - \tan.\theta} = \frac{a}{\cot.\theta' - \cot.\theta}$$

This method would be found in practice to lead to results very inaccurate, on account of the errors arising from the refraction of light, &c. &c. That these errors may be as small as possible, the stations should be taken such that θ and θ' may be nearly of the form $45° + \phi$, $45° - \phi$.

If the summit of the mountain were accessible, and of considerable altitude, that altitude may be found by means of the Barometer.

448. (Fig. 58.) Let O o be the obelisk standing on the declivity o C (which is supposed perfectly gradual,) and θ, θ' be the observed $\angle$ of elevation, at the corresponding distances A o, B o. Call A$o = a$ and B A $= a'$.

Now A O : A B :: sin. θ' : sin. B A O

or A O : a' :: sin. θ' : sin $(\theta - \theta')$

$$\therefore \text{AO} = \frac{a' \sin.\theta'}{\sin.(\theta - \theta')} = b$$

$$\frac{b}{a} = \frac{\sin.}{\sin.x}$$

$$\therefore \frac{b}{a} - 1 = \frac{\sin.y}{\sin.x} - 1 \quad \therefore \frac{b-a}{a} = \frac{\sin.y - \sin.x}{\sin.x}$$

and $\frac{b}{a}+1=\frac{\sin. y}{\sin. x}+1 \quad \therefore \frac{b+a}{a}=\frac{\sin. y+\sin. x}{\sin. x}$

$$\therefore \frac{b+a}{b-a}=\frac{\sin. y+\sin. x}{\sin. y-\sin. x}=\frac{2\sin. \frac{x+y}{2}. \cos. \frac{y-x}{2}}{2\sin. \frac{y-x}{2}. \cos. \frac{y+x}{2}}$$

$$=\frac{\tan. \frac{y+x}{2}}{\tan. \frac{y-x}{2}}$$

$$\therefore \tan. \frac{y-x}{2}=\frac{b-a}{b+a}\times \tan. \frac{y+x}{2}=\frac{b-a}{b+a} \cot. \frac{\theta}{2}$$

$\therefore \frac{y-x}{2}$ is known from the tables
and $\frac{y+x}{2}$ is known $=\frac{\pi}{2}-\frac{\theta}{2}$
} whence we have y and x.

Hence $O o = a \times \frac{\sin. \theta}{\sin. x}$ is known.

Otherwise.

Suppose $O o$ produced to meet the perpendicular $A N$. Then, with the quadrant, measure the $\angle\, o A N = \varphi$ from the station A Also measure the $\angle\, O A N = \theta + \varphi = \varphi'$ by supposition.

Then $\because A o = a$, $o N = a. \sin. \varphi$ is known, and $A N = a. \cos. \varphi$ is;

and $\therefore O N = A N \times \tan. \varphi' = a. \cos. \varphi. \tan. \varphi'$ is known.

$\therefore O o = O N - O N = a(\cos. \varphi. \tan. \varphi' - \sin. \varphi)$ is known.

449. (Fig. 54.) Let the inaccessible object M, as a kite, cloud, &c., be distant from the horizon by the interval $M N = x$. Take three stations A, B, C in the same straight line, such that $A B = B C = m$ a known distance; and at these stations let the respective $\angle$s of elevation be a, b, c.

Then, since $M N$ is perpendicular to the horizon, $\angle\, M N A$, $\angle\, M N B$, and $\angle\, M N C$ are right angles. And, by letting fall

a straight line from M, perpendicular upon C A, it may easily be shewn that $CM^2 + MA^2 = 2CB^2 + 2BM^2$

$$= 2m^2 + 2BN^2 + 2x^2$$

or $CN^2 + x^2 + AN^2 + x^2 = 2m^2 + 2BN^2 + 2x^2$

$$\therefore CN^2 + AN^2 = 2m^2 + 2BN^2$$

$$\left.\begin{array}{l} \text{But } CN = x.\cot. c \\ BN = x.\cot. b \\ AN = x.\cot. a \end{array}\right\}$$

$$\therefore x^2.\cot.^2 c + x^2.\cot.^2 a = 2m^2 + 2x^2.\cot.^2 b$$

$$\therefore x^2 = \frac{2m^2}{\cot.^2 c + \cot.^2 a - 2\cot.^2 b}$$

$$\therefore x = \pm \frac{m\sqrt{2}}{\sqrt{\cot.^2 a + \cot.^2 c - 2\cot.^2 b}}$$

The problem will not be difficult, if AB and BC be *unequal*, as m and n. x in this case $= \pm \dfrac{(m+n) \times mn}{\sqrt{n\cot. 2a + m\cot. 2b - \overline{m+n} \times \cot. 2b}}$

That the errors arising from the observed ∠ may be the less, the ∠ a, b, c should be as nearly = to 45° as possible. Allowance must also be made for the refraction of light, &c. &c.

To adapt the above form to logarithmic computation,

$$x^2 = \frac{2m^2 \tan.^2 a}{1 + \tan.^2 a.(\cot.^2 c - 2\cot.^2 b)} = \frac{2m^2 \tan.^2 a}{1 + \tan.^2 \theta}$$

$= \dfrac{2m^2 \tan.^2 a}{\sec.^2 \theta} = 2m^2 \tan.^2 a \times \cos.^2 \theta$, by assuming $\tan.^2 \theta$

$= \tan.^2 a \times (\cot.^2 c - 2\cot.^2 b)$

whence we have,

$2\log.(\tan.\theta) = 2\log.(\tan. a) + \log.(\cot.^2 c - 2\cot.^2 b)$

and $\therefore \cos.\theta$ will be known from the tables.

Hence $2\log. x = \log. 2 + 2\log. m + 2\log.(\tan. a) + 2\log.(\cos.\theta)$

$$\therefore \log. x = \frac{\log. 2}{2} + \log. m + \log.(\tan. a) + \log.(\cos.\theta)$$

$\therefore \log. x$ is known from the tables

whence x may also be found.

450. $$\cos. A = \frac{\cos. a - \cos. b. \cos. c}{\sin. b. \sin. c} \quad \textit{(Woodhouse.)}$$

$$= \frac{\cos. a - \cos.^2 b}{\sin.^2 b}$$

$$\therefore \sin.^2 b \times \cos. A = \cos. a - 1 + \sin.^2 b$$

$$\therefore \sin.^2 b = \frac{1 - \cos. a}{1 - \cos. A}$$

But $\cos. A = 1 - 2 \sin.^2 \frac{A}{2}$

$$\therefore 1 - \cos. A = 2 \sin.^2 \frac{A}{2}$$

and similarly $1 - \cos. a = 2 \sin.^2 \frac{a}{2}$

$$\therefore \sin.^2 b = \frac{2 \sin.^2 \frac{a}{2}}{2 \sin.^2 \frac{A}{2}} = \frac{\sin.^2 \frac{a}{2}}{\sin.^2 \frac{A}{2}}$$

$$\therefore \sin. b = \frac{\sin. \frac{a}{2}}{\sin. \frac{A}{2}}$$

Again, $$\sin. B = \frac{\sin. b \times \sin. A}{\sin. a} = \frac{\sin. \frac{a}{2}}{\sin. a} \times \frac{\sin. A}{\sin. \frac{A}{2}}$$

$$= \frac{\sin. \frac{a}{2}}{2 \sin. \frac{a}{2}. \cos. \frac{a}{2}} \times \frac{2 \sin. \frac{A}{2}. \cos. \frac{A}{2}}{\sin. \frac{A}{2}}$$

$$= \frac{\cos. \frac{A}{2}}{\cos. \frac{a}{2}}$$

451. Let A and B be the two arcs.

Then $$\tan. A + \tan. B = \frac{\sin. A}{\cos. A} + \frac{\sin. B}{\cos. B} = \frac{\sin. A. \cos. B + \cos. A. \sin. B}{\cos. A. \cos. B}$$

$$= \frac{\sin.(A+B)}{\cos.A.\cos.B} \text{ to radius} = \text{unity}$$

$$\tan.A - \tan.B = \frac{\sin.A}{\cos.A} - \frac{\sin.B}{\cos.B} = \frac{\sin.A.\cos.B - \cos.A.\sin.B}{\cos.A\ \cos.B}$$

$$= \frac{\sin.(A-B)}{\cos.A\cos.B}$$

$\therefore \tan.A + \tan.B : \tan.A - \tan.B :: \sin.(A+B) : \sin.(A-B)$ whatever may be the radius, for every term is of the same number of dimensions.

452. $\text{Tan}.A = r \times \frac{\sin.A}{\cos.A}$, r being the radius

$$\text{and } \cot.A = r \times \frac{\cos.A}{\sin.A} = \frac{r}{\left(\frac{\sin.A}{\cos.A}\right)} = \frac{r}{\frac{\tan.A}{r}}$$

$$= \frac{r^2}{\tan.A}$$

$\therefore \cot.A : r :: r : \tan.A.$

Also, $\sin.A \times \cos.A = \frac{\sin.2A}{2} \propto \sin.2A$, (since 2 is invariable.)

453. Let the $\sin.A = m$.

Then $\sin.2A = 2\sin.A\cos.A = 2\sin.A\sqrt{1-\sin.^2A}$

$= 2m \times \sqrt{1-m^2}$ which is $\therefore$ known.

454. $\because \sin.(B+A) = \sin.B\cos.A + \cos.B\sin.A$

$\sin.(B-A) = \sin.B\cos.A - \cos.B\sin.A$

$\therefore \sin.(B+A) = 2\cos.A\sin.B - \sin.(B-A)$

Let $B = (n-1)A$, and substitute, &c.

Then $\sin.nA = 2\cos.A\sin.(n-1)A - \sin.(n-2)A$

Again, $\cos.(B+A) = \cos.B\cos.A - \sin.B.\sin.A$

$\cos.(B-A) = \cos.B\cos.A + \sin.B\sin.A$

$\therefore$ cos. (B + A) = 2 cos. A $\times$ cos. B − cos. (B − A) in which if for B we substitute (n—1) A, we get

cos. nA = 2 cos. A cos. (n − 1) A − cos. (n − 2) A.

These forms being of great use in constructing tables, &c. &c. ought to be committed to memory.

455. (Fig. 55). Let CA be any radius of the given circle, and AB $\perp$ to it. Take Ab of any magnitude less than AC, and divide in B so that

Ab : AB :: 9 : 4

Draw BN, bn $\perp$Ab and meeting the circle in N,n, and NM, nm $\perp$ CA.

Then nm : NM :: Ab : AB :: 9 : 4

or sin. $\angle$ nCA : sin. $\angle$ NCA :: 9 : 4

$\therefore$ $\angle$ nCA, NCA are such as were required to be found.

It is evident that the problem admits of innumerable solutions.

456. $$\frac{\text{cos. A} + \text{sin. A}}{\text{cos. A} - \text{sin. A}} = \frac{(\text{cos. A} + \text{sin. A})^2}{\text{cos.}^2\text{A} - \text{sin.}^2\text{A}}$$

$$= \frac{\text{cos.}^2\text{A} + \text{sin.}^2\text{A} + 2\,\text{sin. A. cos. A}}{\text{cos. 2A}} = \frac{1 + \text{sin. 2A}}{\text{cos. 2A}}$$

$$= \frac{1}{\text{cos. 2A}} + \frac{\text{sin. 2A}}{\text{cos. 2A}} = \text{sec. 2A} + \text{tan. 2A}.$$

Again, $\text{tan. B} = \text{sec. A} - \text{tan. A} = \dfrac{1}{\text{cos. A}} - \dfrac{\text{sin. A}}{\text{cos. A}}$

$$= \frac{1 - \text{sin. A}}{\text{cos.A}} = \frac{\text{cos.}^2\frac{A}{2} + \text{sin.}^2\frac{A}{2} - 2\,\text{sin.}\frac{A}{2}.\ \text{cos.}\frac{A}{2}}{\text{cos.}^2\frac{A}{2} - \text{sin.}^2\frac{A}{2}}$$

$$= \frac{\left(\text{cos.}\frac{A}{2} - \text{sin.}\frac{A}{2}\right)^2}{\text{cos.}^2\frac{A}{2} - \text{sin.}^2\frac{A}{2}} = \frac{\text{cos.}\frac{A}{2} - \text{sin.}\frac{A}{2}}{\text{cos.}\frac{A}{2} + \text{sin.}\frac{A}{2}}$$ Divide both numerator and denominator, by cos. $\frac{A}{2}$

$$\text{Then tan. B} = \frac{1 - \tan \frac{A}{2}}{1 + \tan. \frac{A}{2}}$$

$$\text{But tan. } \overline{\left(m\pi + 45 - \frac{A}{2}\right)} = \frac{\tan. (m\pi + 45) - \tan. \frac{A}{2}}{1 + \tan. (m\pi + 45) \times \tan. \frac{A}{2}}$$

$$\text{and tan. } (m\pi + 45) = \frac{\tan.\, m\pi + \tan.\, 45}{1 - \tan.\, m\pi \times \tan. 45}$$

$= \frac{\tan.\ 45}{1} = \frac{1}{1} = 1$ (m being any integral number whatever.)

$$\therefore \tan. \left(m\pi + 45 - \frac{A}{2}\right) = \frac{1 - \tan. \frac{A}{2}}{1 + \tan. \frac{A}{2}} = \tan. \text{B}$$

$\therefore$ B $= m\pi + 45 - \frac{A}{2}$ which is a general solution (m being any integer whatever.)

DIFFERENTIAL CALCULUS.

457. Let $\dfrac{x}{(a^2-x^2)^{\frac{1}{2}}} = y$

$\therefore x^2 = y^2 \times (a^2 - x^2)$

$\therefore 2x\,dx = 2y\,dy.\,(a^2-x^2) - 2x\,dx \times y^2$

$\therefore dy = \dfrac{2x\,dx \times (1 + y^2)}{2y \times (a^2 - x^2}$

But $1 + y^2 = 1 + \dfrac{x^2}{a^2-x^2} = \dfrac{a^2}{a^2-x^2}$

$$\therefore d.\,\frac{x}{(a^2-x^2)^{\frac{1}{2}}} = dy = \frac{2x\,dx \times a^2}{\dfrac{2x}{(a^2-x^2)^{\frac{1}{2}}} \times (a^2-x)^2} = \frac{a^2\,dx}{(a^2-x^2)^{\frac{3}{2}}}$$

Again, let $\dfrac{\sqrt{a^2+x^2}}{\sqrt{a^2-x^2}} = z$

Then $a^2 + x^2 = z^2 a^2 - z^2 x^2$

$\therefore 2x\,dx = 2a^2\,zdz - 2zdzx^2 - 2xdx.z^2$

$\therefore (a^2z - zx^2)\,dz = (x+xz^2)\,dx$

$\therefore dz = \dfrac{x.\,(1 + z^2)\,dx}{z \times (a^2 - x^2)}$

But $1 + z^2 = 1 + \dfrac{a^2+x^2}{a^2-x^2} = \dfrac{2a^2}{a^2-x^2}$

and $\therefore \dfrac{1 + z^2}{z} = \dfrac{2a^2}{a^2-x^2} \times \dfrac{\sqrt{a^2-x^2}}{\sqrt{a^2+x^2}} = \dfrac{2a^2}{\sqrt{a^4-x^4}}$

$\therefore dz = \dfrac{2a^2\,xdx}{(a^2-x^2)\sqrt{a^4-x^4}}$

Again, since $d.\log.u = M\frac{du}{u}$ $\left(\frac{1}{M}\right.$ being the modulus). Put $a^x = u$

Then $x\log.a = \log.u$

$$\therefore d.\log.u = M \times \frac{du}{u} = M \times \log.a \times dx$$

$\therefore du = u \times \log.a \times dx = \log.a \times a^x dx$ See *Lacroix*.

This may be found independently of logarithms, by taking for the differential of the function, the second term of a^{x+dx}, developed according to the powers of dx.

458. Let $\sqrt{a^3 + x^3 - \sqrt{a^2 - x^2}} = u$

Then $a^3 + x^3 - \sqrt{a^2 - x^2} = u^2$

$$\therefore 3x^2\,dx + \frac{x\,dx}{\sqrt{a^2 - x^2}} = 2u\,du$$

$$\therefore du = \frac{(3x\sqrt{(a^2 - x^2)} + 1)\,xdx}{2\sqrt{a^2 - x^2} \times \sqrt{a^3 + x^3 - \sqrt{a^2 - x^2}}}$$

459. Let $\frac{x}{\sqrt{1 + x^2}} = u$

$$\therefore x^2 = u^2 + u^2 x^2$$

$$\therefore 2x\,dx = 2u\,du + 2udu \times x^2 + 2xdx \times u^2$$

$$\therefore udu \times (1 + x^2) = xdx \times (1 - u^2)$$

and $du = \frac{xdx}{1 + x^2} \times \frac{1 - u^2}{u}$

But $1 - u^2 = 1 - \frac{x^2}{1+x^2} = \frac{1}{1+x^2}$

$$\therefore \frac{1-u^2}{u} = \frac{1}{1+x^2} \times \frac{\sqrt{1+x^2}}{x} = \frac{1}{x\sqrt{1+x^2}}$$

$$\therefore du = \frac{dx}{(1 + x^2)^{\frac{3}{2}}}$$

460. Let $a + bx + cx^2 = u^2$

$\therefore bdx + 2cxdx = 2u\,du$

$\therefore du = \dfrac{(b + 2cx)\,dx}{2\sqrt{a+bx+cx^2}}$

Again, put $\dfrac{1}{\sqrt{a+x}} = v \therefore 1 = av^2 + xv^2$

$\therefore 0 = 2avdv + v^2dx + 2vdv \times x$

$\therefore 2vdv \times (a + x) = - v^2dx$

$\therefore dv = - \dfrac{dx \times v}{2(a + x)} = - \dfrac{dx}{2(a + x)^{\frac{3}{2}}}$

Again, put $(a^x)^x = u$

$\therefore x \log. a^x = \log. u$

$\therefore x^2 \log. a = \log. u$.

$\therefore 2 \log. a \times xdx = \dfrac{du}{u} \times$ M, $\dfrac{1}{\text{M}}$ being the modulus.

$\therefore du = \dfrac{2 \log. a}{\text{M}} x \times (a^x)^x\,dx$

461. Let $\dfrac{a}{\sqrt{x^2 + y}} = u$

$\therefore a^2 = u^2 x^2 + u^2y$

$\therefore 0 = 2u\,du.\,x^2 + 2x\,dx.\,u^2 + 2u\,du.y + u^2dy$

$\therefore 2u\,du\,(x^2 + y) = - u^2.(2x\,dx + dy)$

$\therefore du = - \dfrac{u}{2} \left(\dfrac{2x\,dx + dy}{x^2 + y}\right)$

$= - \dfrac{a}{2} \times \dfrac{2x\,dx + dy}{(x^2 + y)^{\frac{3}{2}}}$

Again, put $\dfrac{(x + a)^2}{\sqrt{x^2 - a^2}} = u$

Then $(x + a)^4 = u^2 x^2 - u^2a^2$

$\therefore 4\,(x + a)^3\,dx = 2u\,du.\,x^2 + 2x\,dx.\,u^2 - 2udu \times a^2$

$\therefore udu \times (x^2 - a^2) = (2\,(x + a)^3 - x \times u^2)\,dx$

But $2\,(x+ a)^3 - x \times u^2 = 2\,(x + a)^3 - \dfrac{x \times (x + a)^4}{x^2 - a^2}$

$$= \frac{(x+a)^3(2x^2-2a^2-x^2-ax)}{x^2-a^2} = \frac{(x+a)^3 \times (x^2-a.\overline{2a+x})}{x^2-a^2}$$

$$\therefore du = \frac{\sqrt{x^2-a^2}}{(x^2-a^2)\ (x+a)^2} \times \frac{(x+a)^3 \times (x^2-a \times \overline{2a+x})dx}{x^2-a^2}$$

$$\therefore \frac{(x+a)\times(x^2-a.\overline{2\ a+x})}{(x^2-a^2)^{\frac{3}{2}}}dx = \frac{(x+a)^2\times(x-2a)}{(x^2-a^2)^{\frac{3}{2}}}dx$$

for $x^2-ax-2a^2=x^2-a^2-a.\overline{x+a}=\overline{x+a}.\overline{x-2a}$

Otherwise.

$$\log. u = 2\log.(x+a) - \frac{1}{2}\log.(x^2-a^2).$$

$$\therefore \frac{du}{u} = 2\frac{dx}{x+a} - \frac{xdx}{x^2-a^2}$$

$\therefore$ &c. &c. &c.

462. Let $l.\left(\frac{1-x^{\frac{3}{2}}}{1+x^{\frac{3}{2}}}\right) = l.\ (1-x^{\frac{3}{2}}) -$ $l.(1+x^{\frac{3}{2}}) = u$

$$\therefore du = \frac{-\frac{3}{2}x^{\frac{1}{2}}dx}{1-x^{\frac{3}{2}}} - \frac{\frac{3}{2}x^{\frac{1}{2}}dx}{1+x^{\frac{3}{2}}}$$

$$= -\frac{3}{2}x^{\frac{1}{2}}dx\left(\frac{1}{1-x^{\frac{3}{2}}}+\frac{1}{1+x^{\frac{3}{2}}}\right)$$

$$= -\frac{3}{2}x^{\frac{1}{2}}dx \times \frac{2}{1-x^3}$$

$$= \frac{3x^{\frac{1}{2}}dx}{x^3-1}$$

$$d.\frac{x}{1+x^2} = \frac{dx.(1+x^2)-2x^2dx}{(1+x^2)^2} = \frac{1-x^2}{(1+x^2)^2}dx$$

463. Let $l.\ x\ e^{\cos. x}, = u$

$$\therefore\ du = \frac{dx}{x} e^{\cos. x} + d.e^{\cos. x} \times l.\ x$$

Let $e^{\cos. x} = v$

$$\therefore\ l.\ v = \cos.\ x$$

$$\therefore \frac{dv}{v} = -\sin.\ x \times dx$$

$$\therefore\ dv = -\sin.\ x \times e^{\cos. x}\ dx$$

$$\therefore\ du = \frac{dx}{x} e^{\cos. x} - \sin.\ x\ e^{\cos. x}\ dx$$

$$= e^{\cos. x}\ dx \times \left(\frac{1}{x} - \sin.\ x.\ l.\ x\right)$$

$$= \frac{e^{\cos. x}}{x} \times (1 - x \sin.\ x.\ l.\ x)\ dx.$$

464. Let $\dfrac{x}{\sqrt{1+x^2}} = u$

$$\therefore\ \log.\ u = \log.\ x - \frac{1}{2} \log.\ (1 + x^2)$$

$$\therefore\ \frac{du}{u} = \frac{dx}{x} - \frac{xdx}{1+x^2} = \frac{dx}{x \times (1+x^2)}$$

$$\therefore\ du = \frac{dx}{(1+x^2)^{\frac{3}{2}}}$$

465. $d.(1 - x^{\frac{1}{2}} + x^{\frac{1}{3}})^{\frac{4}{5}} = \dfrac{\frac{4}{5} dx \times (\frac{1}{3} x^{-\frac{2}{3}} - \frac{1}{2} x^{-\frac{1}{2}})}{(1 - x^{\frac{1}{2}} + x^{\frac{1}{3}})^{\frac{1}{5}}}$

But $\dfrac{1}{3x^{\frac{2}{3}}} - \dfrac{1}{2x^{\frac{1}{2}}} = \dfrac{2x^{\frac{1}{2}} - 3x^{\frac{2}{3}}}{6x^{\frac{7}{6}}} = \dfrac{2 - 3x^{\frac{1}{6}}}{6x^{\frac{2}{3}}}$

$$\therefore\ d.(1 - x^{\frac{1}{2}} + x^{\frac{1}{3}})^{\frac{4}{5}} = \frac{2}{15} \frac{dx}{x^{\frac{2}{3}}} \times \frac{2 - 3x^{\frac{1}{6}}}{(1 - x^{\frac{1}{2}} + x^{\frac{1}{3}})^{\frac{1}{5}}}$$

$$= \frac{2}{15} \times \frac{2 - 3x^{\frac{1}{6}}}{x^{\frac{2}{3}} (1 - x^{\frac{1}{2}} + x^{\frac{1}{3}})^{\frac{1}{5}}} dx$$

466. $$d.\frac{z}{1+z} = \frac{dz.(1+z) - zdz}{(1+z)^2}$$

$$= \frac{dz}{(1+z)^2}$$

467. Let $\left(\frac{a^2-y^2}{y}\right)^{\frac{1}{2}} = u$

$$\therefore \quad u = \frac{1}{2} \quad (a^2 - y^2) \quad - \frac{1}{2} \quad y$$

$$\therefore \frac{du}{u} = \frac{-y\,dy}{a^2-y^2} - \frac{1}{2}\frac{dy}{y}$$

$$= \frac{-dy}{y.(a^2-y^2)} \times (2y^2+a^2-y^2) = -dy \times \frac{a^2+y^2}{y(a^2-y^2)}$$

$$\therefore du = \frac{-dy \times (a^2+y^2)}{y^{\frac{3}{2}}.(a^2-y^2)^{\frac{1}{2}}}$$

468. Let $ay^x = u$, and l be the characteristic of hyperbolic logarithms.

Then $l.a + x\,l.\,y = l.u$

$$\therefore \frac{du}{u} = l.y \times dx + x.\frac{dy}{y}$$

$$\therefore du = a\,y^x l.y \times dx + ax.y^{x-1}dy$$

$$= ay^{x-1}.(yl.y \times dx + x\,dy)$$

Again let $y = e^z$

Then $z = l.\,y$

$$\therefore du = ay^{x-1}(zy\,dx + x\,dy)$$

469. $$d.\frac{y}{\sqrt{1+y^2}} = \frac{dy.\sqrt{1+y^2} - \frac{y^2\,dy}{\sqrt{1+y^2}}}{1+y^2}$$

$$= dy.\frac{1+y^2-y^2}{(1+y^2)^{\frac{3}{2}}} = \frac{dy}{(1+y^2)^{\frac{3}{2}}}$$

470. Let $\frac{(a^4+x^4)^{\frac{1}{2}}}{x} = u$

$\therefore\ a^4 + x^4 = u^2x^2$

$\therefore\ 4x^3\,dx = 2x^2u\,du + 2u^2\,x\,dx$

$\therefore\ du = \frac{(2x^2 - u^2)\,dx}{xu}$

But $2x^2 - u^2 = 2x^2 - \frac{a^4+x^4}{x^2} = \frac{x^4-a^4}{x^2}$

$\therefore\ du = \frac{x^4-a^4}{x^3} \times \frac{x}{(a^4+x^4)^{\frac{1}{2}}} \times dx = \frac{x^4-a^4}{x^2.(x^4+a^4)^{\frac{1}{2}}} \times dx$

471. Let $\frac{x}{\sqrt{a^3+x^3}} = u$

$\therefore\ l.u = l.x - \frac{1}{2}\,l.(a^3 + a^3)$

$\therefore\ \frac{du}{u} = \frac{dx}{x} - \frac{3}{2} \cdot \frac{x^2\,dx}{a^3+x^3}$

$= \frac{dx}{2x.(a^3+x^3)} \times (2a^3 + 2x^3 - 3x^3)$

$= \frac{dx}{2x\,(a^3+x^3)} \times (2a^3 - x^3)$

$\therefore\ du = \frac{dx}{2\,(a^3+x^3)^{\frac{3}{2}}} \times (2a^3 - x^3)$

472. $d.\frac{\sqrt{ax}-b}{\sqrt{a-x}} = \frac{\frac{a^{\frac{1}{2}}dx}{2x^{\frac{1}{2}}} \times \sqrt{a-x} + \frac{\frac{1}{2}dx \times (\sqrt{ax}-b)}{\sqrt{a-x}}}{a-x} = 0$

$\therefore\ \frac{a^{\frac{1}{2}}\sqrt{a-x}}{2\sqrt{x}} + \frac{\sqrt{ax}-b}{2 \times \sqrt{a-x}} = 0$

$\therefore\ \sqrt{a}.(a-x) + \sqrt{x}.(\sqrt{ax}-b) = 0$

$\therefore\ a^{\frac{3}{2}} - \sqrt{a}.x + \sqrt{a}.x - b\sqrt{x} = 0$

$\therefore\ b\sqrt{x} = a^{\frac{3}{2}}$

$$\therefore \sqrt{x} = \frac{a^{\frac{3}{2}}}{b}$$

$$\text{and } x = \frac{a^3}{b^2}$$

473. Let $z^{yz} = u$

Then $l.u = yzl.z$

$$\therefore \frac{du}{u} = dy.zl.z + dz.yl.z + \frac{dz}{z}yz$$

$$= dy.zl.z + dz.yl.z + ydz$$

$$\therefore du = z^{yz+1}l.z \times dy + yz^{yz}.(l.z+1)\,dz$$

Again, let $x^{y^z} = v$

Then $l.v = y^z\,l.x$

and $l.(l.v) = zl.y + l.(l.x)$

$$\therefore \frac{d.(l.v)}{l.v} = dz.l.y + \frac{zdy}{y} + \frac{d.(l.x)}{l.x}$$

$$\therefore \frac{dv}{vl.v} = dz\,l.y + \frac{zdy}{y} + \frac{dx}{x.l.x}$$

$$\therefore dv = x^{y^z} \times y^z l.x\,\frac{xl.x.l.y \times dz + xz.l.x \times dy + ydx}{xyl.x}$$

$= x^{y^z-1} \times y^{z-1}(xl.xl.y \times dz + xzl.x \times dy + ydx)$, which is $\therefore$ expressed in terms of dz, dy, dx, and functions of z, y, and x.

474. Let $(a^2+x^2) \times \sqrt{a^2-x^2} = u$

$$\therefore l.u = l.(a^2+x^2) + \frac{1}{2}\,l.(a^2-x^2)$$

$$\therefore \frac{du}{u} = \frac{2xdx}{a^2+x^2} - \frac{xdx}{a^2-x^2} = \frac{xdx}{a^4-x^4} \times (2a^2-2x^2-a^2-x^2)$$

$$= \frac{xdx}{a^4-x^4} \times (a^2-3x^2)$$

$$\therefore du = \frac{xdx}{a^2-x^2}(a^2-3x^2)\sqrt{a^2-x^2} = \frac{a^2-3x^2}{\sqrt{a^2-x^2}} \times xdx$$

475. $$d.(a+bx^{\frac{3}{2}}+cx^{\frac{2}{3}})^{\frac{1}{2}} = \frac{\frac{1}{2}(\frac{3}{2}bx^{\frac{1}{2}}+\frac{2}{3}cx^{-\frac{1}{3}})\,dx}{(a+bx^{\frac{3}{2}}+cx^{\frac{2}{3}})^{\frac{1}{2}}}$$

$$= \frac{9\,bx^{\frac{5}{6}}+4c}{12\,x^{\frac{1}{3}}\,(a+bx^{\frac{3}{2}}+cx^{\frac{2}{3}})^{\frac{1}{2}}} \times dx$$

Again, let $u = l.\dfrac{\sqrt{a^2+x^2}}{\sqrt{a^2-x^2}} = \dfrac{1}{2}\,l.(a^2+x^2) - \dfrac{1}{2}\,l.(a^2-x^2)$

$$\therefore du = \frac{xdx}{a^2+x^2} + \frac{xdx}{a^2-x^2}$$

$$= \frac{xdx}{a^4-x^4} \times (a^2-x^2+a^2+x^2)$$

$$= \frac{2a^2}{a^4-x^4} \times dx$$

476. $$d.(a+x)\sqrt{a-x} = \sqrt{a-x}.dx - \frac{dx}{2\sqrt{a-x}} \times$$

$$(a+x) = (2a-2x-a-x)\,\frac{dx}{2\sqrt{a-x}} = \frac{a-3x}{2\sqrt{a-x}} \times dx$$

For $d.a^x$, see *Lacroix*, or *Vince*.

477. Let $(x^m+bx^n)^p = u$

$$\therefore x^m + bx^n = u^{\frac{1}{p}}$$

$$\therefore mx^{m-1}.dx + nb\,x^{n-1}dx = \frac{1}{p}\,u^{\frac{1-p}{p}}\,du$$

$$\therefore du = px^{m-1}.dx \times (m+nb\,x^{n-m}) \times u^{\frac{p-1}{p}}$$

$$\therefore du = px^{m-1}. \times (x^m+bx^n)^{p-1} \times (m+nb\,x^{n-m})\,dx$$

$$= p.x^{mp-1} \times (1+bx^{n-m})^{p-1} \times (m+nbx^{n-m})\,dx$$

478. $d.(a+cz^n)^m \times zp = mncz^{n-1} \times (a+cz^n)^{m-1} \times zpdz$ $+ dz. \times (a+cz^n)^m \times p + dp \times (a+cz^n)^m \times z = pdz \times (a+cz^n)^{m-1}$ $\times (mncz^n + a + cz^n) + dp.(a+cz^n)^m z = p.(a+cz^n)^{m-1}$ $\times (a+\overline{mn+1}.c.z^n)\,dz + z.(a+cz^n)^m\,dp.$

Again, let $x . \sqrt{\dfrac{1+x^2}{1-x^2}} = u$

$$\therefore l.u = l.x + \frac{1}{2} l.(1+x^2) - \frac{1}{2} l.(1-x^2)$$

$$\therefore \frac{du}{u} = \frac{dx}{x} + \frac{xdx}{1+x^2} + \frac{xdx}{1-x^2}$$

$$= \frac{dx}{x} + \frac{2xdx}{1-x^4}$$

$$= \frac{dx.(1-x^4+2x^2)}{x.(1-x^4)}$$

$$\therefore du = \frac{1-x^4+2x^2}{1-x^4} \times \sqrt{\frac{1+x^2}{1-x^2}} \times dx$$

$$= \frac{x^4-2x^2-1}{\sqrt{x^4-1}} \times \frac{dx}{1-x^2}$$

479. Let $\dfrac{(x+a)^2}{\sqrt{x^2-a^2}} = u$

$$\therefore 2\,l.(x+a) - \frac{1}{2} l.(x^2-a^2) = l.u$$

$$\therefore \frac{du}{u} = \frac{2dx}{x+a} - \frac{xdx}{x^2-a^2} = \frac{\overline{x-2a}.dx}{x^2-a^2}$$

$$\therefore du = \frac{(x+a)^2.(x-2a)}{(x^2-a^2)^{\frac{3}{2}}} dx = \frac{\sqrt{x+a}.(x-2a)}{(x-a)^{\frac{3}{2}}} \times dx$$

480. $d. \dfrac{a+x}{a^2+x^2} = \dfrac{dx.(a^2+x^2) - 2x\,dx.(a+x)}{(a^2+x^2)^2}$

$$= \frac{(a^2-2ax-x^2)}{(a^2+x^2)^2} dx$$

Again, let $x.(a^2+x^2).\sqrt{a^2-x^2}) = u$

$$\therefore l.x + l.(a^2+x^2) + \frac{1}{2}.l.(a^2-x^2) = l.u.$$

$$\therefore \frac{du}{u} = \frac{dx}{x} + \frac{2xdx}{a^2+x^2} - \frac{xdx}{a^2-x^2}$$

$$= \frac{dx}{x} + xdx. \frac{2a^2-2x^2-a^2-x^2}{a^4-x^4}$$

$$= \frac{dx}{x} + xdx. \frac{a^2-3x^2}{a^4-x^4}$$

$$= \frac{dx}{x.(a^4-x^4)} \times (a^4 - x^4 + a^2 x^2 - 3x^4)$$

$$= \frac{a^4 + a^2 x^2 - 4x^4}{x.(a^4-x^4)} . dx$$

Again $d. \sec. x = d. \frac{1}{\cos. x} = \frac{-d. \cos. x}{\cos.^2 x}$

$$= \frac{\sin. x. dx}{\cos.^2 x} \text{ (*Lacroix* or *Vince.*)}$$

$$= \frac{\sin. x. dx}{1-\sin^2 x} \text{ or } = \tan. x \sec. x. dx.$$

481. Let $u = x \times e^{\tan. x}$

$\therefore l. u = l. x + \tan. x$ (since $l. e^{\tan. x} = \tan. x \log. e = \tan. x$)

$$\therefore \frac{du}{u} = \frac{dx}{x} + d. \tan. x.$$

But $d. \tan. x = d. \frac{\sin. x}{\cos. x} = \frac{\cos. x \times \cos. x + \sin. x. \sin. x}{\cos. ^2x}$

$\times dx = \frac{dx}{\cos.^2 x}$

$$\therefore du = u \times \frac{dx}{x} + u. \frac{dx}{\cos. ^2x}$$

$$= \left(e^{\tan. x} + \frac{xe^{\tan. x}}{\cos.^2 x}\right) dx = \frac{e^{\tan. x}}{\cos.^2 x} \times (\cos.^2 x + x)\, dx$$

482. Let z be the arc.

Then $\tan. z = \sqrt{\frac{1-x}{1+x}}$

But $d.\ \tan.\ z = \frac{dz}{\cos.^2 z} = dz.\ \sec.^2 z = dz.\ (1 + \tan.^2 z)$

$$= dz.\left(1 + \frac{1-x}{1+x}\right) = \frac{2dz}{1+x}$$

$$\therefore\ dz = \frac{1+x}{2} \times d.\ \tan.\ z = \frac{1+x}{2} \times d.\ \sqrt{\frac{1-x}{1+x}}$$

Put $u = \sqrt{\frac{1-x}{1+x}}$

$$\therefore\ l.\ u = \frac{1}{2} l.\ (1-x) - \frac{1}{2} l.\ (1+x)$$

$$\therefore\ \frac{du}{u} = -\frac{1}{2} \cdot \frac{dx}{1-x} - \frac{1}{2} \frac{dx}{1+x} = \frac{-dx}{1-x^2}$$

$$\therefore\ du = \frac{-dx}{1-x^2} \times \sqrt{\frac{1-x}{1+x}}$$

Hence $dz = \frac{-dx}{2(1-x)} . \sqrt{\frac{1-x}{1+x}} = \frac{-dx}{2.\sqrt{1-x^2}}$.

INTEGRAL CALCULUS.

483. $$\int\frac{dx}{1+x} = l.\ (1+x)$$

$$\therefore \int\frac{dx}{1+x} = l.\ (1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \&c.$$

$$\therefore \int\frac{dx}{x}\int\frac{dx}{1+x} = \int dx - \int\frac{xdx}{2} + \int\frac{x^2dx}{3} - \&c.$$

$$= x - \frac{x^2}{2^2} + \frac{x^3}{3^2} - \&c. + C$$

$$= x - \frac{x^2}{2^2} + \frac{x^3}{3^2} - \&c.$$

(since C $=$ 0 when $x = 0$)

Let $x = 1$

Then between $x = 0$ and $= 1$,

$$\int\frac{dx}{x}\int\frac{dx}{1+x} = 1 - \frac{1}{2^2} + \frac{1}{3^2} - \&c....$$

Again, sin. $x = x.\ (x-\pi).(x+\pi).(x-2\pi).(x+2\pi)$.... &c. to infinity, because if sin. $x = 0$, and 0, π, $-\pi$ &c. being substituted for x satisfies the equation.

$\therefore$ sin. $x = x.(x^2-\pi^2)(x^2-\overline{2\pi}|^2)\ (x^2-\overline{3\pi}|^2)\times$&c. to infinity.

But sin. x also $= x - \dfrac{x^3}{1.2.3} + \&c.$ ....

$$\therefore (x^2-\pi^2).(x^2-2^2\pi^2).(x^2-3^2.\pi^2)\times\&c. = 1 - \frac{x^2}{1.2.3} + \&c.$$

$\therefore\ \pi^2,\ 2^2\pi^2$, &c. are roots or the values of x^2 of the equation $1 - \dfrac{x^2}{1.2.3}$ &c.

Now, by the theory of equations, the last coefficient but one divided by the last ~~but one~~ = the sum of the reciprocals of the roots.

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$$\therefore\ \frac{1}{1.2.3} = \frac{1}{\pi^2} + \frac{1}{2^2\pi^2} + \frac{1}{3^2\pi^2} + \text{\&c.}$$

$$\therefore\ \frac{2Q^2}{3} = \frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \text{\&c.}$$

Now ~~but~~ $2 \times \left(\frac{1}{2^2} + \frac{1}{4^2} + \text{\&c.}\right) = \frac{1}{2}\left(1+\frac{1}{2^2} + \frac{1}{3^2} + \text{\&c.}\right)$

$$= \frac{Q^2}{3}$$

$$\therefore\ \frac{2Q^2}{3} - \frac{Q^2}{3} = 1 + \frac{1}{2^2}+\frac{1}{3^2} + \text{\&c.} - 2\left(\frac{1}{2^2}+\frac{1}{4^2}+ \text{\&c.}\right)$$

$$= 1 - \frac{1}{2^2} + \frac{1}{3^2} - \text{\&c.}$$

$\displaystyle\int\frac{dx}{x}\int\frac{dx}{1+x} = \frac{2Q^2}{3} - \frac{Q^2}{3} = \frac{Q^2}{3}$ the limits of x being 0 and 1.

484. Let $z^{\frac{n}{2}} = x$

Then $\dfrac{n}{2}.\ z^{\frac{n}{2}-1}\ dz = dx$

and $\dfrac{z^{\frac{n}{2}-1}\,dz}{\sqrt{a^n+z^n}} = \dfrac{2}{n}\dfrac{dx}{\sqrt{x^2+a^n}}$

Again, put $\sqrt{x^2 + a^n} = v$

$\therefore\ xdx = vdv$

$\therefore\ dx : dv :: v : x$

$\therefore\ dx : dx + dv :: v : v + x$

$$\therefore\ \frac{dx}{v} = \frac{dx+dv}{x+v} = d.\ l.\ (x+v)$$

$$= d.\ l.\ (x + \sqrt{x^2+a^n})$$

$$\therefore \int \frac{z^{\frac{n}{2}-1}\,dz}{\sqrt{a^n+z^n}} = \frac{2}{n}.\,l.\,(x + \sqrt{x^2+a^n})$$

$$= \frac{2}{n}\,.\,l.\,(z^{\frac{n}{2}} + \sqrt{z^n+a^n})$$

Again, since $dpq = pdq + qdp$

$$\therefore \int pdq = pq - \int qdp$$

$$\therefore \int v^2\,dx = v^2\,x - \int xd.v^2$$

But $d.v^2 = 2vdv = 2\,l.\,x.\,\dfrac{dx}{x}$

$$\therefore \int v^2\,dx = v^2\,x - 2\int vdx$$

Similarly, $\int vdx = vx - \int v^0\,dx = vx - x$

$$\therefore \int v^2\,dx = v^2\,x - 2vx + 2x$$

$$= x.\,(v^2 - 2v + 2)$$

485. Let $\displaystyle\int \frac{dx}{x^3.(a+x)^2} = u$, and assume $\dfrac{1}{x^2.(a+x)} = v$

$$\therefore dv = d.x^{-2}.(a+x)^{-1} = -2x^{-3}.(a+x)^{-1}\,dx - (a+x)^{-2}x^{-2}dx$$

$$= -2x^{-3}.(a+x)\,(a+x)^{-2}\,dx - (a+x)^{-2}\,x^{-2}\,dx$$

$$= -2a\,x^{-3}.(a+x)^{-2}\,dx - 3\,(a+x)^{-2}\,x^{-2}\,dx$$

$$\therefore v = -2a.u - 3\int (a+x)^{-2}\,x^{-3}\,dx$$

$$\therefore u = -\frac{v}{2a} - \frac{3}{2a}\int (a+x)^{-2}\,x^{-2}\,dx$$

Again, put $\dfrac{1}{x.(a+x)} = v_1$

Then $dv_1 = d.x^{-1}.(a+x)^{-1}$

$$= -x^{-2}.(a+x)^{-1}\,dx - (a+x)^{-2}\,x^{-1}\,dx$$

$$= -a.x^{-2}.(a+x)^{-2}\,dx - 2.(a+x)^{-2}\,x^{-1}\,dx$$

$$\therefore \int x^{-2}.(a+x)^{-2}\,dx = \frac{-v_1}{a} - \frac{2}{a}\int (a+x)^{-2}\,x^{-1}\,dx$$

$$\therefore u = -\frac{v}{2a} + \frac{3v_1}{2a^2} + \frac{3}{a^2}\int (a+x)^{-2}\,x^{-1}\,dx$$

Now, to find $\int (a+x)^{-2}\,x^{-1}\,dx$, put $\dfrac{1}{a+x} = \dfrac{w}{a}$

$$\therefore \frac{-dx}{(a+x)^2} = \frac{dw}{a} \text{ and } x = \frac{a}{w} - a = \frac{a}{w}.\overline{1-w}$$

$$\therefore \int (a+x)^{-2} x^{-1}\, dx = \int \frac{-dw}{a} \cdot \frac{w}{a.(1-w)} = \int \frac{-w\,dw}{a^2(1-w)}$$

$$= -\frac{1}{a^2}\int dw + \frac{1}{a^2}\int \frac{dw}{1-w} = -\frac{1}{a^2}w - \frac{1}{a^2}.\, l.(1-w) = -\frac{1}{a^2} \times$$

$$l.\left(1-\frac{a}{a+x}\right) - \frac{1}{a^2}w = -\frac{1}{a^2}l.\frac{x}{(a+x)} - \frac{1}{a^2} \times \frac{a}{(a+x)}.$$

Hence, we get $u = -\frac{v}{2a} + \frac{3v_1}{2a^2} + \frac{3}{a^2}\left(-\frac{1}{a^2}.\, l.\, \frac{x}{a+x}\right.$

$$\left.-\frac{1}{a^2}.\frac{a}{a+x}\right) = \frac{-1}{2a.\, x^2.(a+x)} + \frac{3}{2a^2.x.(a+x)} - \frac{3}{a^4.(a+x)}$$

$$-\frac{3}{a^4}.\, l.\frac{x}{(a+x)} = \frac{-a^3+3a^2x-6x^2}{2a^4.(a+x)} - \frac{3}{a^4}\, l.\, \frac{x}{(a+x)}$$

Otherwise.

Assume $\frac{1}{x^3.(a+x)^2} = \frac{A}{x^3} + \frac{B}{x^2} + \frac{C}{x} + \frac{P}{(a+x)^2} + \frac{Q}{(a+x)}$ which fractions being reduced to a common denominator, equate the numerators on both sides of the equation. Then let $x = -a$, and we get the value of A; let $x = 0$, and we get P. Next, differentiate, divide by dx, and repeat the operation, which will give B and Q. Differentiate again, and let $x = -a$. Thence we have C. Having found A, B, C, &c., the sum of the integrals of $\frac{A\,dx}{x^3}$, $\frac{B\,dx}{x^2}$, &c., will be the integral required.

Secondly, put $\frac{1}{1-y} = v$, $\therefore y = 1 - \frac{1}{v} = \frac{v-1}{v}$

$$\therefore y^2 + 1 = \frac{2v^2-2v+1}{v^2}$$

$$\text{and } \frac{dy}{(1-y)^2} = dv$$

$$\therefore \int \frac{dy}{(1-y)^2.\sqrt{1+y^2}} = \int \frac{v\,dv}{\sqrt{2v^2-2v+1}} = \frac{1}{\sqrt{2}}\int \frac{v\,dv}{\sqrt{v^2-v+\frac{1}{2}}}$$

Again, let $v - \frac{1}{2} = z$, $\therefore dv = dz$

and $v^2 - v + \frac{1}{4} = z^2$

and $\therefore \sqrt{v^2-v+\frac{1}{2}} = \sqrt{z^2+\frac{1}{4}}$

Hence $\int \frac{dy}{(1-y)^2.\sqrt{1+y^2}} = \frac{1}{\sqrt{2}} \int \frac{(z+\frac{1}{2})\,dz}{\sqrt{z^2+\frac{1}{4}}}$

$$= \frac{1}{\sqrt{2}} \int \frac{zdz}{\sqrt{z^2+\frac{1}{4}}} + \frac{1}{2^{\frac{3}{2}}} \int \frac{dz}{\sqrt{z^2+\frac{1}{4}}}$$

$$= \frac{1}{\sqrt{2}} \sqrt{z^2+\tfrac{1}{4}} + \frac{1}{2^{\frac{3}{2}}} l.\,(z + \sqrt{z^2+\tfrac{1}{4}})$$

But $z^2 + \frac{1}{4} = v^2 - v + \frac{1}{2} = \frac{1}{(1-y)^2} - \frac{1}{1-y} + \frac{1}{2}$

$$= \frac{2 - 2.\overline{1-y} + (1-y)^2}{2.(1-y)^2} = \frac{y^2+1}{2(1-y)^2}$$

and $z = v - \frac{1}{2} = \frac{1}{1-y} - \frac{1}{2} = \frac{1+y}{2.(1-y)}$

$$\therefore \int \frac{dy}{(1-y)^2\sqrt{1+y)^2}} = \frac{1}{2}.\frac{\sqrt{1+y^2}}{1-y} + \frac{1}{2^{\frac{3}{2}}} l.\,\frac{1+y+\sqrt{1+y^2}}{2(1-y)}$$

We learn from the process, that one assumption, viz.,

$z = \frac{1+y}{2.(1-y)}$, would have been sufficient.

Thirdly, let $= v = \frac{1}{x}$

Then $dv = \frac{-dx}{x^2}$

$$\therefore \sqrt{a^2-ax+x^2} = \sqrt{a^2 - \frac{a}{v} + \frac{1}{v^2}} = \frac{\sqrt{a^2v^2-av+1}}{v}$$

$$\therefore \int \frac{adx}{x\sqrt{a^2-ax+x^2}} = \int \frac{-adv}{\sqrt{a^2v^2-av+1}}$$

Again, put $av - \frac{1}{2} = w$

Then $a^2v^2 - av + \frac{1}{4} = w^2$

$$\therefore \sqrt{a^2v^2-av+1} = \sqrt{w^2+\frac{3}{4}}$$

Also $-adv = -dw$

$$\therefore \int \frac{adx}{x\sqrt{a^2-ax+x^2}} = \int \frac{-dw}{\sqrt{w^2+\frac{3}{4}}} = -l.(w+\sqrt{w^2+\tfrac{3}{4}})$$

But $w = av - \frac{1}{2} = \frac{a}{x} - \frac{1}{2} = \frac{2a-x}{2x}$

$$\therefore w^2 + \frac{3}{4} = \frac{4a^2-4ax+x^2}{4x^2} + \frac{3}{4} = \frac{a^2-ax+x^2}{x^2}$$

$$\therefore \int \frac{adx}{x\sqrt{a^2-ax+x^2}} = -l.\frac{2a-x+2\sqrt{a^2-ax+x^2}}{2x}$$

486. Let $y = \frac{1}{x} \therefore x = \frac{1}{y}$

$$\therefore dx = \frac{-dy}{y^2}$$

$$\therefore \frac{dx}{x} = \frac{-dy}{y}$$

Also $a^2 - x^2 = a^2 - \frac{1}{y^2} = \frac{a^2y^2-1}{y^2}$.

$$\therefore \int \frac{d \times dx}{x.(a^2-x^2)} = -d\int \frac{ydy}{a^2y^2-1} = \frac{-d}{2a^2}\,l.(a^2y^2-1)$$

$$= \frac{-d}{2a^2}.l.\frac{a^2-x^2}{x^2} = \frac{d}{a^2}\,l\,x - \frac{d}{2a^2}.l.(a^2-x^2)$$

Again, let $\frac{1}{(a+y)^{\frac{1}{2}}} = u$

$$\therefore \frac{-\frac{1}{2}dy}{(a+y)^{\frac{3}{2}}} = du, \text{ and } y = \frac{1}{u^2} - a = \frac{1-au^2}{u^2}$$

$$\therefore \frac{hdy}{y.(a+y)^{\frac{3}{2}}} = \frac{-2hdu}{y} = \frac{2hu^2du}{au^2-1} = \frac{2h}{a}du + \frac{2hdu}{a(au^2-1)}$$

by division.

$$\therefore \int \frac{hdy}{y.(a+y)^{\frac{3}{2}}} = \frac{2h}{a}u + \frac{2h}{a^2} \times \int \frac{du}{u^2-\frac{1}{a}}$$

Put $\frac{1}{u^2-\frac{1}{a}} = \frac{A}{u+\sqrt{\frac{1}{a}}} + \frac{B}{u-\sqrt{\frac{1}{a}}}$ (since the factors of $u^2 - \frac{1}{a}$ are $u + \sqrt{\frac{1}{a}}$ and $u - \sqrt{\frac{1}{a}}$)

$$\therefore \overline{A+B}\,.\,u + (B-A)\,.\,\sqrt{\frac{1}{a}} = 1$$

$$\therefore A + B = 0$$

$$\therefore A - B = -\sqrt{a}$$

$$\therefore A = -\frac{\sqrt{a}}{2} \text{ and } B = \frac{\sqrt{a}}{2}$$

$$\therefore \int \frac{du}{u^2 - \frac{1}{a}} = \int \frac{\frac{\sqrt{a}}{2}\,du}{u - \sqrt{\frac{1}{a}}} - \int \frac{\frac{\sqrt{a}}{2}\,du}{u + \sqrt{\frac{1}{a}}} = \frac{\sqrt{a}}{2}\,l\,.\,\frac{u - \sqrt{\frac{1}{a}}}{u + \sqrt{\frac{1}{a}}}$$

$$\therefore \int \frac{hdy}{y.(a+y)^{\frac{3}{2}}} = \frac{2h}{a}\,u + \frac{h}{a^{\frac{3}{2}}}\,l\,.\,\frac{u - \sqrt{\frac{1}{a}}}{u + \sqrt{\frac{1}{a}}},$$ which, by substitution, may easily be expressed in terms of y.

487. Let $z + a = u$

Then $z^2 + 2az + 1 = u^2 + 1 - a^2 = u^2 + r^2$, by supposition.

Also $z^2 = u^2 - 2au + a^2$

$$\therefore zdz = udu - adu$$

$$\therefore \frac{zdz}{1 + 2az + z^2} = \frac{udu - adu}{u^2 + r^2} = \frac{udu}{u^2 + r^2} - \frac{adu}{u^2 + r^2}$$

$$= \frac{\frac{udu}{r^2}}{\frac{u^2}{r^2} + 1} - \frac{a}{r} \times \frac{\frac{du}{r}}{\frac{u^2}{r^2} + 1}$$

$$\therefore \int \frac{zdz}{1 + 2az + z^2} = \frac{1}{2}\,l\,.\,\left(\frac{u^2}{r^2} + 1\right) - \frac{a}{r}\,.\,\tan.^{-1}\frac{u}{r}$$

If a be $>$ unity

$1 - a^2$ is negative $= -r^2$, and the integral will be

$$\therefore \int \frac{zdz}{1 + 2az + z^2} = \frac{1}{2}\,l\,.\left(1 - \frac{u^2}{r^2}\right) - \int \frac{adu}{u^2 - r^2}$$

$$\text{Let } \therefore \frac{1}{u^2 - r^2} = \frac{A}{u + r} + \frac{B}{u - r}$$

$$\therefore \overline{A+B}\,.\,u + (B - A)\,r = 1$$

$$\left.\begin{array}{l}\text{or } A + B = 0 \\ \text{and } B - A = \dfrac{1}{r}\end{array}\right\} \quad \therefore A = -\frac{1}{2r} \text{ and } B = \frac{1}{2r}$$

$$\therefore \int \frac{adu}{u^2 - r^2} = -\int \frac{\frac{a}{2r}du}{u+r} + \int \frac{\frac{a}{2r}du}{u-r} = \frac{a}{2r}\, l\,.\, \frac{u-r}{u+r}$$

$$\therefore \text{ the integral in this case} = \frac{1}{2}\, l\,.\left(1 - \frac{u^2}{r^2}\right) - \frac{a}{2r}\, l.\, \frac{u-r}{u+r}$$

See Demoivre's *Miscellanea Analytica*, Lib. III.

Again, put $a^2 + x^2 = u^2$

$$\therefore xdx = udu$$

$$\therefore \int \frac{xdx}{\sqrt{a^2+x^2}} = \int \frac{udu}{u} = \int du = u = \sqrt{a^2+x^2}$$

488. Let $(a + cz^n)^m \times z^{pn+n-1}\, dz = d.P$

and $(a + cz^n)^{m+1} \times z^{pn-1}\, dz = d.Q$

Assume $(a + cz^n)^{m+1} \times z^{pn} = u$

$\therefore du = (m+1)\, n.c\, (a + cz^n)^m\, z^{pn+n-1}\, dz + pn(a+cz^n)^{m+1} \times z^{pn-1}\, dz = m.\, \overline{m + 1}.\, c\, d.P + np.\, d.Q$

$$\therefore dQ = \frac{1}{np}\, du - \frac{m+1}{p}\, cd.P$$

$$\therefore Q = \frac{u}{np} - \frac{m+1}{p}\, cP$$

$$= \frac{(a + c\, z^n)^{m+1} \times z^{pn}}{np} - \frac{m+1}{p}\, c\, P, \text{ whence}$$

Q is known.

Again, put $\dfrac{\sqrt{a^2 + x^2}}{x^2} = u$

$$\therefore du = \frac{x^3\, dx}{x^4 \sqrt{a^2+x^2}} - \frac{2xdx\,.\,\sqrt{a^2 + x^2}}{x^4}$$

$$\therefore \int \frac{dx}{x^3} \sqrt{a^2+x^2} = \frac{1}{2} \int \frac{dx}{x\sqrt{a^2+x^2}} - \frac{u}{2}$$

But $\int \frac{2adx}{x\sqrt{a^2+x^2}} = l.\ \frac{\sqrt{a^2+x^2}-a}{\sqrt{a^2+x^2}+a}$ (*Vince or Lacroix*)

$$\therefore \int \frac{dx}{x^3}\cdot\sqrt{a^2+x^2} = \frac{1}{4a}\ l.\ \frac{\sqrt{a^2+x^2}-a}{\sqrt{a^2+x^2}+a} - \frac{u}{2}$$

$$= \frac{1}{4a}\cdot l.\ \frac{(\sqrt{a^2+x^2}-a)^2}{x^2} - \frac{u}{2}$$

$$= \frac{1}{2a}\ l.\ \frac{\sqrt{a^2+x^2}-a}{x} - \frac{\sqrt{a^2+x^2}}{2x^2}$$

Again, $\frac{z^\theta}{1+mz} = \frac{z^\theta}{mz+1} = \frac{z^{\theta-1}}{m} - \frac{z^{\theta-2}}{m^2} + \frac{z^{\theta-3}}{m^3}$

$-$ &c. $\pm \frac{1}{m^\theta} \mp \frac{1}{m^\theta.(1+mz)}$ by actual division.

$$\therefore \int \frac{z^\theta dz}{1+mz} = \int \frac{z^{\theta-1}dz}{m} - \int \frac{z^{\theta-2}\,dz}{m^2} + \&c. \pm \int \frac{dz}{m^\theta}$$

$$\mp \int \frac{dz}{m^\theta\ .(1+mz)}$$

$$= \frac{z^\theta}{m\theta} - \frac{z^{\theta-1}}{m^2.(\theta-1)} + \frac{z^{\theta-2}}{m^3.(\theta-2)} \ldots \pm \frac{z}{m^\theta}$$

$$\mp \frac{1}{m^{\theta+1}}\ l.\ (1+mz)$$

488. Since $\int u dw = uw - \int w du$

$$\int v.xdx = v \times \frac{x}{2} - \int \frac{x^2}{2}\,dv$$

$$= v \times \frac{x^2}{2} - \int \frac{x^2}{2} \times \frac{dx}{\sqrt{x^2+a^2}}$$

Again put $x\sqrt{x^2+a^2} = y$

$$\therefore dy = dx.\sqrt{x^2+a^2} + \frac{x^2dx}{\sqrt{x^2+a^2}}$$

$$= \frac{2x^2dx}{\sqrt{x^2+a^2}} + \frac{a^2dx}{\sqrt{x^2+a^2}}$$

$$\therefore \frac{x^2dx}{2\sqrt{x^2+a^2}} = \frac{dy}{4} - \frac{a^2}{4} \times \frac{dx}{\sqrt{x^2+a^2}}$$

Hence $\int v.\, xdx = \frac{vx^2}{2} - \frac{y}{4} + \frac{a^2}{4} v$

$$= \frac{vx^2}{2} - \frac{x\sqrt{x^2+a^2}}{4} + \frac{a^2}{4} v$$

$$= \frac{1}{4} (v.\, \overline{2x^2 + a^2} - x\sqrt{x^2+a^2})$$

489. Let $x\sqrt{a-x} = u$

$$\therefore dx.\sqrt{a-x} - \frac{xdx}{2\sqrt{a-x}} = du$$

$$\therefore \frac{xdx}{\sqrt{a-x}} = 2dx.\sqrt{a-x} - 2\,du$$

$$\therefore \int \frac{xdx}{\sqrt{a-x}} = -\frac{4}{3}(a-x)^{\frac{3}{2}} - 2\,u$$

$$= -\frac{4}{3}(a-x)^{\frac{3}{2}} - 2x\sqrt{a-x}$$

$$= -2\sqrt{a-x}\left(\frac{2}{3}a - \frac{2}{3}x + x\right)$$

$$= -2\sqrt{a-x}\left(\frac{2}{3}a + \frac{1}{3}x\right)$$

$$= -\frac{2}{3}\sqrt{a-x}\,(2a+x)$$

Again, $\frac{x^2dx}{a-x} = -\frac{x^2dx}{x-a} = -dx\left(x + a + \frac{a^2}{x-a}\right)$

$$= -xdx - adx - \frac{a^2dx}{x-a}$$

$$\therefore \int \frac{x^2dx}{a-x} = -\frac{x^2}{2} - ax + a^2.\, l.\,(a-x)$$

490. Let $x^{-1}.\sqrt{a-x} = u$

$$\therefore du = -\frac{dx}{x^2}.\sqrt{a-x} - \frac{x^{-1}dx}{2\sqrt{a-x}} = -\frac{adx}{x^2\sqrt{a-x}} + \frac{dx}{x\sqrt{a-x}}$$

$$-\frac{dx}{2x\sqrt{a-x}} = \frac{-adx}{x^2\sqrt{a-x}} + \frac{dx}{2x\sqrt{a-x}}$$

$$\therefore \int\frac{dx}{x^2\sqrt{a-x}} = \frac{1}{2a}\int\frac{dx}{x\sqrt{a-x}} - \frac{u}{a}$$

Let now $a - x = z^2$

$$\therefore dx = -2zdz$$

$$\frac{1}{x} = \frac{1}{a-z^2}$$

$$\therefore \int\frac{dx}{x\sqrt{a-x}} = \int\frac{-2zdz}{(a-z^2)z} = \int\frac{-2dz}{(a-z^2)}$$

$$= \int\frac{2dz}{z^2-a} = \frac{1}{\sqrt{a}}\,l.\frac{z-\sqrt{a}}{z+\sqrt{a}} \text{ by a common form.}$$

$$\therefore \int\frac{dx}{x^2\sqrt{a-x}} = \frac{1}{2a^{\frac{3}{2}}}\,l.\,\frac{z-\sqrt{a}}{z+\sqrt{a}} - \frac{\sqrt{a-x}}{ax}$$

$$= \frac{1}{a^{\frac{3}{2}}}\,l.\,\frac{\sqrt{a}-\sqrt{a-x}}{x} - \frac{\sqrt{a-x}}{ax}$$

Again, $y^{\frac{n}{2}} = u$

$$\therefore \frac{n}{2}\,y^{\frac{n}{2}-1}\,dy = du$$

$$\therefore y^{\frac{3n}{2}-1}\,dy = \frac{2}{n}\,u^2du$$

And $\sqrt{a^n-y^n} = \sqrt{a^n-u^2} = \sqrt{b^2-u^2}$ (by putting $b^2 = a^n$)

$$\therefore \int y^{\frac{3n}{2}-1}dy\,\sqrt{a^n-y^n} = \frac{2}{n}\int u^2\,du\,\sqrt{b^2-u^2}$$

Let $u\,(b^2-u^2)^{\frac{3}{2}} = w$

Then $du \times (b^2-u^2)^{\frac{3}{2}} - 3\,(b^2-u^2)^{\frac{1}{2}}\,u^2du = dw$

or $b^2\,du\,(b^2-u^2)^{\frac{1}{2}} - 4\,(b^2-u^2)^{\frac{1}{2}}\,u^2du = dw$

$$\therefore \int u^2du\,\sqrt{b^2-u^2} = \frac{b^2}{4}\int du\,\sqrt{b^2-u^2} - \frac{w}{4}$$

But if b be the radius of a circle, and u the abscissa measured from the centre (u being supposed less than b), then $\sqrt{b^2 - u^2}$ = the corresponding ordinate, and $\int du \sqrt{b^2 - u^2}$ = that part of the quadrant comprised between the radius $\perp$ line of abscissæ, and the ordinate $\sqrt{b^2 - u^2}$; which being put = A, we have

$$\int y^{\frac{3n}{2}-1} dy \sqrt{a^n - y^n} = \frac{2}{n} \times \frac{b^2}{4} A - \frac{2}{n} \times \frac{w}{4} = \frac{b^2 A}{2n}$$

$$- \frac{u.(b^2 - u^2)^{\frac{3}{2}}}{2n} = \frac{1}{2n} \left\{ a^n \times A - y^{\frac{n}{2}} (a^n - y^n)^{\frac{3}{2}} \right\}$$

Again, put $v = a \dfrac{1-u}{1+u}$

Then $dv = a \times \dfrac{-du.(1+u) - du.(1-u)}{(1+u)^2} = \dfrac{-2adu}{(1+u)^2}$

and $v + a = a.\dfrac{1 - u + 1 + u}{1 + u} = \dfrac{2a}{1+u}$

$\therefore \dfrac{dv}{v+a} = \dfrac{-du}{1+u}$

But $v^2 + a^2 = \dfrac{4a^2}{(1+u)^2} - 2av = \dfrac{4a^2}{(1+u)^2} - 2a^2 \dfrac{1-u}{1+u}$

$$= \frac{4a^2 - 2a^2 + 2a^2 u^2}{(1+u)^2} = \frac{2a^2 (1 + u^2)}{(1+u)^2}$$

$$\therefore \frac{vdv}{(a+v).(a^2+v^2)} = a.\frac{1-u}{1+u} \times \frac{-du}{1+u} \times \frac{(1+u)^2}{2a^2 \times (1+u^2)}$$

$$= -\frac{1}{2a} \times \frac{(1-u)du}{(1+u^2)} = \frac{udu}{2a.(1+u^2)} - \frac{du}{2a.(1+u^2)}$$

$$\therefore \int \frac{vdv}{(a+v).(a^2+v^2)} = \frac{1}{4a} l.(1+u^2) - \frac{1}{2a} \tan.^{-1} u$$

$$= \frac{1}{4a} l. \frac{2.(a^2+v^2)}{(a+v)^2} - \frac{1}{2a} \tan.^{-1} \frac{a-v}{a+v}$$

$$= \frac{1}{4a} l. 2. (a^2 + v^2) - \frac{1}{2a} l.(a+v) -$$

$$\frac{1}{2a} \tan.^{-1} \frac{a-v}{a+v}$$

Otherwise.

Assume $\dfrac{1}{(a+v).(a^2+v^2)} = \dfrac{A + Bv}{a^2 + v^2} + \dfrac{C}{a+v}$, reduce to a

common denominator, equate the sums of the coefficients of the same powers of v in the numerators to zero, and thence determine A, B, and C. Then, multiplying by vdv, and integrating each fractional function, the sum of the integrals will be the integral required.

491. $$xdx\sqrt{a^2-x^2} = -\frac{1}{3} \times (-3\,xdx\sqrt{a^2-x^2})$$
$$= -\frac{1}{3}\,d.\,(a^2-x^2)^{\frac{3}{2}}$$
$$\therefore \int xdx\sqrt{a^2-x^2} = -\frac{1}{3}(a^2-x^2)^{\frac{3}{2}}$$

Again, let $\frac{1}{x} = u^2$

$$\therefore \frac{dx}{x^2} = -2udu$$

And $\frac{dx}{x} = -udu \times x = \frac{-2du}{u}$

Also $x = \frac{1}{u^2}$

$$\therefore \sqrt{x-a} = \sqrt{\frac{1}{u^2}-a} = \sqrt{\frac{1-au^2}{u^2}} = \frac{\sqrt{1-au^2}}{u}$$

$$\therefore \int\frac{dx}{x\sqrt{x-a}} = \int\frac{-2du}{\sqrt{1-au^2}} = \frac{2}{\sqrt{a}}\int\frac{-\sqrt{a}du}{\sqrt{1-au^2}} =$$
$$\frac{2}{\sqrt{a}}\cos.^{-1} a^{\frac{1}{2}} u.$$

Let $a^x = u$

$\therefore x\,l.\,a = l.\,u$

and $dx\,l.\,a = \frac{du}{u}$. $\therefore du = a^x dx \times l.\,a$

Hence $\int a^x dx = \int\frac{du}{l.\,a} = \frac{u}{l.a}$ since $l.\,a$ is constant

$$= \frac{a^x}{l.a}$$

492. Let $(a+x)^{-\frac{1}{2}} = u$

$$\therefore -\frac{1}{2}(a+x)^{-\frac{3}{2}}dx = du.$$

Also $\frac{1}{a+x} = u^2$. $\therefore a + x = \frac{1}{u^2}$

And $\frac{1}{x} = \frac{u^2}{1-au^2}$. $\therefore x^{-\frac{1}{2}} = \frac{u}{\sqrt{1-au^2}}$

$$\therefore \int \frac{a^2 x^{-\frac{1}{2}}dx}{(a+x)^{\frac{3}{2}}} = \int \frac{-2a^2udu}{\sqrt{1-au^2}} = 2a\int \frac{-au\,du}{\sqrt{1-au^2}} =$$

$$2a\sqrt{1-au^2} = 2a\sqrt{\frac{x}{a+x}}$$

Again, let $x^2 + ax + 1 = 0$

Then $x = \frac{-a \pm \sqrt{a^2-4}}{2}$

And $x^2 + ax + 1 = \left(x + \frac{a}{2} - \frac{\sqrt{a^2-4}}{2}\right) \times \left(x + \frac{a}{2} + \frac{\sqrt{a^2-4}}{2}\right)$

Assume $\therefore \frac{1}{x^2+ax+1} = \frac{A}{x+\frac{a}{2}-\frac{\sqrt{a^2-4}}{2}} + \frac{B}{x+\frac{a}{2}+\frac{\sqrt{a^2-4}}{2}}$

$$= \frac{\overline{A+B}.x + A.\left(\frac{a}{2}+\frac{\sqrt{a^2-4}}{2}\right) + B.\left(\frac{a}{2}-\frac{\sqrt{a^2-4}}{2}\right)}{x^2+ax+1}$$

$\therefore A + B = 0$ or $B = -A$

$$\therefore A.\left(\frac{a}{2}+\frac{\sqrt{a^2-4}}{2}\right) - A.\left(\frac{a}{2}-\frac{\sqrt{a^2-4}}{2}\right) = 1$$

Or $A\sqrt{a^2-4} = 1$

$$\therefore A = \frac{1}{\sqrt{a^2-4}}$$

Hence $B = -A = -\frac{1}{\sqrt{a^2-4}}$

$$\therefore \frac{x^n dx}{x^2 + ax + 1} = \frac{x^n dx}{\sqrt{a^2-4}\,.\left(x + \frac{a}{2} - \frac{\sqrt{a^2-4}}{2}\right)} -$$

$$\frac{x^n dx}{\sqrt{a^2-4}\,.\left(x + \frac{a}{2} \frac{\sqrt{a^2-4}}{2}\right)}$$

$$\text{Let } \left.\begin{array}{l} \frac{a}{2} + \sqrt{a^2-4} = b \\ \frac{a}{2} - \sqrt{a^2-4} = c \end{array}\right\} \quad \therefore \sqrt{a^2-4} = \frac{b-c}{2}$$

$$\therefore \frac{x^n dx}{x^2 + ax + 1} = \frac{2}{b-c} \times \frac{x^n dx}{x+c} - \frac{2}{b-c}\,\frac{x^n dx}{x+b}$$

But, by common division $\frac{x^n}{x+c} = x^{n-1} - cx^{n-2} + c^2 x^{n-3} - c^3 x^{n-4} + \&c. \pm c^{n-1} \mp \frac{c^{n-1}}{x+c}$; and $\frac{x^n}{x+b} = x^{n-1} - bx^{n-2} + b^2 x^{n-3} - b^3 x^{n-4} + \&c. \pm b^{n-1} \mp \frac{b^{n-1}}{x+b}$

$$\therefore \frac{x^n dx}{x^2 + ax + 1} = \frac{2}{b-c}\Big\{(b-c)\, x^{n-2} dx - (b^2-c^2)\, x^{n-3} dx +$$

$$(b^3 - c^3)\, x^{n-4} dx + \&c. \mp (b^{n-1} - c^{n-1})\, dx \pm \frac{b^{n-1} dx}{x+b} \mp \frac{c^{n-1} dx}{x-c}\Big\}$$

$$\therefore \int \frac{x^n dx}{x^2 + ax + 1} = \frac{2}{b-c}\left\{\frac{b-c}{n-1} x^{n-1} - \frac{b^2-c^2}{n-2} x^{n-2} + \right.$$

$$\frac{b^3-c^3}{n-3} x^{n-3} - \frac{b^4-c^4}{n-4} x^{n-4} + \&c. \mp (b^{n-1} - c^{n-1})\, x \pm b^{n-1} \times$$

$$\left. l.(x+b) \mp c^{n-1}\, l.(x+c)\right\}$$

493. Assume $\frac{x}{(1+x^2)^{\frac{1}{2}}} = u$

$$\therefore \frac{dx\,.(1+x^2)^{\frac{1}{2}}}{(1+x^2)} - \frac{x^2 dx}{(1+x^2)^{\frac{3}{2}}} = du$$

$$\therefore \int \frac{x^2 dx}{(1+x^2)^{\frac{3}{2}}} = \int \frac{dx}{(1+x^2)^{\frac{1}{2}}} - \int du$$

$= l.(x + \sqrt{1+x^2}) - u$ (see any Elementary Treatise.) $= l.(x + \sqrt{1+x^2}) - \dfrac{x}{(1+x^2)^{\frac{1}{2}}}$

Again, let $x^4 \sqrt{1-x^2} = u$

Then $4x^3dx\sqrt{1-x^2} - \dfrac{x^5dx}{\sqrt{1-x^2}} = \dfrac{4x^3dx}{\sqrt{1-x^2}} - \dfrac{5x^5dx}{\sqrt{1-x^2}} = du$

$$\therefore \frac{x^5dx}{\sqrt{1-x^2}} = \frac{4}{5} \times \frac{x^3dx}{\sqrt{1-x^2}} - \frac{du}{5}$$

Now let $x^2 \sqrt{1-x^2} = w$

Then $2xdx\sqrt{1-x^2} - \dfrac{x^3dx}{\sqrt{1-x^2}} = \dfrac{2xdx}{\sqrt{1-x^2}} - \dfrac{3x^3dx}{\sqrt{1-x^2}} = dw$

$$\therefore \frac{x^3dx}{\sqrt{1-x^2}} = \frac{2}{3}\frac{xdx}{\sqrt{1-x^2}} - \frac{dw}{3}$$

$$\therefore \frac{x^5dx}{\sqrt{1-x^2}} = \frac{8}{15}\frac{xdx}{\sqrt{1-x^2}} - \frac{4}{15}dw - \frac{du}{5}$$

$$\therefore \int \frac{x^5dx}{\sqrt{1-x^2}} = -\frac{8}{15}\sqrt{1-x^2} - \frac{4}{15}x^2\sqrt{1-x^2} - \frac{x^4\sqrt{1-x^2}}{5} + C$$

Let $x = 0$, then $C = \dfrac{8}{15}$

$$\therefore \int \frac{x^5dx}{\sqrt{1-x^2}} = \frac{8}{15} - \frac{8}{15}\sqrt{1-x^2} - \frac{4}{15}x^2\sqrt{1-x^2} - \frac{x^4}{5}\sqrt{1-x^2} = \frac{8}{15}$$

when $x = 1$. i.e. its value between the values of x, 0, and 1.

494. $\dfrac{dy}{(1+y^2)^{\frac{3}{2}}} = \dfrac{dy}{y^3\left(\frac{1}{y^2}+1\right)^{\frac{3}{2}}} = \dfrac{y^{-3}dy}{(y^{-2}+1)^{\frac{3}{2}}}$ where the index of y without the radical is less by unity than that within the radical.

$$\therefore \int \frac{dy}{(1+y^2)^{\frac{3}{2}}} = \frac{1}{(y^{-2}+1)^{\frac{1}{2}}} = \frac{y}{(1+y^2)^{\frac{1}{2}}}.$$

495. Let $\dfrac{1}{\sqrt{a-x}} = u$

Then $\dfrac{\frac{1}{2}\,dx}{(a-x)^{\frac{3}{2}}} = du$

Also $\dfrac{1}{a-x} = u^2$ and $\therefore x = a - \dfrac{1}{u^2}$

$$\therefore x + a = \frac{2au^2 - 1}{u^2}$$

$$\therefore \frac{a^2\,dx}{(a+x).(a-x)^{\frac{3}{2}}} = \frac{2a^2 u^2\,du}{2au^2-1} = \frac{a \times u^2\,du}{u^2 - \frac{1}{2a}}$$

$$= a\,du + \frac{1}{2a} \times \frac{du}{u^2 - \frac{1}{2a}} = a\,du + \frac{b^2\,du}{u^2 - b^2} \left(\frac{1}{2a} = b^2\right)$$

$$\therefore \int \frac{a^2\,dx}{(a+x).(a-x)^{\frac{3}{2}}} = au + \frac{b}{2}\int \frac{2b\,du}{u^2-b^2} = au + \frac{b}{2}\,l.\,\frac{u-b}{u+b}$$

Again, if x be the versed sine z to radius r, $\sqrt{2rx - x^2} = \sin. z$, and vers. $z = r - \cos. z$

$$\therefore dx = -d\cos. z = dz \times \sin. z$$

Hence, by substitution, we get

$$\int \frac{z\,dx}{\sqrt{2rx - x^2}} = \int \frac{z\,dz \times \sin. z}{\sin. z} = \int z\,dz = \frac{z^2}{2}$$

496. Assume $\dfrac{\sqrt{1-x^2}}{x^2} = u$

$$\therefore l.u = \frac{1}{2}\,l.(1-x^2) - 2l.x$$

and $\dfrac{du}{u} = -\dfrac{x\,dx}{1-x^2} - \dfrac{2dx}{x}$

$$= -\frac{2dx}{x.(1-x^2)} + \frac{x\,dx}{1-x^2}$$

$$\therefore\ du = \frac{-2dx}{x^3\sqrt{1-x^2}} + \frac{dx}{x\,.\sqrt{1-x^2}}.$$

Hence $\dfrac{dx}{x^3\sqrt{1-x^2}} = \dfrac{1}{2}\,\dfrac{dx}{x\,.\sqrt{1-x^2}} - \dfrac{du}{2}$

$$\therefore \int \frac{dx}{x^3\sqrt{1-x^2}} = \frac{1}{4}\,l.\frac{1-\sqrt{1-x^2}}{1+\sqrt{1-x^2}} - \frac{\sqrt{1-x^2}}{2x^2}$$

$$= \frac{1}{2}\left(l.\frac{x}{1+\sqrt{1-x^2}} - \frac{\sqrt{1-x^2}}{2x^2}\right)$$

Again, let $y = \sin. z$

Then $dy = dz.\cos. z = dz\sqrt{1-y^2}\quad \therefore\ dz = \dfrac{dy}{\sqrt{1-y^2}}$

$$\therefore\ y^4\,dz = \frac{y^4 dy}{\sqrt{1-y^2}}$$

Assume $y^3\sqrt{1-y^2} = u$

Then $du = 3y^2\,dy\sqrt{1-y^2} - \dfrac{y^4 dy}{\sqrt{1-y^2}} = \dfrac{3y^2\,dy}{\sqrt{1-y^2}}$

$- \dfrac{4y^4\,dy}{\sqrt{1-y^2}}$.

$$\therefore\ \frac{y^4\,dy}{\sqrt{1-y^2}} = \frac{3}{4}.\ \frac{y^2\,dy}{\sqrt{1-y^2}} - \frac{du}{4}$$

Now, let $y\sqrt{1-y^2} = v$

$$\therefore\ dv = dy\sqrt{1-y^2} - \frac{y^2\,dy}{\sqrt{1-y^2}} = \frac{dy}{\sqrt{1-y^2}} - \frac{2y^2\,dy}{\sqrt{1-y^2}}$$

Hence $\dfrac{3}{4}.\ \dfrac{y^2\,dy}{\sqrt{1-y^2}} = \dfrac{3}{8}\,\dfrac{dy}{\sqrt{1-y^2}} - \dfrac{3}{8}\,dv$

And $\displaystyle\int \frac{y^4\,dy}{\sqrt{1-y^2}} = \frac{3}{8}\int \frac{dy}{\sqrt{1-y^2}} - \frac{3}{8}\,v - \frac{u}{4}$

$$= \frac{3}{8}z - \frac{3}{8}\,y\sqrt{1-y^2} - \frac{1}{4}\,y^3\sqrt{1-y^2}$$

$$= \frac{3}{8}z - \frac{1}{16}.\ (3\sin. 2z - \cos. 2z + 1).$$

497. $$\frac{dx}{\sqrt{1-x^4}} = \frac{dx}{\sqrt{1-x^2}\times\sqrt{1+x^2}} = \frac{dx}{\sqrt{1-x^2}}$$
$$\times (1+x^2)^{-\frac{1}{2}}.$$

But $(1+x^2)^{-\frac{1}{2}} = 1 - \frac{1}{2}x^2 + \frac{1}{2}\times\frac{3}{4}x^4 - \frac{1}{2}\times\frac{3}{4}\times\frac{5}{6}x^6 + \&c.\ldots$

$$\therefore \frac{dx}{\sqrt{1-x^4}} = \frac{dx}{\sqrt{1-x^2}} - \frac{1}{2}\times\frac{x^2\,dx}{\sqrt{1-x^2}} + \frac{1}{2}\times\frac{3}{4}\times\frac{x^4\,dx}{\sqrt{1-x^2}} - \&c.\ldots$$

Let now
$$\left.\begin{array}{r}
\dfrac{dx}{\sqrt{1-x^2}} = F_0 \\
\dfrac{x^2\,dx}{\sqrt{1-x^2}} = F_2 \\
\&c. = \&c. \\
\dfrac{x^{2n}\,dx}{\sqrt{1-x^2}} = F_{2n}
\end{array}\right\}
\quad \text{Also} \quad
\left.\begin{array}{r}
\dfrac{1}{2} = C_2 \\
\dfrac{1}{2}\times\dfrac{3}{4} = C_4 \\
\dfrac{1}{2}\times\dfrac{3}{4}\times\dfrac{5}{6} = C_6 \\
\&c. = \&c. \\
\dfrac{1}{2}\times\dfrac{3}{4}\times\ldots\dfrac{2n-1}{2n} = C_{2n}
\end{array}\right\}$$

And assume
$$\left.\begin{array}{r}
x\sqrt{1-x^2} = P_1 \\
x^3\sqrt{1-x^2} = P_3 \\
\&c. = \&c. \\
x^{2n-1}\sqrt{1-x^2} = P_{2n-1}
\end{array}\right\}$$

Then $dP_1 = dx\sqrt{1-x^2} - \dfrac{x^2\,dx}{\sqrt{1-x^2}} = \dfrac{dx}{\sqrt{1-x^2}} - \dfrac{2x^2\,dx}{\sqrt{1-x^2}}$
$= F_0 - 2F_2$

$$\therefore F_2 = \frac{1}{2}\times F_0 - \frac{1}{2}\times dP_1$$

Similarly, $F_4 = \dfrac{3}{4}\times F_2 - \dfrac{1}{4}\times dP_3$

$$F_6 = \frac{5}{6}\times F_4 - \frac{1}{6}\times dP_5$$

$\&c. = \&c.$

$$F_{2n} = \frac{2n-1}{2n}\times F_{2n-2} - \frac{1}{2n}\times dP_{2n-1}$$

Hence, by substituting the different values of F_2, F_4 &c. in $\frac{dx}{\sqrt{1-x^4}}$ expanded, so as to express it in terms of constants, and F_0 (which is known to be the differential of an arc whose sine is x), and dP_1, dP_3, &c, we get

$$F_0 = F_0$$

$$-C_2.F_2 = -\frac{1}{2^2} \times F_0 + C_2 \times \frac{dP_1}{2}$$

$$C_4.F_4 = \frac{1^2 \times 3^2}{2^2 \times 4^2} \times F_0 - \frac{3^2}{4^2} \times C_2 \times \frac{dP_1}{2} - C_4 \times \frac{dP_3}{4}$$

$$-C_6.F_6 = -\frac{1^2 \times 3^2 \times 5^2}{2^2 \times 4^2 \times 6^2} \times F_0 + \frac{3^2 \times 5^2}{4^2 \times 6^2} \times C_2 \times \frac{dP_1}{2} - \frac{5^2}{6^2} \times C_4 \times \frac{dP_3}{4} + C_6 \times \frac{dP_5}{6}$$

&c. = &c.

$\therefore \int \frac{dx}{\sqrt{1-x^4}}$ (= sum of the integrals of the differentials into which it is expanded, or of F_0, $- C_2 F_2 + C_4 F_4 -$ &c.)

$$= \left(1 - \frac{1}{2^2} + \frac{1^2 \times 3^2}{2^2 \times 4^2} - \text{\&c.}\right) \times \int F_0 + \frac{C_2}{2} \times \left(1 - \frac{3^2}{4^2} + \frac{3^2 \times 5^2}{4^2 \times 6^2} - \text{\&c.}\right) \times P_1 - \frac{C_4}{4} \times \left(1 - \frac{5^2}{6^2} + \frac{5^2 \times 7^2}{6^2 \times 8^2} - \text{\&c.}\right) \times P_3 + \frac{C_6}{6} \times \left(1 - \frac{7^2}{8^2} + \frac{7^2 \times 9^2}{8^2 \times 10^2} - \text{\&c.}\right) \times P_5 - \text{\&c.}$$

But since $\int F_0 \left(= \int \frac{dx}{\sqrt{1-x^2}}\right)$ = the arc whose sine $= x$ and radius $= 1$, or $= \sin.^{-1} x$,) when $x = 0$, $\int F_0 = 0$, or $\pm \pi'$, or $\pm 2\pi'$, or $\pm 3\pi'$, &c. $\pm m\pi'$, π' being $= 3.14159$, &c. = the semi-circumference of the circle whose radius is unity. Also, when $x = 0$; P_1, P_2, P_3, &c., being of the form $x^p \sqrt{1-x^2}$, are each $= 0$.

$\therefore$ the correct integral from $x = 0$, to $x = x$, is

$$\left(1 - \frac{1}{2^2} + \text{\&c.}\right) \times \sin.^{-1} x + \frac{C_2}{2} \times \left(1 - \frac{3^2}{4^2} + \text{\&c.}\right) P_1 - \frac{C_4}{4} \times \left(1 - \frac{5^2}{6^2} + \text{\&c.}\right) P_3 + \text{\&c.}$$

or $(1-\frac{1}{2^2}+\&c.)\times(\sin.^{-1}x\mp\pi')+\frac{C_2}{2}(1-\&c.)\ P_1-\frac{C_4}{4}\times$

$(1-\&c.)\ P_3+\&c.$

or $(1-\frac{1}{2^2}+\&c.)\ (\sin.^{-1}x\mp 2\pi')+\frac{C_2}{2}(1-\&c.)\ P_1-\frac{C_4}{4}\times$

$(1-\&c.)\ P_3+\&c.$

or &c. + &c. − &c.

or $(1-\frac{1}{2^2}+\&c.)\ (\sin.^{-1}x\mp m\pi')+\frac{C_2}{2}(1-\&c.)\ P_1-\frac{C_4}{4}\times$

$(1-\&c.)\ P_3+\&c.$

a general value of the integral, which may be expressed in terms of x, by substituting for P_1, P_3, &c. their values, and expanding $\sin.^{-1}x$ according to the powers of x.

To find the particular value required by the problem (that between $x=0$ and 1), let $x=1$, then $\sin.^{-1}x=\frac{\pi'}{2}$, or $=2\pi'+\frac{\pi'}{2}$, $4\pi'+\frac{\pi'}{2}$, &c., or $2r\pi'+\frac{\pi'}{2}$, and P_1, P_3, &c. vanish.

$\therefore$ the general integral of $\frac{dx}{\sqrt{1-x^4}}$ between $x=0$ and 1, is expressed by $(1-\frac{1}{2^2}+\frac{1\times 3^2}{2^2\times 4^2}-\&c.)\times(2r\pi'+\frac{\pi'}{2}\mp m\pi')$

$=(1-\frac{1}{2^2}+\frac{1\times 3^2}{2^2\times 4^2}-\&c.)\times\frac{4r\mp 2m+1}{2}\times\pi'$ (r and m being any integers whatever.)

But $\pi'=3.14159\ \&c.=2\pi$

$$\therefore\ \frac{\pi'}{2}=\pi$$

$\therefore$ the general integral of $\frac{dx}{\sqrt{1-x^4}}$ taken between $x=0$ and 1

$=(1-\frac{1^2}{2^2}+\frac{1^2\times 3^2}{2^2\times 4^2}-\&c.)\times(4r\mp 2m+1)\times\pi$

Let $r=0$ and $m=0$

Then $\int\frac{dx}{\sqrt{1-x^4}}$ between $x=0$ and $1=(1-\frac{1^2}{2^2}+\frac{1^2\times 3^2}{2^2\times 4^2}-\&c.)\ \pi$ the value required.

From a further investigation of the general value of $\int \frac{dx}{\sqrt{1-x^4}}$ many important results may be derived, which, for want of room, we leave to the reader.

498. $\int \frac{2adx}{x\sqrt{a^2+x^2}} = l.\ \frac{\sqrt{a^2+x^2}-a}{\sqrt{a^2+x^2}+a}$ (see *Vince*, *Lacroix*, or any elementary book on the subject.)

Again, to integrate $\frac{rx^{\frac{1}{2}}dx}{\sqrt{r^3-x^3}}$

Let $\frac{x^{\frac{3}{2}}}{r^{\frac{3}{2}}} = u$

Then $\frac{3}{2r^{\frac{3}{2}}} \times x^{\frac{1}{2}}\,dx = du$, and $rx^{\frac{1}{2}}\,dx = \frac{2}{3}\,\gamma^{\frac{5}{2}}\,du$

And $\sqrt{\gamma^3-x^3} = r^{\frac{3}{2}}\sqrt{1-\frac{x^3}{r^3}} = \gamma^{\frac{3}{2}}\sqrt{1-u^2}$

$$\therefore \frac{rx^{\frac{1}{2}}dx}{\sqrt{r^3-x^3}} = \frac{\frac{2}{3}r^{\frac{5}{2}}du}{r^{\frac{3}{2}}\sqrt{1-u^2}} = \frac{2}{3r} \times \frac{du}{\sqrt{1-u^2}}$$

$$\therefore \int \frac{rd^{\frac{1}{2}}dx}{\sqrt{r^3-x^3}} = \frac{2}{3r}\sin.^{-1}u$$ (*Vince*, *Lacroix*, &c.)

499. $\frac{dx}{x} \times (a^2+x^2)^{\frac{5}{2}} = \frac{dx}{x} \times (a^2+x^2)(a^2+x^2)^{\frac{3}{2}} =$

$a^2\frac{dx}{x}(a^2+x^2)^{\frac{3}{2}} + xdx\,(a^2+x^2)^{\frac{3}{2}}$

Again, $\frac{dx}{x}.(a^2+x^2)^{\frac{3}{2}} = \frac{dx}{x}.(a^2+x^2).(a^2+x^2)^{\frac{1}{2}} =$

$a^2\,\frac{dx}{x}.\ (a^2+x^2)^{\frac{1}{2}} + x\,dx\ .\ (a^2+x^2)^{\frac{1}{2}}$

Again, $\frac{dx}{x}.(a^2+x^2)^{\frac{1}{2}} = \frac{dx}{x}.\frac{(a^2+x^2)}{(a^2+x^2)^{\frac{1}{2}}} = \frac{a^2\,dx}{x\sqrt{a^2+x^2}}$

$+ \frac{xdx}{\sqrt{a^2+x^2}}$

$$\therefore \int \frac{dx}{x} \times (a^2+x^2)^{\frac{5}{2}} = \int xdx\,(a^2+x^2)^{\frac{3}{2}} + a^2 \int xdx\,.\,(a^2+x^2)^{\frac{1}{2}}$$

$$+\ a^4 \int \frac{dx}{x\sqrt{a^2 \times x^2}} + a^2 \int \frac{xdx}{\sqrt{a^2+x^2}} = \frac{1}{5}.\,(a^2+x^2)^{\frac{5}{2}} + \frac{1}{3}a^2 \times$$

$$(a^2+\ x^2)^{\frac{3}{2}} + \frac{a^4}{2a}\,l.\,\frac{\sqrt{a^2+x^2}-a}{\sqrt{a^2+x^2}+a} + a^2\ \sqrt{a^2+x^2} = \sqrt{a^2+x^2}$$

$$\times \left\{\frac{1}{5} \times (a^2+x)^2 + \frac{1}{3}\,a^2.\,(a^2+x^2) + a^2\right\} + a^3 \times l.\,\frac{x}{\sqrt{a^2+x^2}+a}$$

$$= \left(\frac{8}{16}a^4 + \frac{11}{15}\,a^2x^2 + a^2 + \frac{1}{5}x^4\right)\sqrt{a^2+x^2} + a^3 \times l.\frac{x}{\sqrt{a^2+x^2}+a}$$

To integrate $V^2\,dx$, let $1+x=u$

Then $dx = du$

And $V^2 dx = (l.\,u)^2 \times du$

Hence $\int V^2 dx = u \times (l.\,u)^2 - \int u \times d.\,(l.\,u)^2$

But $ud.\,(l.\,u)^2 = 2u\,l.\,u \times d.\,(l.\,u) = 2\,l.\,u \times d.\,u$

$$\therefore \int ud.\,(l.u)^2 = 2\int l.\,u \times du = 2u\,l.\,u - 2\int ud.\,l.\,u$$

$$= 2u.\,lu - 2\int du = 2u\,.\,lu - 2u$$

$$\therefore \int V^2 dx = u\,(l.\,u)^2 - 2u.\,l.\,u + 2u$$

$$= u\,(l.\,u \times \overline{l.\,u-2} + 2)$$

$$= (1+x)\{l.\,(1+x) \times \overline{l.\,(1+x)-2} + 2\}$$

To integrate $X\,x^2dx$, we have

$$\tan.\,X = \sqrt{\frac{x}{r}}$$

$$\therefore \frac{1}{2}\,\frac{dx}{\sqrt{rx}} = d\,\tan.\,X = \frac{dX}{1+\tan.^2 X} = \frac{rdX}{r+x}$$

Hence $dX = \dfrac{r+x}{2r^{\frac{3}{2}}} \times \dfrac{dx}{\sqrt{x}} = \dfrac{dx}{2r^{\frac{1}{2}}\sqrt{x}} + \dfrac{\sqrt{x}\,dx}{2r^{\frac{3}{2}}}$

But, by the form $\int udv = uv - \int vdu$, we have $\int Xx^2dx =$

$$X \times \frac{x^3}{3} - \int \frac{x^3}{3}\,dX$$

$$\therefore \int Xx^2dx = \frac{x^3 X}{3} - \int \frac{x^3}{3}\left(\frac{dx}{2r^{\frac{1}{2}}\sqrt{x}} + \frac{\sqrt{x}\,dx}{2r^{\frac{3}{2}}}\right)$$

$$= \frac{x^3 X}{3} - \int \frac{x^{\frac{5}{2}}dx}{6r^{\frac{1}{2}}} - \int \frac{x^{\frac{7}{2}}dx}{6r^{\frac{3}{2}}}$$

$$= \frac{x^3 X}{3} - \frac{x^{\frac{7}{2}}}{3 \times 7r^{\frac{1}{2}}} - \frac{x^{\frac{9}{2}}}{3 \times 9r^{\frac{3}{2}}}$$

$$= \frac{x^3}{3} \times \left(X - \frac{x^{\frac{1}{2}}}{7r^{\frac{1}{2}}} - \frac{x^{\frac{3}{2}}}{9r^{\frac{3}{2}}}\right)$$

$$= r^3 \tan.^6 X \times \left(X - \frac{\tan. X}{7} - \frac{\tan.^3 X}{9}\right)$$

500. Let $x\sqrt{a^2+x^2} = u$

Then $du = dx\sqrt{a^2+x^2} + \dfrac{x^2dx}{\sqrt{a^2+x^2}} = \dfrac{a^2dx}{\sqrt{a^2+x^2}} + \dfrac{2x^2dx}{\sqrt{a^2+x^2}}$

$$\therefore \int \frac{x^2dx}{\sqrt{a^2+x^2}} = \int \frac{du}{2} - \int \frac{a^2dx}{2\sqrt{a^2+x^2}} = \frac{u}{2} -$$

$$\frac{a^2}{2}\int \frac{dx}{\sqrt{a^2+x^2}} = x\frac{\sqrt{a^2+x^2}}{2} - \frac{a^2}{2}\,l.\,(x + \sqrt{a^2+x^2})$$

To integrate $\dfrac{dz}{1+2az+z^2}$ assume

$z + a = u$

$\therefore z^2 + 2az + 1 = u^2 + 1 - a^2$

Put $1 - a^2 = b^2$, a positive quantity, because a is less than unity.

Then $\dfrac{dz}{1+2az+z^2} = \dfrac{du}{u^2+b^2}$ whose integral, by a common form is $\dfrac{1}{b^2} \times \tan.^{-b} u$, i.e.

$\displaystyle\int \frac{dz}{1+2az+z^2} = \frac{1}{b^2} \times$ (the arc whose radius is b and tangent $a+z$)

501. $$\frac{dz}{z + 2az^2 + z^3} = \frac{dz}{z} \times \frac{1}{1 + 2az + z^2}$$

$$= \frac{dz}{z} \times \left\{1 - \frac{2az + z^2}{1 + 2az + z^2}\right\} = \frac{dz}{z} - dz\,\frac{a + a + z}{1 + 2az + z^2}$$

$$= \frac{dz}{z} - \frac{adz}{1 + 2az + z^2} + \frac{adz + zdz}{1 + 2az + z^2}$$

$$\therefore \int \frac{dz}{z + 2az^2 + z^3} = l.z - \frac{a}{b^2} \times \tan.^{-b} (a + z) - \frac{1}{2} \times$$

$l.\ (1 + 2az + z^2)$ by the preceding problem, and common forms.

$$\therefore \int \frac{dz}{z + 2az^2 + z^3} = l.z - l.\sqrt{1 + 2az + z^2} - \frac{a}{1 - a^2} \times$$

$\tan.^{-\sqrt{1-a^2}} (a + z)$.

To construct this integral, let AMm, Fig. 56, be an hyperbola whose asymptotes are CR, CT.

Take CB $=$ BA (BA being $\perp$ CT) $= 1$

CP $= x$

PM $= y$

Then, by the property of the hyperbola, we have

$$\text{CP} \times \text{PM} = \text{CB} \times \text{BA} = 1 \;\therefore\; y = \frac{1}{x}$$

Hence the differential of the area AB PM $= ydx = \dfrac{dx}{x}$, and $\therefore$

the area itself (ABPM) $= \displaystyle\int \frac{dx}{x} = l.\,x + \text{C}$

But when ABPM $= 0$, $x = 1$, and $l.\,1 = 0$

$\therefore$ C $= 0$

$\therefore$ the area ABPM $= l.\,x$

Let $x = z$, and ABPM will represent the $l.\,z$

Let $x = \sqrt{1 + 2az + z^2} =$ Cp which is greater than z or CP since a is less than unity.

And the area AB $pm = l.\,\sqrt{1 + 2az + z^2}$

$\therefore l.\,z - l.\,\sqrt{1 + 2az + z^2} =$ ABPM $-$ AB $pm = -$ MP pm.

Again, with C as centre and radius $=$ CB $= 1$ describe the semicircle bNB; take C$n = a$ and draw nN $\perp$ CT; join NP, and with N as centre and radius Nn describe the circular arc nr.

Then N$n^2 = bn \times n$B $= \overline{1 - a} \,.\, \overline{1 + a} = 1 - a^2$

And $nP = nC + CP = a + z = Nn \times \tan.\ nr$

$$\therefore\ nr = \tan.^{-\sqrt{1-a^2}}(a+z)$$

$$\therefore\ \frac{a}{1-a^2} \times \tan.^{-\sqrt{1-a^2}}(a+z) = \frac{Cn}{Nn^2} \times nr$$

$$\therefore \int \frac{dz}{z + 2az^2 + z^3} = \frac{Cn}{Nn^2} \times nr - MP\,pm$$

502. $$\int dx = x + c_1$$

$$\int dx \int dx = \frac{x^2}{2} + c_1 x + c_2$$

$$\int dx \int dx \int dx = \frac{x^3}{2.3} + c_1 . \frac{x}{2} + c_2 x + c_3$$

$$\&c. = \&c.$$

and, generally,

$$\int dx \int dx \ldots . n \text{ factors} = \frac{x^n}{2.3.4 \ldots . n} + c_1 . \frac{x^{n-1}}{2.3 \ldots . n-1}$$

$$+ c_2 . \frac{x^{n-2}}{2.3 \ldots . n-2} + \&c. + c_{n-1} x + c_n.$$

To integrate $x^a a^x dx$ we have the form $\int u dv = uv - \int v du$, which gives us, by separating the algebraic from the exponential part,

$$\int x^a \times a^x dx = x^a \times \int a^x dx - \int a x^{a-1} dx \int a^x dx$$

But since $d . a^x = l . a \times a^x du$

$$\therefore\ a^x dx = \frac{d . a^x}{l . a}$$

And $\int a^x dx = \frac{a^x}{l . a}$, hence, and by substitution, we have

$$\int x^a \times a^x dx = \frac{x^a \times a^x}{l . a} - \int \frac{a}{l . a} x^{a-1} a^x dx$$

Similarly $\int x^{a-1} \times a^x dx = \frac{x^{a-1} \times a^x}{l . a} - \frac{a-1}{l . a} \int x^{a-2} a^x dx$

$$\&c. = \&c.$$

$\therefore$ by successive substitutions, we get

$$\int x^a \times a^x dx = \frac{x^a \times a^x}{l.a} - \frac{a}{(l.a)^2} \times x^{a-1}a^x + \frac{a \times (a-1)}{(l.a)^3} x^{a-2}a^x$$

$$-\frac{a \times (a-1)(a-2)}{(l.a)^4} x^{a-3}a^x + \ldots \pm \frac{a \times (a-1) \times (a-2) \times \ldots a - \overline{p-2}}{(l.a)^p}$$

$\times\, x^{a-p+1}\, a^x \mp$ &c., which we must continue until we arrive at a differential whose integral is known, and thus obtain the integral required.

If a be a positive integer, the series will terminate in the $(a+1)^{th}$ term; because that term will be

$$\pm \frac{a.\,(a-1).\,(a-2) \times \ldots\ldots a - (a+1-2)}{(l.a)^{a+1}} \times x^{a-(a+1)+1}\, a^x$$

$$= \frac{a.(a-1)\ldots.2 \times 1}{(l.a)^{a+1}}\, x^0 a^x = \frac{a.(a-1)\ \ldots.\ (2 \times 1)}{(l.a)^{a+1}}\, a^x, \text{ and the}$$

coefficients of the subsequent terms have each a factor $= 0$.

To integrate $\dfrac{dx}{(x-a)^2 \,.\, (x-b)^3}$

Assume $\dfrac{1}{(x-b)^2} = y^2$

Then $\dfrac{-2dx}{(x-b)^3} = 2y\, dy$

Also $x - b = \dfrac{1}{y}$

$$\therefore x - a = b - a + \frac{1}{y} = \frac{(b-a)\,.\,y + 1}{y}$$

$$\text{And } \therefore \frac{1}{x-a} = \frac{y}{(b-a).y+1} = \frac{1}{b-a} \cdot \frac{y}{y + \frac{1}{b-a}}$$

$$\therefore \int \frac{dx}{(x-a)^2.(x-b)^3} = \int \frac{-y^2\, dy}{(b-a).\left(y + \frac{1}{b-a}\right)} = \int \frac{y dy}{a-b}$$

$$-\int \frac{dy}{(a-b)^2} + \frac{1}{(a-b)^3} \times \int \frac{dy}{y - \frac{1}{a-b}} \text{ (by common division)}$$

which are known forms, giving $\displaystyle\int \frac{dx}{(x-a)^2 .\, (x-b)^3} = \frac{y^2}{2\,.\,(a-b)}$

$$-\frac{y}{(a-b)^2} + \frac{1}{(a-b)^3} \times l.\left(y - \frac{1}{a-b}\right)$$

$$=\frac{1}{2(a-b).(x-b)^2}-\frac{1}{(a-b)^2.(x-b)}+\frac{1}{(a-b)^3}\times l.\left(\frac{1}{x-b}-\frac{1}{a-b}\right)$$

$$=\frac{a+b-2x}{2.(a-b)^2.(x-b)^2}+\frac{1}{(a-b)^3}.l.\frac{a-x}{(x-b).(a-b)},$$ which may be further reduced if necessary.

Otherwise.

Assume $\frac{1}{(x-a)^2.(x-b)^3}=\frac{A}{(x-a)^2}+\frac{B}{x-a}+\frac{P}{(x-b)^3}$ $\frac{Q}{(x-b)^2}+\frac{R}{x-b}=\{A\times(x-b)^3+(x-a).(x-b)^3.B+(x-a)^2.P+(x-a)^2.(x-b)Q+(x-a)^2.(x-b)^2.R\}\times\frac{1}{(x-a)^2.(x-b)^3}$

$\therefore$ (1.) $A.(x-b)^3+(x-a).(x-b)^3B+(x-a)^2.P+(x-a)^2\times(x-b)Q+(x-a)^2.(x-b)^2.R=1$

$$\left.\begin{aligned}&\text{Let } x=a\\&\text{Then } A(a-b)^3=1\\&\therefore A=\frac{1}{(a-b)^3}\end{aligned}\right\}\qquad\left.\begin{aligned}&\text{Let } x=b\\&\text{Then } (b-a)^2.P=1\\&\therefore P=\frac{1}{(b-a)^2}\end{aligned}\right\}$$

Again, differentiate the above equation (1) and divide by dx; the result will give, by the substitutions a and b for x, the values of B and Q respectively. Differentiate again, and substitute b for x in the result; we hence obtain R.

Now, having found A, B, P, Q, R, we shall have split the differential into five others, whose integrals are each known; $\therefore$ &c.

503. $$\frac{xdx}{\sqrt{2ax-x^2}}=\frac{adx-adx+xdx}{\sqrt{2ax-x^2}}=\frac{adx}{\sqrt{2ax-x^2}}-\frac{adx-xdx}{\sqrt{2ax-x^2}}=d.(\text{vers.}^{-1}x)-d.\sqrt{2ax-x^2}$$

$$\therefore\int\frac{xdx}{\sqrt{2ax-x^2}}=\text{vers.}^{-1}x-\sqrt{2ax-x^2}$$

To integrate $\frac{dx}{1-x^4}$

Assume $\frac{1}{x^4-1}\left(=\frac{1}{(x^2-1)(x^2+1)}=\frac{1}{(x-1).(x+1).(x^2+1)}\right)$

$$=\frac{A}{x-1}+\frac{B}{x+1}+\frac{Px+Q}{x^2+1}$$

$$=\frac{A.(x+1).(x^2+1)+B.(x-1).(x^2+1)+Px.(x-1).(x+1)+Q.(x-1)(x+1)}{x^4-1}$$

$$\therefore A(x+1).(x^2+1)+B.(x-1).(x^2+1)+P.x.(x^2-1)+Q.(x^2-1)=1$$

$$\left.\begin{array}{r}\text{Let } x=1\\ \text{Then } 4A=1\\ \text{And } A=\frac{1}{4}\end{array}\right\}\left.\begin{array}{r}\text{Let } x=-1\\ \text{Then } -4B=1\\ \therefore B=-\frac{1}{4}\end{array}\right\}\left.\begin{array}{r}\text{Let } x=\sqrt{-1}\\ \text{Then } -2P\sqrt{-1}-2Q=1\\ \therefore P=0 \text{ and } Q=-\frac{1}{2}\end{array}\right\}$$

$$\therefore \int\frac{dx}{1-x^4}=-\frac{1}{4}\int\frac{dx}{x-1}+\frac{1}{4}\int\frac{dx}{x+1}+\frac{1}{2}\int\frac{dx}{x^2+1}$$

$$=-\frac{1}{4}.l.(x-1)+\frac{1}{4}l\ (x+1)+\frac{1}{2}\tan.^{-1}x$$

$$=-\frac{1}{4}l.\frac{x+1}{x-1}+\frac{1}{2}\tan.^{-1}x$$

For the third integral and construction, see *Demoivre*, or *Dealtry*, p. 297.

504. $x.\sqrt{a^2+x^2}=u$

$$\therefore dx.\sqrt{a^2+x^2}+\frac{x^2\,dx}{\sqrt{a^2+x^2}}=du$$

$$\text{or }\frac{a^2\,dx}{\sqrt{a^2+x^2}}+\frac{2x^2\,dx}{\sqrt{a^2+x^2}}=du$$

$$\therefore \int\frac{x^2dx}{\sqrt{a^2+x^2}}=\frac{u}{2}-\frac{a^2}{2}\int\frac{dx}{\sqrt{a^2+x^2}}$$

$$=\frac{x.\sqrt{a^2+x}}{2}-\frac{a^2}{2}.l.(x+\sqrt{x^2+a^2})$$

505. Let $x^2 . (a^2+x^2)^{\frac{5}{2}} = u$

$$\therefore\ du = 2xdx . (a^2+x^2)^{\frac{5}{2}} + 5x^3dx . (a^2+x^2)^{\frac{3}{2}}$$

$$= 2a^2xdx . (a^2+x^2)^{\frac{3}{2}} + 7x^3dx\, (a^2+x^2)^{\frac{3}{2}}$$

$$\therefore \int x^3dx . (a^2+x^2)^{\frac{3}{2}} = \int \frac{du}{7} - \int \frac{2a^2}{7} xdx . (a^2+x^2)^{\frac{3}{2}}$$

$$= \frac{u}{7} - \frac{2a^2}{5\times 7} (a^2+x^2)^{\frac{5}{2}}$$

$$= \frac{(a^2+x^2)^{\frac{5}{2}}}{7} \times \left(x^2 - \frac{2a^2}{5}\right)$$

To integrate $\dfrac{dz}{z^n . (a+bz)}$, let $z = \dfrac{1}{u}$

Then $dz = \dfrac{-du}{u^2}$

$$z^n = \frac{1}{u^n}$$

$$a + bz = a + \frac{b}{u} = \frac{au+b}{u}$$

$$\therefore\ \frac{dz}{z^n . (a+bz)} = \frac{-du}{u^2} \times u^n \times \frac{u}{au+b}$$

$$= \frac{-u^{n-1}du}{au+b} = -\frac{1}{a} \times \frac{u^{n-1}du}{u+\frac{b}{a}}$$

$$= -\frac{1}{a}\left(u^{n-2}du - \frac{b}{a} . u^{n-3}du + \frac{b^2}{a^2} . u^{n-4}du + \ldots . \pm \frac{b^{n-2}}{a^{n-2}} . du \right.$$

$\left. \mp \dfrac{b^{n-1}}{a^{n-1}} \dfrac{du}{u+\frac{b}{a}} \right)$ by actual division.

$$\therefore \int \frac{dz}{z^n.(a+bz)} = -\frac{1}{a}\left(\frac{u^{n-1}}{n-1} - \frac{b}{a} . \frac{u^{n-2}}{n-2} + \frac{b^2}{a^2} . \frac{u^{n-3}}{n-3} \ldots\ldots\right.$$

$$\left. \pm \frac{b^{n-2}}{a^{n-2}} . u \mp \frac{b^{n-1}}{a^{n-1}} . \ l . \left(u + \frac{b}{a}\right)\right) = -\frac{1}{a} \left\{ \frac{1}{(n-1)z^{n-1}} - \right.$$

$$\left. \frac{b}{(n-2) . az^{n-2}} + \frac{b^2}{(n-3)a^2 . z^{n-3}} \ldots . \pm \frac{b^{n-2}}{a^{n-2}z} \mp \frac{b^{n-1}}{a^{n-1}} l . \left(\frac{a+bz}{az}\right) \right\}$$

506. $\int y dx = xy - \int x dy$ (for $d(xy) = ydx + xdy$) and $\int xdy = \int xdx \times \frac{dy}{dx}$, where, if x and y are functions of the same variable, $\frac{dy}{dx}$ is an integral.

$\therefore$ *in this case,* $\int xdy = \frac{x^2}{2} \cdot \frac{dy}{dx} - \int \frac{x^2}{2} \times d \cdot \frac{dy}{dx} = \frac{x^2}{2} \cdot \frac{dy}{dx}$
$- \int \frac{x^2}{2} \times \frac{d^2y}{dx}$ since dx is constant.

Again $\int \frac{x^2}{2} \cdot \frac{d^2y}{dx} = \int \frac{x^2dx}{2} \times \frac{d^2y}{dx^2}$

$$= \frac{x^3}{2\times 3} \times \frac{d^2y}{dx^2} - \int \frac{x^3}{2\times 3} \times \frac{d^3y}{dx^2}$$

Similarly $\int \frac{x^3}{2\times 3} \times \frac{d^3y}{dx^2} = \frac{x^4}{2\times 3\times 4} \times \frac{d^3y}{dx^3} - \int \frac{x^4}{2\times 3\times 4} \times \frac{d^4y}{dx^3}$

&c. = &c.

$\therefore$ by substitution, we get

$$\int ydx = xy - \frac{dy}{dx} \times \frac{x^2}{1.2} + \frac{d^2y}{dx^2} \times \frac{x^3}{1.2.3} - \text{\&c.} \ldots\ldots$$

507. Assume $x^2 . \sqrt{a^2 - x^2} = u$

$$\therefore 2xdx \sqrt{a^2 - x^2} - \frac{3x^3dx}{\sqrt{a^2 - x^2}} = du$$

$$\text{or } \frac{2a^2xdx}{\sqrt{a^2 - x^2}} - \frac{3x^3dx}{\sqrt{a^2 - x^2}} = du$$

$$\therefore \int \frac{x^3dx}{\sqrt{a^2 - x^2}} = \frac{2a^2}{3} \int \frac{xdx}{\sqrt{a^2 - x^2}} - \int \frac{du}{3}$$

$$= -\frac{2a^2}{3} . \sqrt{a^2 - x^2} - \frac{u}{3}$$

$$= -\sqrt{a^2 - x^2} . \frac{2a^2 + x^2}{3}$$

508. To integrate $\frac{bxdx}{\sqrt{a+x}}$, put $x\sqrt{a+x} = u$

Then $du = dx.\sqrt{a+x} + \dfrac{\frac{1}{2}xdx}{\sqrt{a+x}}$

$$= \frac{adx}{\sqrt{a+x}} + \frac{\frac{3}{2}xdx}{\sqrt{a+x}}$$

$$\therefore \frac{bxdx}{\sqrt{a+x}} = \frac{2bdu}{3} - \frac{2abdx}{3\sqrt{a+x}}$$

$$\therefore \int \frac{bxdx}{\sqrt{a+x}} = \frac{2bx\sqrt{a+x}}{3} - \frac{4}{3}ab\sqrt{a+x}$$

$$= \frac{26}{3}.\sqrt{a+x} \times (x-2a)$$

To integrate $\dfrac{2ax^2dx}{a^2-x^2} = -2a.\dfrac{x^2dx}{x^2-a^2}$, divide the numerator by the denominator

Whence $\displaystyle\int \frac{2ax^2dx}{a^2-x^2} = -2a\int dx - a^2\int\frac{2adx}{x^2-a^2}$.

$$= -2ax - a^2.l.\frac{x-a}{x+a}, \text{ by a common form.}$$

509. Let $(a+cz^n)^{m+1} \times z^{pn} = P_0$

Then $\dfrac{dP_0}{dz} = (m+1)\,n.cz^{n+pn-1} \times (a+cz^n)^m + pn\,z^{pn-1} \times$ $(a+cz^n)^{m+1} = n.(m+1).cz^{pn+n-1} \times (a+cz^n)^m. + npa\,z^{pn-} \times$ $(a+cz^n)^m + pnc\,z^{p.+n-1}\,(a+ez)^m$

$\therefore \{n.(m+1)c + pnc\}\,z^{pn+n-1}.(a+cz^n)^m dz = dP_0 - pna. \times$ $z^{pn-1}.(a+cz^n)^m dz$

For the greater perspicuity, let

$$\left.\begin{aligned} \int z^{pn-1}.(a+cz^n)^m dz &= F_0 \\ \int z^{pn+n-1}.(a+cz^n)^m dz &= F_1 \\ \int z^{pn+2n-1}.(a+cz^n)^m dz &= F_2 \\ \&c. &= \&c. \\ \int z^{pn+vn-1}.(a+cz^n)^m dz &= F_v \end{aligned}\right\} \text{and } \left.\begin{aligned} (a+cz^n)^{m+1}.z^{pn} &= P_0 \\ (a+cz^n)^{m+1}.z^{pn+n} &= P_1 \\ (a+cz^n)^{m+1}z^{pn+m} &= P_2 \\ \&c. &= \&c. \\ (a+cz^n)^{m+1}z^{pn+\overline{v-1}.n} &= P_{v-1} \end{aligned}\right\}$$

Then, we have, $cn.(m+p+1)\,F_1 = P_0 - nap.F_0$

Again $\frac{dP_1}{dz} = n.(m+1).c.z^{p+2n-1} \times (a+cz^n)^m + (pn+n) \times$
$z^{pn+n-1}(a+cz^n)^{m+1} = \{n.(m+1)c + (pn+n)c\} z^{pn+2n-1} \times$
$\times (a+cz^n)^m + n.(p+1).az^{pn+n-1}(a+cz^n)^m$

Hence $cn.(m+p+2)F_2 = P_1 - na(p+1).F_1$

Similarly $cn.(m+p+3)F_3 = P_2 - na(p+2).F_2$

$$\&c. = \&c.$$

$$cn.(\overline{m+p+v-1})F_{v-1} = P_{v-2} - na.(p+v-2).F_{v-2}$$

$$cn.(m+p+v)\ F_v = P_{v-1} - na.(p+v-1)\ F_{v-1}$$

Hence $F_v = \frac{P_{v-1}}{cn.(m+p+v)} - \frac{a}{c}.\frac{p+v-1}{m+p+v}.F_{v-1}$; (let $m+p+v=w$ and $p+v=s$)

$$= \frac{P_{v-1}}{cn \times w} - \frac{a.(s-1)}{c.w}.(P_{v-2} - na.\overline{s-2}.F_{v-2}) \times \frac{1}{cn \times (w-1)}$$

$$= \frac{P_{v-1}}{cn \times w} - \frac{a.(s-1)}{c^2.nw(w-1)} \times P_{v-2} + \frac{a^2.(s-1).(s-2)}{c^2w.(w-1)}F_{v-2}$$

But $F_{v-2} = \frac{P_{v-3}}{cn.(w-2)} - \frac{a.(s-3)}{c.(w-2)}F_{v-3},\quad F_{v-3} = \&c.$

$$\therefore F_v = \frac{1}{cnw}\left\{\frac{P_{v-1}}{1} - \frac{a.(s-1)}{c.(w-1)}P_{v-2} + \frac{a^2(s-1)(s-2)}{c^2.(w-1).(w-2)}P_{v-3}\right.$$

$$- \frac{a^3.(s-1).(s-2).(s-3)}{c^3.(w-1).(w-2).(w-3)}.P_{v-4} + \&c.....$$

$$\pm \frac{a^{v-1}.(s-1).(s-2)....(s-v+1)}{c^{v-1}.(w-1).(w-2)....(w-v+1)}.P_0$$

$$\left.\mp \frac{na^v.(s-1).(s-2)....(s-v)}{c^{v-1}.(w-1).(w-2)...(w-v+1)}.F_0\right\}$$ the law of continuation being evident.

But P_{v-1}, P_{v-2}, &c. are known, and F_0 is given

$\therefore F_v = \int z^{pn+n-1}(a+cz^n)^m dz$ is expressed in known terms.

Again, let $pn + vn - 1 = q$

And assume

$$\left.\begin{aligned} \int z^q.(a+cz^n)^{m+1}dz &= M_1 \\ \int z^q.(a+cz^n)^{m+2}dz &= M_2 \\ \&c. &= \&c. \\ \int z^q.(a+cz^n)^{m+r}dz &= M_r \end{aligned}\right\} \left\{\begin{aligned} z^{q+1}.(a+cz^n)^{m+1} &= Q_1 \\ z^{q+1}.(a+cz^n)^{m+2} &= Q_2 \\ \&c. &= \&c. \\ z^{q+1}.(a+cz^n)^{m+r} &= Q_r \end{aligned}\right.$$

Then $dQ_1 = (q+1).(a+cz^n)^{\overline{m+1}}z^q dz + n.(m+1).cz^{q+n}dz(a+cz^n)^m$

But $cz^{q+n}(a+cz^n)^m = z^q(a+cz^n)^m.cz^n = z^q.(a+cz^n)^m(cz^n+a-a)$
$= z^q.(a+cz^n)^{\overline{m+1}} - a.z^q(a+cz^n)^m$

$\therefore dQ_1 = (q+1+n.\overline{m+1}).(a+cz^n)^{\overline{m+1}}z^q dz - an.(m+1)\times$
$(a+cz^n)^m z^q dz$

$\therefore Q_1 = (n.\overline{m+1}+q+1)M_1 - an.(m+1).F_v$

Similarly $Q_2 = (n.\overline{m+2}+q+1)M_2 - an.(m+2)M_1$

$Q_3 = (n.\overline{m+3}+q+1)M_3 - an.(m+3)M_2$

&c. = &c.

$Q_r = (n.\overline{m+r}+q+1)M_r - an.(m+r)M_{r-1}$

Let $nm+nr+q+1 = w'$ and $m+r = s'$

$$\text{Then } M_r = \frac{Q_r}{w'} + an.\frac{s'}{w'}M_{r-1}$$

$$M_{r-1} = \frac{Q_{r-1}}{w'-n} + an.\frac{s'-1}{w'-n}M_{r-2}$$

$$M_{r-2} = \frac{Q_{r-2}}{w'-2n} + an.\frac{s'-2}{w'-2n}M_{r-3}$$

&c. = &c.

$$M_1 = \frac{Q_1}{w'-\overline{r-1}.n} + an.\frac{s'-\overline{r-1}}{w'-\overline{r-1}.n}\times F_v$$

Hence by successive substitutions M_r may be expressed in terms of F_v and other known quantities; which, from having already extended this problem beyond its due limits, we leave to the reader.

Continuations of this kind are chiefly of use in exhibiting integrals in the form of series, which in certain cases are very remarkable.

To integrate v^2dx.

$\int v^2dx = v^2x - \int 2xvdv$

But $dv = dl.(x+\sqrt{a^2+x^2}) = \dfrac{dx}{\sqrt{a^2+x^2}}$

$$\therefore \int v^2 dx = v^2 x - \int \frac{2xdx}{\sqrt{a^2+x^2}} \times v$$

$$= v^2 x - \left(2\sqrt{a^2+x^2} \times v - \int 2\sqrt{a^2+x^2} \times \frac{dx}{\sqrt{a^2+x^2}}\right)$$

$$= v^2 x - 2v\sqrt{a^2+x^2} + 2x$$

To integrate $\dfrac{dx}{\sqrt{1+a^x}}$

Put $\sqrt{1+a^x} = u$

Then $a^x = u^2 - 1$

And $l.\,a \times a^x dx = 2udu$

Or $dx = \dfrac{2udu}{l.a.a^x} = \dfrac{2udu}{l.a.(u^2-1)}$

$$\therefore \frac{dx}{\sqrt{1+a^x}} = \frac{dx}{u} = \frac{2du}{l.a(u^2-1)}$$

$$= \frac{1}{la}\int \frac{2du}{u^2-1} = \frac{1}{l.a}\, l.\frac{u-1}{u+1}$$

510. Let $\dfrac{x}{x-a} = u$

Then, $du = \dfrac{dx.(x-a) - dx \times x}{(x-a)^2} = \dfrac{-adx}{(x-a)^2}$

$$\therefore \frac{dx}{(x-a)^2} = -\frac{du}{a}$$

Also $x = -\dfrac{au}{1-u}$

$$\therefore \frac{xdx}{(x-a)^2} = \frac{udu}{1-u} = du - du + \frac{udu}{1-u}$$

$$= \frac{du}{1-u} - du$$

$$\therefore \int \frac{xdx}{(x-a)^2} = -l.(1-u) - u$$

$$= -l.\left(1 - \frac{x}{x-a}\right) - \frac{x}{x-a}$$

$$= -l.\frac{-a}{x-a} - \frac{x}{x-a} = l.(a-x) - l.a - \frac{x}{x-a}$$

To integrate $\dfrac{xdx\sqrt{1-x^2}}{\sqrt{1+x^2}}$

Assume $\sqrt{1+x^2} = u$

Then $\dfrac{xdx}{\sqrt{1+x^2}} = du$

And $\sqrt{1-x^2} = \sqrt{2-u^2}$

$$\therefore \int \frac{xdx\sqrt{1-x^2}}{\sqrt{1+x^2}} = \int du . \sqrt{2-u^2}$$

Now if $\sqrt{2}$ be the radius of a circle, and u the abscissa measured from the centre, then $\sqrt{2-u^2}$ = the corresponding ordinate; and it is known that $\int d.$ (abscissa) $\times$ (ordinate) = the area comprised between the ordinates at the extremities of the abscissa, which call A.

$$\therefore \int \frac{xdx}{\sqrt{1+x^2}} = \text{A}.$$

Otherwise.

Assume $\sqrt{\dfrac{1-x^2}{1+x^2}} = u$

and the integral is reduced to $\displaystyle\int \frac{-u^2du}{(1+u^2)^2}$

To integrate $\dfrac{dx}{\sqrt{1-e^{-mx}}}$

Put $1 - e^{-mx} = u^2$

$$\therefore mdx \times e^{-mx} = 2udu$$

$$\therefore \frac{dx}{u} = \frac{2du}{me^{-mx}} = \frac{2du}{m.(1-u^2)}$$

$$\therefore \int \frac{dx}{\sqrt{1-e^{-mx}}} = \frac{1}{m}\int \frac{2du}{1-u^2} = \frac{1}{m}\ l.\ \frac{1+u}{1-u}$$

$$= \frac{1}{m}.\ l.\ (2e^{mx} + 2\sqrt{e^{2mx} - e^{mx}} - 1)$$

511. Assume $x\,(1+x)^{\frac{3}{2}} = u$

Then $dx.\ (1+x)^{\frac{3}{2}} + \frac{3}{2}\ xdx.\ \sqrt{1+x} = du$

$$\therefore \int xdx\ \sqrt{1+x} = \frac{2}{3}\,u - \frac{4}{3\times 5}\,(1+x)^{\frac{5}{2}}$$

$$= \frac{2}{3}(1+x)^{\frac{3}{2}}.\ (x - \frac{2}{5}\,.\,\overline{1+x})$$

$$= \frac{2}{3}.(1+x)^{\frac{3}{2}}\,.\,\frac{3x-2}{5}$$

512. Let $1 + ax = u$

Then $adx = du$

And $dx = \frac{du}{a}$

$$\therefore \int xdx.\ (1+ax)^{-\frac{3}{2}} = \int \frac{u^{-\frac{3}{2}}du}{a} = -\frac{2u^{-\frac{1}{2}}}{a} = -\frac{2}{a\sqrt{1+ax}}$$

513. Assume $\frac{x}{\sqrt{1+x}} = u$

Then $du = \frac{dx\sqrt{1+x}}{1+x} - \frac{1}{2}\,\frac{xdx}{(1+x)^{\frac{3}{2}}}$

$$= \frac{dx}{\sqrt{1+x}} - \frac{1}{2}\,\frac{xdx}{(1+x)^{\frac{3}{2}}}$$

$$\therefore \frac{xdx}{(1+x)^{\frac{3}{2}}} = \frac{2dx}{\sqrt{1+x}} - 2du$$

And $\int \frac{xdx}{(1+x)^{\frac{3}{2}}} = \frac{1}{4}\sqrt{1+x} - u$

$$= \frac{1}{4}\sqrt{1+x} - \frac{x}{\sqrt{1+x}}$$

$$= \frac{1}{4} \times \frac{1-3x}{\sqrt{1+x}}$$

514. $$\frac{dz}{\cos.^2(nz)} = dz.\sec.^2(nz) = dz.(1 + \tan.^2(nz))$$

$$= \frac{1}{n}.d(nz).(1 + \tan.^2(nz))$$

$$= \frac{1}{n}.\frac{d.\tan.(nz)}{1+\tan.^2(nz)} \times (1 + \tan^2.(nz))$$

$$= \frac{1}{n}.d.\tan.(nz)$$

$$\therefore \int\frac{dz}{\cos.^2(nz)} = \frac{1}{n}.\tan.(nz)$$

Otherwise.

$$d.\tan.nz = \frac{d.nz}{\cos.^2(nz)} = \frac{ndz}{\cos.^2(nz)}$$

$$\therefore \int\frac{dz}{\cos.^2(nz)} = \frac{1}{n}\int d\tan.(nz)$$

$$= \frac{1}{n}\tan.nz$$

515. Let $x^2.(1 - x^2)^{\frac{3}{2}} = u$

Then $du = 2xdx.(1 - x^2)^{\frac{3}{2}} - 3x^3dx.\sqrt{1 - x^2}$

$$= 2xdx\ \sqrt{1 - x^2} - 5x^3dx\ \sqrt{1 - x^2}$$

$$\therefore x^3dx\ \sqrt{1 - x^2} = \frac{2}{5}xdx\ \sqrt{1 - x^2} - \frac{du}{5}$$

$$\therefore \int x^3dx\ \sqrt{1 - x^2} = -\frac{4}{15}.(1 - x^2)^{\frac{3}{2}} - \frac{u}{5}$$

$$= -\frac{(1 - x^2)^{\frac{3}{2}}}{5}.\left(\frac{4}{3} + x^2\right)$$

$$= -\frac{(1 - x^2)^{\frac{3}{2}}}{15}.(4 + 3x^2)$$

To integrate $\dfrac{xdx}{\sqrt{1 - x^4}}$

Put $1 + x^2 = u^2$

$\therefore xdx = udu$

And $1 - x^2 = 1 - (u^2 - 1) = 2 - u^2$

$$\therefore 1 - x^4 = u^2.(2 - u^2)$$

$$\therefore \frac{xdx}{\sqrt{1-x^4}} = \frac{udu}{u.\sqrt{2-u^2}} = \frac{du}{\sqrt{2-u^2}}$$

$$\therefore \int\frac{xdx}{\sqrt{1-x^4}} = \int\frac{\sqrt{2}du}{\sqrt{2}.\sqrt{2-u^2}} = \frac{1}{\sqrt{2}} \times \sin.^{-\sqrt{2}}u$$

516. $$\int v^3x^2dx = v^3\,\frac{x^3}{3} - \int\frac{x^3}{3}.\,3v^2dv$$

But $\int\frac{x^3}{3} \times 3v^2dv = \int v^2 \times \left(x^3\frac{dx}{x}\right) = \int v^2(x^2dx) = \frac{x^3}{3}v^2 - \int\frac{x^3}{3}2vdv$

And $\int\frac{x^3}{3}\times 2vdv = \int\frac{2}{3}\,vx^3.\frac{dx}{x} = \int\frac{2}{3}vx^2dx = \frac{2x^3}{9}v - \int\frac{2x^3}{9}dv$

And $\int\frac{2x^3}{9}\,dv = \int\frac{2}{9}\,x^3\,\frac{dx}{x} = \int\frac{2}{9}\,x^2dx = \frac{2}{27}\,x^3$

$$\therefore \int v^3.x^2dx = \frac{v^3x^3}{3} - \frac{v^2x^3}{3} + \frac{2vx^3}{9} - \frac{2x^3}{27}$$

$$= \frac{x^3}{3}.\left(v^3 - v^2 + \frac{2v}{3} - \frac{2}{9}\right)$$

To integrate $\frac{dx}{x\sqrt{1+\sqrt{x}}}$, let $\sqrt{x} = y$

Then $x = y^2$

And $dx = 2ydy$

$$\therefore \frac{dx}{x.\sqrt{1+\sqrt{x}}} = \frac{2ydy}{y^2.\sqrt{1+y}} = \frac{2dy}{y\sqrt{1+y}}$$

Again, let $1 + y = u^2$

Then $2dy = 4\,udu$

$$\therefore \frac{2dy}{u} = 4du$$

And $\int\frac{dx}{x.\sqrt{1+\sqrt{x}}} = \int\frac{2dy}{y.\sqrt{1+y}} = \int\frac{4du}{u^2-1} = 2\int\frac{2du}{u^2-1}$

$$= 2\;l.\,\frac{u-1}{u+1}$$

But $u = \sqrt{1+y} = \sqrt{1+\sqrt{x}}$

$$\therefore \int \frac{dx}{x\sqrt{1+\sqrt{x}}} = 2\, l.\ \frac{\sqrt{1+\sqrt{x}}-1}{\sqrt{1+\sqrt{x}}+1}$$

$$= 2\, l.\ \frac{\sqrt{x}}{(\sqrt{1+\sqrt{x}}+1)^2}$$

$$= l.\, x - 4\, l.(\sqrt{1+\sqrt{x}}+1)$$

To integrate $vxdx$, we have

$$\int vxdx = v.\frac{x^2}{2} - \int \frac{x^2}{2}.\, dv$$

$$= \frac{vx^2}{2} - \int \frac{x^2}{2}.\ \frac{2adx}{a^2-x^2}$$

$$\text{Now} \int \frac{ax^2dx}{a^2-x^2} = -a\int \frac{x^2dx}{x^2-a^2} = -a\int dx + \int \frac{a^3dx}{a^2-x^2}$$

$$\text{(by division)} = -ax + \frac{a^2}{2}.\, v$$

$$\therefore \int vxdx = \frac{vx^2}{2} + ax - \frac{a^2v}{2} = \frac{v.\overline{x^2-a^2}+2ax}{2}$$

517. Instead of $(\sin.\, z)^{\frac{1}{4}}\, dz$, the first Edition has it $(\sin.\, z)^4\, dz$, which we will suppose the correct form.

Now $dz = \dfrac{d \sin.\, z}{\sqrt{1-\sin.^2 z}}$; $\therefore$ if we put $\sin.\, z = y$, there results

$$(\sin.\, z)^4\, dz = \frac{y^4\, dy}{\sqrt{1-y^2}}$$

Assume $y^3 . \sqrt{1-y^2} = P_1$

$$\text{Then } dP_1 = 3y^2dy\sqrt{1-y^2} - \frac{y^4dy}{\sqrt{1-y^2}}$$

$$= \frac{3y^2dy}{\sqrt{1-y^2}} - \frac{4y^4dy}{\sqrt{1-y^2}}$$

$$\therefore \int \frac{y^4dy}{\sqrt{1-y^2}} = \frac{3}{4}\int \frac{y^2dy}{\sqrt{1-y^2}} - \frac{1}{4}.\, P_1$$

Again, assume $y\sqrt{1-y^2} = P_2$

Then $dP_2 = dy.\sqrt{1-y^2} - \dfrac{y^2dy}{\sqrt{1-y^2}} = \dfrac{dy}{\sqrt{1-y^2}} - \dfrac{2y^2dy}{\sqrt{1-y^2}}$

$$\therefore \int\frac{y^2dy}{\sqrt{1-y^2}} = \frac{1}{2}\int\frac{dy}{\sqrt{1-y^2}} - \frac{1}{2}.P_2 = \frac{1}{2}z - \frac{1}{2}P_2$$

$$\text{Hence }\int\frac{y^4dy}{\sqrt{1-y^2}} = \frac{3}{8}z - \frac{3}{8}P_2 - \frac{1}{4}.P_1$$

$$= \frac{3}{8}z - \frac{1}{4}.y\sqrt{1-y^2}.\left(\frac{3}{2} - y^2\right)$$

But $y.\sqrt{1-y^2} = \sin. z. \cos. z = \dfrac{1}{2}\sin. 2z$

$$\frac{3}{2} - y^2 = \frac{3-2\sin.^2z}{2} = \frac{3-\cos.2z+1}{2} = \frac{4+\cos.2z}{2}$$

$$\therefore \int(\sin. z)^4\,dz = \frac{1}{16}\times(6z - 4\sin.2.z + \sin.2z.\cos.2z)$$

$$= \frac{1}{32}(12z - 8\sin.2.z + \sin.4z)$$

This might have been obtained, by expanding $(\sin. z)^4$ according to the functions of the multiple arcs.

Thus, let $2\cos. z = x + \dfrac{1}{x}$

Then $2.\cos. 4z = x^4 + \dfrac{1}{x^4}$

$2\sqrt{-1}\sin. z = x - \dfrac{1}{x}$

&c. = &c.

Hence $16\sin.^4z = x^4 - 4x^2 + 6 - 4.\dfrac{1}{x^2} + \dfrac{1}{x^4}$

$$= x^4 + \frac{1}{x^4} - 4.\left(x^2 + \frac{1}{x^2}\right) + 6$$

$$= 2\cos. 4z - 8\cos. 2z + 6$$

$$\therefore \sin.^4z = \frac{1}{8}\cos. 4z - \frac{1}{2}\cos. 2z + \frac{3}{8}$$

$$= \frac{1}{32}.(4\cos. 4z - 8.2\cos. 2z + 12)$$

$$\therefore \int \sin.^4 z . dz = \frac{1}{32} \left(\int 4dz \cos.\ 4z - 8 \int 2dz \cos.\ 2z + 12z \right)$$

$$= \frac{1}{32} (\sin.\ 4z - 8 \sin.\ 2z + 12z)$$

518. $$\frac{dx + \frac{1}{3} x dx}{\sqrt{1 + x^2}} = \frac{dx}{\sqrt{1 + x^2}} + \frac{1}{3} . \frac{x dx}{\sqrt{1 + x^2}}$$

$$\therefore \int \frac{dx + \frac{1}{3} x dx}{\sqrt{1 + x^2}} = \int \frac{dx}{\sqrt{1 + x^2}} + \frac{1}{3} \int \frac{x dx}{\sqrt{1 + x^2}}$$

$$= l. (x + \sqrt{1 + x^2}) + \frac{1}{3} \sqrt{1 + x^2}$$

both being common forms.

519. $$\left.\begin{aligned} \cos.\ (A - B) &= \cos.\ A . \cos.\ B + \sin.\ A . \sin.\ B \\ \cos.\ (A + B) &= \cos.\ A . \cos.\ B - \sin.\ A . \sin.\ B \end{aligned}\right\}$$

$$\therefore \cos.\ (A - B) - \cos.\ (A + B) = 2 \sin.\ A . \sin.\ B$$

Let $A = m\theta$, $B = n\theta$

Then $d\theta \sin.\ m\theta . \sin.\ n\theta = \frac{d\theta}{2} . \cos.\ (m - n)\theta - \frac{d\theta}{2} \cos.\ (m + n)\theta$

$$= \frac{1}{2 . (m - n)} . d . (m - n)\theta . \cos.\ (m - n)\theta$$

$$- \frac{1}{2 . (m + n)} . d . (m + n)\theta . \cos.\ (m + n)\theta$$

$$\therefore \int d\theta . \sin.\ m\theta . \sin.\ n\theta = \frac{1}{2 (m - n)} \sin.\ (m - n)\theta - \frac{1}{2 . (m + n)} \times$$

$$\sin.\ (m + n)\theta$$

$$= \frac{1}{m^2 - n^2} \times (n . \sin.\ m\theta . \cos.\ n\theta - m \cos.\ m\theta \sin.\ n\theta)$$

520. $$x^{\frac{3}{5}} dx (1 + a x^{\frac{1}{5}})^{-\frac{1}{3}} = \frac{16}{15} a \frac{x^{\frac{8}{5} - 1} dx}{\frac{16}{15} a} . (1 + a x^{\frac{8}{5}})^{-\frac{1}{3}}$$

$$\therefore \int x^{\frac{3}{5}} dx (1 + a x^{\frac{8}{5}})^{-\frac{1}{3}} = \frac{15}{16} a . (1 + a x^{\frac{8}{5}})^{\frac{2}{3}}$$

521. $$\int \frac{dx}{1+x^2} = \tan.^{-1} x \quad (\textit{Vince, Lacroix}).$$

To integrate $x^3\, dx\, (1+x^2)^{\frac{1}{2}}$

Assume $x^2 .\, (1+x^2)^{\frac{3}{2}} = \mathrm{P}$

$$\therefore\ d\mathrm{P} = 2x dx\, .\, (1+x^2)^{\frac{3}{2}} + 3x^3\, dx\, .\, \sqrt{1+x^2}$$

$$= 2x dx \sqrt{1+x^2} + 5x^3\, dx \sqrt{1+x^2}$$

$$\therefore\ \int x^3 dx \sqrt{1+x^2} = \frac{\mathrm{P}}{5} - \frac{2}{5} \int x dx \sqrt{1+x^2}$$

$$= \frac{\mathrm{P}}{5} - \frac{2}{15} \times (1+x^2)^{\frac{3}{2}}$$

$$= \frac{1}{5}\, .\, (1+x^2)^{\frac{3}{2}}\, .\, \left(x^2 - \frac{2}{3}\right)$$

$$= \frac{1}{15}\, .\, (1+x^2)^{\frac{3}{2}}.\, (3x^2-2)$$

To integrate $\dfrac{\sqrt{a^2+x^2}\, .\, dx}{a}$

We have $dx \sqrt{a^2+x^2} = \dfrac{dx\, .\, (a^2+x^2)}{\sqrt{a^2+x^2}} = \dfrac{a^2 dx}{\sqrt{a^2+x^2}} + \dfrac{x^2\, dx}{\sqrt{a^2+x^2}}$

Now, let $x \sqrt{a^2+x^2} = \mathrm{P}$

$$\therefore\ d\mathrm{P} = dx\, .\, \sqrt{a^2+x^2} + \frac{x^2 dx}{\sqrt{a^2+x^2}}$$

$$= \frac{a^2 dx}{\sqrt{a^2+x^2}} + \frac{2x^2 dx}{\sqrt{a^2+x^2}}$$

$$\therefore\ \frac{x^2 dx}{\sqrt{a^2+x^2}} = \frac{d\mathrm{P}}{2} - \frac{a^2}{2} . \frac{dx}{\sqrt{a^2+x^2}}$$

Hence $\displaystyle\int \frac{dx \sqrt{a^2+x^2}}{a} = a \int \frac{dx}{\sqrt{a^2+x^2}} + \frac{\mathrm{P}}{2a} - \frac{a}{2} \int \frac{dx}{\sqrt{a^2+x^2}}$

$$= \frac{a}{2} \int \frac{dx}{\sqrt{a^2+x^2}} + \frac{\mathrm{P}}{2a}$$

$$= \frac{a}{2}\, l.\, (x + \sqrt{a^2+x^2}) + \frac{x\sqrt{a^2+x^2}}{2a}.$$

$$\int a^x x^2 dx \left(= \int x^2 . a^x dx\right) = x^2 \frac{a^x}{l.a} - \int \frac{a^x}{l.a} 2x dx$$

But $\displaystyle\int \frac{a^x}{l.a} dx \times 2x = \frac{a^x}{(l.a)^2} \times 2x - \int 2 \frac{a^x dx}{(l.a)^2}$

$$= \frac{2a^x x}{(l.a)^2} - \frac{2a^x}{(l.a)^3}$$

$$\therefore \int a^x x^2 dx = \frac{x^2 a^x}{la} - \frac{2x a^x}{(l.a)^2} + \frac{2a^x}{(l.a)^3}$$

$$= \frac{a^x}{la}\left(x^2 - \frac{2x}{l.a} + \frac{2}{(l.a)^2}\right) = \frac{a^x}{(l.a)^3} \times \left\{x^2 (l.a)^2 - 2x\, la + 2\right\}.$$

522. Since $dy = d.(1+y)$

$$\int \frac{dy}{1+y} = \int \frac{d.(1+y)}{1+y} = l.(1+y)$$

Again, $\displaystyle\int \frac{2b dx}{x\sqrt{x^2 - b^2}} = \frac{2}{b} \int \frac{b^2 dx}{x\sqrt{x^2 - b^2}} = \frac{2}{b} . \sec.^{-1} x$

523. To integrate $\dfrac{ax^2 dx}{\sqrt{1 - x^2}}$, let $x . \sqrt{1 - x^2} = \text{P}$

$$\therefore d\text{P} = dx \sqrt{1 - x^2} - \frac{x^2 dx}{\sqrt{1 - x^2}} = \frac{dx}{\sqrt{1 - x^2}} - \frac{2x^2 dx}{\sqrt{1 - x^2}}$$

$$\therefore \frac{ax^2 dx}{\sqrt{1 - x^2}} = \frac{a}{2} \frac{dx}{\sqrt{1 - x^2}} - \frac{a}{2} . d\text{P}$$

$$\therefore \int \frac{ax^2 dx}{\sqrt{1 - x^2}} = \frac{a}{2} . \sin.^{-1} x - \frac{a}{2} x . \sqrt{1 - x^2}$$

To integrate $\dfrac{dv}{(a^2 + v^2)^2}$, let $v . (a^2 + v^2)^{-1} = \text{P}$

$$\therefore d\text{P} = dv . (a^2 + v^2)^{-1} - \frac{2v^2 dv}{(a^2 + v^2)^2}$$

But $\displaystyle\frac{v^2 dv}{(a^2 + v^2)^2} = \frac{dv}{(a^2 + v^2)^2} \times (v^2 + a^2 - a^2) = \frac{dv}{a^2 + v^2} - \frac{a^2 dv}{(a^2 + v^2)^2}$

$$\therefore d\text{P} = \frac{2a^2 dv}{(a^2 + v^2)^2} - \frac{dv}{a^2 + v^2}$$

$$\therefore \int \frac{dv}{(a^2+v^2)^2} = \frac{P}{2a^2} - \frac{1}{2a^4}\int \frac{a^2dv}{a^2+v^2}$$
$$= \frac{v}{a^2+v^2} - \frac{1}{2a^4} \times \tan.^{-1} v$$

To integrate z^3dy, we have the form $\int udv = uv - \int vdu$.

$\int z^3dy = z^3y - \int 3yz^2dz$; but $dz = \dfrac{ady}{\sqrt{a^2-y^2}}$ (*Lacroix, Vince.*)

$$\therefore \int 3yz^2dz = 3a\int z^2 \frac{ydy}{\sqrt{a^2-y^2}} = -\ 3az^2 \cdot \sqrt{a^2-y^2} +$$
$3a \int \sqrt{a^2-y^2} \times 2zdz$

But $6a\int \sqrt{a^2-y^2} \times zdz = 6a\int \sqrt{a^2-y^2} \times z\dfrac{ady}{\sqrt{a^2-y^2}} = 6a^2\int zdy$

$$= 6a^2zy - 6a^2\int ydz$$
$$= 6a^2zy - 6a^2\int \frac{aydy}{\sqrt{a^2-y^2}}$$
$$= 6a^2zy + 6a^3 . \sqrt{a^2-y^2};$$ hence by substitution we get

$$\int z^2dy = z^3y + 3az^2\sqrt{a^2-y^2} - 6a^2zy - 6a^3\sqrt{a^2-y^2}$$
$$= zy . (z^2 - 6a^2) + 3a\sqrt{a^2-y^2} \times (z^2 - 2a^2)$$
$$= z . (z^2 - 6a^2) \sin. z + 3\, a . (z^2 - 2a^2) \cos. z$$

524. To integrate $\dfrac{dx}{x^2 . (x+a)}$, put $\dfrac{1}{x} = u$

$$\therefore \frac{dx}{x^2} = -\ du \text{ and } x + a = \frac{1}{u} + a = \frac{1+au}{u}.$$

$$\therefore \frac{dx}{x^2(x+a)} = \frac{-udu}{1+au} = \frac{-udu}{au+1}$$
$$= \frac{-du}{a} + \frac{du}{a.(au+1)} \text{ by division}$$
$$= \frac{-du}{a} + \frac{1}{a^2} . \frac{adu}{au+1}$$

$$\therefore \int \frac{dx}{x^2(x+a)} = -\frac{1}{a}u + \frac{1}{a^2} l. (au+1)$$
$$= -\frac{1}{ax} + \frac{1}{a^2} . l. \frac{a+x}{x}.$$

Otherwise.

Assume $\dfrac{1}{x^2.(x+a)} = \dfrac{A}{x^2} + \dfrac{B}{x} + \dfrac{C}{x+a}$ reduce to a common denominator, and equate the like powers of x in the numerator, &c. &c.

To integrate $\dfrac{x^2 dx}{(x-2).(x-3)}$, assume $\dfrac{1}{x-2.(x-3)}$

$$= \frac{A}{x-2} + \frac{B}{x-3} = \frac{(A+B).x - 3A - 2B}{(x-2).(x-3)}$$

$$\left.\begin{aligned} \therefore\ 3A + 2B &= -1 \\ 3A + 3B &= \ \ 0 \end{aligned}\right\}$$

$$\therefore\ B = 1 \text{ and } A = -1$$

$$\therefore \int \frac{x^2 dx}{(x-2)(x-3)} = \int \left(\frac{x^2 dx}{x-3} - \frac{x^2 dx}{x-2}\right)$$

Now $\dfrac{x^2 dx}{x-3} = x dx + 3dx + \dfrac{9dx}{x-3}$ by division.

And $\dfrac{x^2 dx}{x-2} = x dx + 2dx + \dfrac{4dx}{x-2}$

$$\therefore \int \frac{x^2 dx}{(x-2).(x-3)} = \int dx + 9\int \frac{dx}{x-3} - 4\int \frac{dx}{x-2}$$

$$= x + 9\ l.\ (x-3) - 4\ l.\ (x-2)$$

Otherwise.

$$\frac{x^2 dx}{(x-2).(x-3)} = \frac{x^2 dx}{x^2 - 5x + 6}$$

Assume $x - \dfrac{5}{2} = u$

Then $x^2 - 5x + 6 = u^2 + 6 - \dfrac{25}{4} = u^2 - \dfrac{1}{4}$

And $x^2 = \left(u + \dfrac{5}{2}\right)^2 = u^2 + 5u + \dfrac{25}{4}$

And $dx = du$

$$\therefore \int \frac{x^2 dx}{(x-2).(x-3)} = \int \frac{u^2 du}{u^2 - \frac{1}{4}} + \int \frac{5udu}{u^2 - \frac{1}{4}} + \frac{25}{4} \int \frac{du}{u^2 - \frac{1}{4}}$$

$$= \int du + \frac{1}{4} \int \frac{du}{u^2 - \frac{1}{4}} + \int \frac{5udu}{u^2 - \frac{1}{4}} + \frac{25}{4} \int \frac{du}{u^2 - \frac{1}{4}}$$

$$= \int du + \frac{13}{2} \int \frac{du}{u^2 - \frac{1}{4}} + \int \frac{5udu}{u^2 - \frac{1}{4}}$$

$$= u + \frac{13}{2}. \; l. \; \frac{u - \frac{1}{2}}{u + \frac{1}{2}} + \frac{5}{2} l. \left(u^2 - \frac{1}{4}\right)$$

But $\frac{13}{2} l. \frac{u-\frac{1}{2}}{u+\frac{1}{2}} = \frac{13}{2} l.\left(u - \frac{1}{2}\right) - \frac{13}{2}. l. \left(u + \frac{1}{2}\right)$

And $\frac{5}{2}. l.\left(u^2 - \frac{1}{4}\right) = \frac{5}{2} l. \left(u - \frac{1}{2}\right) + \frac{5}{2}. l. \left(u + \frac{1}{2}\right)$

$$\therefore \int \frac{x^2 dx}{(x-2).(x-3)} = u + 9 \; l. \left(u - \frac{1}{2}\right) - 4 \; l. \left(u + \frac{1}{2}\right)$$

$$= x - \frac{5}{2} + 9. \; l. \; (x-3) - 4 \; l. \; (x-2)$$

a result less than the former by $\frac{5}{2}$ which was introduced by the principle of $du = d. \left(x - \frac{5}{2}\right) = dx.$

$$\int \frac{xdx}{\sqrt{a^2 + x^2}} \times \log. x = \sqrt{a^2 + x^2} \times \log. x - \int \sqrt{a^2 + x^2} \times d. \log. x$$

But $d. \log. x = \frac{dx}{x. \; l. \; r}$ (r being the radix of the system.)

$$\therefore \int \sqrt{a^2 + x^2} \times \frac{dx}{x.l.r} = \frac{1}{l.r} \int \frac{dx}{x}. \sqrt{a^2 + x^2}$$

$$= \frac{1}{l.r} \int \frac{a^2 dx}{x\sqrt{a^2 + x^2}} + \frac{1}{l.r} \int \frac{xdx}{\sqrt{a^2 + x^2}}$$

$$= \frac{a}{2l.r} \int \frac{2adx}{x\sqrt{a^2 + x^2}} + \frac{1}{l.r}. \sqrt{a^2 + x^2}$$

$$= \frac{a}{2l.r}. l. \frac{\sqrt{a^2 + x^2} - a}{\sqrt{a^2 + x^2} + a} + \frac{1}{l.r}. \sqrt{a^2 + x^2}$$

$$\therefore \int \frac{xdx}{\sqrt{a^2+x^2}} \log. x = \sqrt{a^2+x^2} \log. x - \frac{a}{l.r}.l. \frac{x}{\sqrt{a^2+x^2}+a} -$$

$$\frac{1}{l.r}\sqrt{a^2+x^2}$$

$$= \frac{lx}{lr}\sqrt{a^2+x^2} - \frac{alx}{lr} + \frac{a}{lr}.\ l.(\sqrt{a^2+x^2}+a)$$

$$- \frac{1}{r}.\sqrt{a^2+x^2}$$

$$= \frac{1}{l.r}\left\{lx.\ \overline{\sqrt{a^2+x^2}-a} + a.l.(\sqrt{a^2+x^2}+a)\right.$$

$$\left. - \sqrt{a^2+x^2}\right\}$$

525. $$\frac{xdx}{\sqrt{2ax-x^2}} = \frac{adx-(adx-xdx)}{\sqrt{2ax-x^2}}$$

$$= \frac{adx}{\sqrt{2ax-x^2}} - \frac{adx - xdx}{\sqrt{2ax-x^2}}$$

$$\therefore \int \frac{xdx}{\sqrt{2ax-x^2}} = \text{vers.}^{-1}x - \sqrt{2ax-x^2}$$

526. $$\int xdx.(a^2+x^2)^3 = \int \frac{1}{8} \times 8xdx.\ (a^2+x^2)^3$$

$$= \frac{1}{8}.(a^2+x^2)^4$$

Again, $$\int \frac{xdx}{a^2+x^2} = \frac{1}{2}\int \frac{2xdx}{a^2+x^2}$$

$$= \frac{1}{2}.\ l.(a^2+x^2)$$

Also $$\int \frac{xdx}{\sqrt{2ax-x^2}} = \text{vers.}^{-1}x$$ (*Lacroix* or *Vince.*)

527. Let $\frac{1}{x-a} = u$

Then $\frac{-dx}{(x-a)^2} = du$ and $x-a = \frac{1}{u}$

$$\therefore\ x-b = \frac{1}{u} + \overline{a-b} = \frac{+\ \overline{a-b}.u}{u}$$

$$\therefore\ \frac{dx}{(x-a)^2 \times (x-b)} = -\ \frac{udu}{(a-b).u+1} = -\ \frac{du}{a-b} +$$

$$\frac{du}{(a-b).(\overline{a-b}.u+1)} = \frac{-du}{a-b} + \frac{(a-b).du}{(a-b)^2.\,(\overline{a-b}.u+1)}$$

$$\therefore \int \frac{dx}{(x-a)^2 \times (x-b)} = -\ \frac{u}{a-b} + \frac{1}{(a-b)^2}.\, l.(\overline{a-b}.u+1)$$

$$= -\ \frac{1}{(x-a).(a-b)} + \frac{1}{(a-b)^2}.\, l. \frac{x-b}{x-a}$$

Otherwise.

Assume $\dfrac{1}{(x-a)^2.(x-b)} = \dfrac{Ax+B}{(x-a)^2} + \dfrac{C}{(x-b)}$; reduce these fractions to a common denominator, and equate the coefficients of like powers of x. Hence obtaining A, B, C, &c. the integral required may be found.

To integrate $\dfrac{a+\sqrt{a^2-x^2}}{x}.\, dx$, we have

$$a + \sqrt{a^2-x^2} = \frac{a^2-(a^2-x^2)}{a-\sqrt{a^2-x^2}} = \frac{x^2}{a-\sqrt{a^2+x^2}}$$

$$\therefore\ \frac{a+\sqrt{a^2-x^2}}{x} \times dx = \frac{xdx}{a-\sqrt{a^2-x^2}}$$

Again, let $a^2 - x^2 = u^2$

$\therefore\ xdx = -\ udu$ and we get

$$\frac{a+\sqrt{a^2-x^2}}{x}.\, dx = -\ \frac{udu}{u-a} = -\ du - \frac{adu}{u-a}$$

$$\therefore \int \frac{a+\sqrt{a^2-x^2}}{x} \times dx = -\ u - a.\, l.\,(u-a)$$

$$= -\ \sqrt{a^2-x^2} - a.\, l.\,(\sqrt{a^2-x^2} - a)$$

To integrate $X^2dx = (l.x)^2\, dx$, we have,

$$\int (l.x)^2 dx = (l.x)^2 x - \int x \times 2\, l.x \times d.\,l.\,x$$
$$= (l.x)^2 x - \int 2l.x \times dx \ \left(d.l.x = \frac{dx}{x}\right)$$

Similarly $\int l.x \times 2dx = 2xl.x - \int 2x \times d.lx$
$$= 2xl.x - 2\int dx$$
$$= 2xl.x - 2x$$
$$\therefore \int (l.x)^2 dx = x(l.x)^2 - 2xlx + 2x$$
$$= x.\{(l.x)^2 - 2l.x + 2\}$$

To integrate $\dfrac{dx}{x^4 - 2x^3 - x^2 + 2x}$

Assume $\dfrac{1}{x^4 - 2x^3 - x^2 + 2x} \left(= \dfrac{1}{x.(x+1).(x-1).(x-2)}\right) =$ $\dfrac{A}{x} + \dfrac{B}{x+1} + \dfrac{C}{x-1} + \dfrac{D}{x-2}$, which being reduced to the same denominator, we shall have

$A \times (x^2 - 1).(x - 2) + B \times x.(x - 1)(x-2) + C \times x.(x + 1).(x - 2) + D \times x.(x^2 - 1) = 1$

Let $x = 0$

Then $A \times (-1) \times (-2) = 1$

$\therefore A = \dfrac{1}{2}$

Let $x = -1$

Then $B \times (-1).(-2).(-3) = 1$

$\therefore B = -\dfrac{1}{6}$

Let $x = 1$

Then $C \times 2 \times (-1) = 1$

$\therefore C = -\dfrac{1}{2}$

Let $x = 2$

Then $D \times 2 \times 3 = 1$

$\therefore D = \dfrac{1}{6}$

Hence $\displaystyle\int \frac{dx}{x^4 - 2x^3 - x^2 + 2x} = \frac{1}{2}\int \frac{dx}{x} - \frac{1}{6}.\int \frac{dx}{x+1} -$

$$\frac{1}{2}\int\frac{dx}{x-1}+\frac{1}{6}.\int\frac{dx}{x-2}=\frac{1}{2}\,l.x-\frac{1}{6}\,l.(x+1)-\frac{1}{2}.l.(x-1)$$
$$+\frac{1}{6}\,l.(x-2)$$
$$=\frac{1}{2}.\,l.\frac{x}{x-1}-\frac{1}{6}.\,l.\frac{x+1}{x-2}$$

528. Assume $P_1 = x^{m-1} \times \sqrt{a^2-x^2}$

$$\therefore dP_1 = (m-1).x^{m-2}dx \times \sqrt{a^2-x^2} - \frac{x^m dx}{\sqrt{a^2-x^2}}$$
$$= \frac{(m-1)\,a^2x^{m-2}dx}{\sqrt{a^2-x^2}} - \frac{mx^m dx}{\sqrt{a^2-x^2}}$$

Hence appears the use of the following assumptions:

$$\left.\begin{aligned} P_1 &= x^{m-1}\sqrt{a^2-x^2} \\ P_3 &= x^{m-3}\sqrt{a^2-x^2} \\ P_5 &= x^{m-5}\sqrt{a^2-x^2} \\ \&c. &= \&c. \\ P_{m-1} &= x\sqrt{a^2-x^2} \end{aligned}\right\} \quad \left\{\begin{aligned} F_0 &= \int\frac{x^m dx}{\sqrt{a^2-x^2}} \\ F_2 &= \int\frac{x^{m-2}dx}{\sqrt{a^2-x^2}} \\ \&c. &= \&c. \\ F_m &= \int\frac{dx}{\sqrt{a^2-x^2}} \text{ (m being even)} \end{aligned}\right.$$

Whence $P_1 = (m-1)\,a^2 \times F_2 - m\,F_0$

Similarly $P_3 = (m-3).a^2 \times F_4 - (m-2)\,F_2$

$P_5 = (m-5).a^2 \times F_6 - (m-4)\,F_4$

&c. = &c.

$P_{m-1} = a^2 \times F_m - 2 \times F_{m-2}$

$$\therefore F_0 = a^2.\frac{m-1}{m}.\,F_2 - \frac{P_1}{m}$$
$$F_2 = a^2.\frac{m-3}{m-2}.\,F_4 - \frac{P_3}{m-2}$$
$$F_4 = a^2.\frac{m-5}{m-4}.\,F_6 - \frac{P_5}{m-4}$$

&c. = &c.

$$F_{m-2} = a^2 \times \frac{1}{2} \times F_m - \frac{P_{m-1}}{2}$$

Hence, by substitution, we get

$$F_0 = -\frac{P_1}{m} - \frac{a^2.(m-1)}{m} \times \frac{P_3}{m-2} - \frac{a^4.(m-1).(m-3)}{m.(m-2)}$$

$$\times \frac{P_5}{m-4} - \frac{a^6 \times (m-1).(m-3) \times (m-5)}{m.(m-2).(m-4)} \times \frac{P_7}{m-6} - \&c.$$

$$-\frac{a^{m-2} \times (m-1) \ldots. 3}{m.(m-2) \ldots. 2} \times \frac{P_{m-1}}{2} + a^m \times \frac{(m-1).(m-3) \ldots. 5 \times 3}{m.(m-2) \ldots. 4 \times 2} \times \frac{F_m}{2}$$

Let now $x = 0$; then, because $F_m = \int \frac{dx}{\sqrt{a^2 - x^2}} = \frac{1}{a} \times \int \frac{adx}{\sqrt{a^2 - x^2}} = \frac{1}{a}. \sin.^{-1} x$, when $x = 0$, we have $F_m = 0$, $\pm \pi'$, $\pm 2\pi'$, &c., generally $\pm p\pi'$ (p being any integer, and π' the semi-circumference of the circle whose radius is a)

Also P_1, P_3, &c. P_{m-1}, having each a factor $= x$, vanish when $x = 0$.

$\therefore$ when $x = 0$ and m is an even number,

$$\int \frac{x^m dx}{\sqrt{a^2 - x^2}} = F_0 = \pm \frac{p\pi'}{2} \times \frac{1 \times 3 \times 5 \times \ldots.(m-5).(m-3).(m-1)}{2 \times 4 \times 6 \times \ldots. (m-4).(m-2).m} \times a^m$$

But $\pi : 1 :: \pi' : a$

$\therefore \pi' = a\pi$

and $2 \times 4 \times 6 \times \&c \ldots. (m-4) \times (m-2) \times m = 2^{\frac{m}{2}} \times \left(1 \times 2 \times 3 \times 4 \times \ldots. \frac{m}{2}\right)$

$$\therefore \int \frac{x^m dx}{\sqrt{a^2 - x^2}} = \pm p\pi \times \frac{1 \times 3 \times 5 \times \ldots.(m-3) \times (m-1)}{1.2.3 \ldots. (\frac{m}{2}-1).\frac{m}{2}} \times \frac{a^{m+1}}{2^{\frac{m}{2}+1}}$$

If m be an odd number, the same process will apply, as in this case, we have only to continue the operation, reducing m until we arrive at $F_{m-1} = \int \frac{xdx}{\sqrt{a^2 - x^2}} = -\sqrt{a^2 - x^2}$

To integrate $\frac{dz}{\sin. z}$, we have

$$dz = \frac{d. \sin. z}{\sqrt{1 - \sin.^2 z}}$$

$$\therefore \frac{dz}{\sin. z} = \frac{d.\sin. z}{\sin. z.\sqrt{1-\sin.^2 z}}, \text{ which is a common form.}$$

$$\therefore \int\frac{dz}{\sin. z} = \frac{1}{2}.l.\frac{1-\sqrt{1-\sin.^2 z}}{1+\sqrt{1-\sin.^2 z}} = l.\frac{\sin. z}{1+\sqrt{1-\sin.^2 z}}$$

$$= l.\frac{\sin. z}{1+\cos. z} = l.\tan. z$$

Otherwise.

$$\int\frac{dz}{\sin. z} = \int\frac{dz}{2\sin.\frac{z}{2}.\cos.\frac{z}{2}}\int\frac{\left(\frac{d.\frac{z}{2}}{\cos.^2\frac{z}{2}}\right)}{\left(\frac{\sin.\frac{z}{2}}{\cos.\frac{z}{2}}\right)}$$

$$= \int\frac{d.\tan.\frac{z}{2}}{\tan.\frac{z}{2}} = l.\tan.\frac{z}{2}$$

To integrate $\dfrac{y^{\frac{1}{2}}dy}{\sqrt{r^3-y^3}}$, let $\dfrac{y^{\frac{3}{2}}}{r^{\frac{3}{2}}} = u$

Then $\dfrac{y^{\frac{1}{2}}dy}{r^{\frac{3}{2}}} = \dfrac{2}{3}du$

And $\dfrac{y^3}{r^3} = u^2$

$$\text{Hence} \int\frac{y^{\frac{1}{2}}dy}{\sqrt{r^3-y^3}} = \int\frac{y^{\frac{1}{2}}dy}{r^{\frac{3}{2}}\sqrt{1-\frac{y^3}{r^3}}} = \frac{2}{3}\int\frac{du}{\sqrt{1-u^2}}$$

$$= \sin.^{-1}u = \sin.^{-1}\frac{y^{\frac{3}{2}}}{r^{\frac{3}{2}}}$$

529. $\dfrac{x^3dx}{x-a} = x^2dx + axdx + a^2dx + \dfrac{a^3dx}{x-a}$ by division.

$$\therefore \int\frac{x^3dx}{x-a} = \frac{x^3}{3} + \frac{ax^2}{2} + a^2x + a^3\,l.(x-a)$$

To integrate $y^2 dy\ (a^2 - y^2)^{\frac{1}{2}}$, assume $y(a^2 - y^2)^{\frac{3}{2}} = \text{P}$

Then $d\text{P} = dy\,.(a^2 - y^2)^{\frac{3}{2}} - 3y^2 dy\ \sqrt{a^2 - y^2}$

$$= a^2 dy\,.\sqrt{a^2 - y^2} - 4y^2 dy\ \sqrt{a^2 - y^2}$$

$$\therefore \int y^2 dy\ \sqrt{a^2 - y^2} = \frac{a^2}{4}\int dy\,\sqrt{a^2 - y^2} - \frac{\text{P}}{4}$$

But if a be the radius of a circle, y an abscissa measured from the centre, then the corresponding ordinate is $\sqrt{a^2 - y^2}$, and $\int dy \times \sqrt{a^2 - y^2} =$ the area comprised between the ordinates at the extremities of the abscissa y; which put $= \text{A}$

$$\therefore \int y^2 dy\ \sqrt{a^2 - y^2} = \frac{a^2}{4} \times \text{A} - \frac{\text{P}}{4}$$

Otherwise.

Let $y = \sin.\ \theta$, to radius $= a$.

$$\therefore y^2 dy\,.\ (a^2 - y^2)^{\frac{1}{2}} = \sin.^2\theta\ d\theta.\ \cos.\ \theta \times (a^2 - \sin.^2\theta)^{\frac{1}{2}}$$

$$= d\theta.\ \sin.^2\theta.\ \cos.^2\theta$$

$$= \frac{d\theta}{4} \times \sin.^2(2\theta)$$

But since $\cos.\,4\theta = 1 - 2\sin.^2 2\theta$, we have

$$\sin.^2 2\theta = \frac{1}{2} - \frac{\cos.\ 4\theta}{2}$$

$$\therefore \int y^2 dy\,.\ (a^2 - y^2)^{\frac{1}{2}} = \int\frac{d\theta}{8} - \int\frac{d\theta}{8}.\ \cos.\ 4\theta = \frac{\theta}{8} - \frac{\sin.\ 4\theta}{32}.$$

$\int a^x x dx = x.\int a^x dx - \int dx \int a^x dx$, by the form $\int u dv = uv - \int v du$

But $a^x dx = \dfrac{d\,.\,a^x}{l.a}$

$$\therefore \int a^x x dx = \frac{x}{l a}.\ a^x - \int\frac{dx}{l a}.\ a^x$$

$$= \frac{x}{l a}.\ a^x - \int\frac{1}{l.a}.\ a^x dx$$

$$= \frac{x}{l a}.\ a^x - \frac{a^x}{(l.a)^2}$$

$$= \frac{a^x}{l a} \times \left(x - \frac{1}{l a}\right) = \frac{a^x}{(la)^2} \times (xla - 1)$$

530. $$\int \frac{2adx}{a^2+x^2} = \frac{2}{a}\int\frac{a^2dx}{a^2+x^2} = \frac{2}{a}\tan.^{-1}x$$

531. $$\frac{x^6dx}{x^2+a^2} = x^4dx - a^2x^2dx + a^4dx - \frac{a^6dx}{x^2+a^2}$$ by actual division.

$$\therefore \int\frac{x^6dx}{x^2+a^2} = \frac{x^5}{5} - \frac{a^2x^3}{3} + a^4x - a^4\int\frac{a^2dx}{a^2+x^2}$$

$$= \frac{x^5}{5} - \frac{a^2x^3}{3} + a^4x - a^4\tan.^{-1}x$$

To integrate, $\dfrac{dx}{x^5\sqrt{a^2+x^2}}$, assume $\dfrac{\sqrt{a^2+x^2}}{x^4} = \mathrm{P}_1$

Then $$d.\mathrm{P}_1 = \frac{-4dx}{x^5}\times\sqrt{a^2+x^2} + \frac{dx}{x^3.\sqrt{a^2+x^2}}$$

$$= \frac{-4a^2dx}{x^5.\sqrt{a^2+x^2}} - \frac{3dx}{x^3.\sqrt{a^2+x^2}}$$

$$\therefore \frac{dx}{x^5\sqrt{a^2+x^2}} = -\frac{d\mathrm{P}_1}{4a^2} - \frac{3}{4a^2}\cdot\frac{dx}{x^3\sqrt{a^2+x^2}}$$

Again, assume $\mathrm{P}_2 = \dfrac{\sqrt{a^2+x^2}}{x^2}$

Then $$d.\,\mathrm{P}_2 = \frac{-2dx}{x^3}\cdot\sqrt{a^2+x^2} + \frac{dx}{x\sqrt{a^2+x^2}}$$

$$= \frac{-2a^2dx}{x^3\sqrt{a^2+x^2}} - \frac{dx}{x\sqrt{a^2+x^2}}$$

$$\therefore \frac{dx}{x^3\sqrt{a^2+x^2}} = \frac{-d.\mathrm{P}_2}{2a^2} - \frac{1}{2a^2}\cdot\frac{dx}{x\sqrt{a^2+x^2}}$$

Hence, by substitution, we get

$$\frac{dx}{x^5\sqrt{a^2+x^2}} = -\frac{d.\mathrm{P}_1}{4a^2} + \frac{3}{8a^4}d.\mathrm{P}_2 + \frac{3}{8a^4}\cdot\frac{dx}{x\sqrt{a^2+x^2}}$$

$$\therefore \int\frac{dx}{x^5\sqrt{a^2+x^2}} = \frac{1}{8a^4}\times\left(3\mathrm{P}_2 - 2a^2\mathrm{P}_1 + \frac{3}{2a}\,l.\,\frac{\sqrt{a^2+x^2}-a}{\sqrt{a^2+x^2}+a}\right)$$

$$= \frac{1}{4a^4}\times\left(3\,\mathrm{P}_2 - 2a^2\mathrm{P}_1 + \frac{3}{a}.l.\frac{\sqrt{a^2+x^2}-a}{x}\right)$$

which is ∴ known.

To integrate $\dfrac{dz}{\sin. z\cos. z}$, we have $\sin. z. \cos. z = \dfrac{\sin. 2z}{2} =$

$$\therefore \frac{dz}{\sin. z\cos. z} = \frac{2dz}{\sin. 2z}; \text{ put } \sin. 2z = y$$

$$\text{Then } d.(2z)\ (= 2dz) = \frac{d.\sin.(2z)}{\sqrt{1-\sin.^2 2z}} = \frac{dy}{\sqrt{1-y^2}}$$

$$\therefore \int\frac{dz}{\sin. z. \cos. z} = \int\frac{dy}{y.\sqrt{1-y^2}} = \frac{1}{2}\, l. \frac{1-\sqrt{1-y^2}}{1+\sqrt{1-y^2}}$$

$$= l. \frac{y}{1+\sqrt{1-y^2}} = l. \frac{\sin. 2z}{1+\cos. 2z}$$

Now $1 + \cos. 2z = 2\cos.^2 z$

And $\sin. 2z = 2\cos. z. \sin. z$

$$\therefore \frac{\sin. 2z}{1+\cos. 2z} = \frac{\sin. z}{\cos. z} = \tan. z$$

$$\therefore \int\frac{dz}{\sin. z. \cos. z} = l. (\tan. z)$$ (See page 301.)

532. $\displaystyle\int\frac{d\times x^{3n-1}dx}{\sqrt{a^n - x^n}}$ may be found thus:

$$\left.\begin{array}{l} \text{Assume } x^{2n}. \sqrt{a^n - x^n} = P_1 \\ \qquad x^n. \sqrt{a^n - x^n} = P_1 \end{array}\right| \left.\begin{array}{l} \displaystyle\int\frac{x^{3n-1}dx}{\sqrt{a^n - x^n}} = F_0 \\ \displaystyle\int\frac{x^{2n-1}dx}{\sqrt{a^n - x^n}} = F_1 \end{array}\right\}$$

$$\text{Then } dP_1 = 2n. x^{2n-1}dx\sqrt{a^n - x^n} - \frac{n}{2}. \frac{x^{3n-1}dx}{\sqrt{a^n - x^n}}$$

$$= \frac{2na^n. x^{2n-1}dx}{\sqrt{a^n - x^n}} - \frac{5n}{2}. \frac{x^{3n-1}dx}{\sqrt{a^n - x^n}}$$

$$\therefore P_1 = 2na^n. F_1 - \frac{5n}{2}. F_0$$

$$\text{Similarly } P_2 = na^n\int\frac{x^{n-1}dx}{\sqrt{a^n - x^n}} - \frac{3n}{2}. F_1$$

$$= -2a^n. \sqrt{a^n - x^n} - \frac{3n}{2} F_1$$

Hence, $F_0 = \frac{4a^n}{5}. F_1 - \frac{5n}{2} P_1 = \frac{4a^n}{5} (-\frac{4}{3n} a^n \sqrt{a^n - x^n}$

$- \frac{2}{3n} P_2) - \frac{5n}{2}. P_1 = - \frac{16a^{2n}}{15n} \sqrt{a^n - x^n} - \frac{8a^n}{15n} P_2 - \frac{5n}{2} P_1$

$$\therefore \int \frac{d \times x^{3n-1} dx}{\sqrt{a^n - x^n}} = -d \times \sqrt{a^n - x^n} \left(\frac{16a^{2n}}{15^n} + \frac{8a^n x^n}{15n} + \frac{5nx^{2n}}{2} \right)$$

To $\int \frac{d \times x^{3n-1} dx}{a^n - x^n} = -d \times \int \frac{x^{3n-1} dx}{x^n - a^n}$ we have, by division,

$$\frac{x^{3n-1} dx}{x^n - a^n} = x^{2n-1} dx + a^n x^{n-1} dx + \frac{a^{2n} x^{n-1} dx}{x^n - a^n}$$

$$\therefore \int \frac{d \times x^{3n-1} dx}{a^n - x^n} = \frac{-d \times x^{2n}}{2n} - \frac{d \times a^n x^n}{n} - \frac{d \times a^{2n}}{n} \times l.(x^n - a^n)$$

533. Let $y = \sin. \theta.$ to radius (a).

Then $ady = ad\theta. \cos. \theta = ad\theta. \sqrt{a^2 - y^2}$

$$\therefore \int \frac{ady}{\sqrt{a^2 - y^2}} = \int ad\theta = a\theta = a \sin.^{-1} y$$

534. To integrate $\frac{d \times dx}{x^3. \sqrt{a^2 + x^2}}$, assume $\frac{\sqrt{a^2 + x^2}}{x^2} = P$

$$\therefore dP = \frac{xdx}{x^2. \sqrt{a^2 + x^2}} - \frac{2xdx. \sqrt{a^2 + x^2}}{x^4}$$

$$= \frac{dx}{x. \sqrt{a^2 + x^2}} - \frac{2a^2 dx}{x^3. \sqrt{a^2 + x^2}} - \frac{2dx}{x \sqrt{a^2 + x^2}}$$

$$\therefore \int \frac{d \times dx}{x^3. \sqrt{a^2 + x^2}} = - \int \frac{d \times dx}{2a^2 x \sqrt{a^2 + x^2}} - \int \frac{d}{2a^2} \times dP$$

$$= - \frac{d}{4a^3} \times l. \frac{\sqrt{a^2 + x^2} - a}{\sqrt{a^2 + x^2} + a} - \frac{d}{2a^2} \times P$$

$$= - \frac{d}{2a^3} \times l. \frac{\sqrt{a^2 + x^2} - a}{x} - \frac{d}{2a^2} \times \frac{\sqrt{a^2 + x^2}}{x^2}$$

To integrate $\frac{x^2dx}{(x-a).(x-b).(x-c)}$, assume

$$\frac{1}{(x-a)\times(x-b)\times(x-c)} = \frac{A}{x-a} + \frac{B}{x-b} + \frac{C}{x-c}$$

$$= \frac{A(x-b).(x-c)+B.(x-a)(x-c)+C.(x-a).(x-b)}{(x-a).(x-b).(x-c)}$$

$\therefore$ A. $(x-b).(x-c) + B.(x-a)(x-c) + C.(x-a)(x-b) =$

Let $x = a$

Then A. $(a-b).(a-c) = 1$

$\therefore A = \frac{1}{(a-b).(a-c)}$

Let $x = b$

Then B. $(b-a).(b-c) = 1$

$\therefore B = \frac{1}{(b-a).(b-c)}$

Let $x = c$

Then C.$(c-a).(c-b)$

$\therefore C = \frac{1}{(c-a).(c-b)}$

$\therefore$ A, B and C are known.

Hence $\frac{x^2dx}{(x-a).(x-b).(x-c)} = \frac{Ax^2dx}{(x-a)} + \frac{Bx^2dx}{(x-b)} + \frac{Cx^2dx}{x-c}$
$= (A+B+C).xdx + (Aa+Bb+Cc)dx$
$+ \frac{Aadx}{x-a} \frac{Bbdx}{x-b} + \frac{Ccdx}{x-c}$

$\therefore \int \frac{x^2dx}{(x-a).(x-b).(x-c)} = \frac{A+B+C}{2} \times x^2 + (Aa+Bb+$
$Cc)\,x + Aa.\,l.\,(x-a) + Bb.\,l.\,(x-b) + Cc.\,l.\,(x-c)$.

To integrate $(a^2+x^2)^m dx$, when $\int (a^2+x^2)^{m+r} \times dx = A$.

$$\left.\begin{aligned} \text{Let } (a^2+x^2)^{m+1} \times x &= P_1 \\ (a^2+x^2)^{m+2} \times x &= P_2 \\ \&c. &= \&c. \\ (a^2+x^2)^{m+r} \times x &= P_r \end{aligned}\right\} \text{ and } \left.\begin{aligned} \int (a^2+x^2)^m\,dx &= F_0 \\ \int (a^2+x^2)^{m+1}\,dx &= F_1 \\ \int (a^2+x^2)^{m+2}\,dx &= F_2 \\ \&c. &= \&c. \\ \int (a^2+x^2)^{m+r}\,dx = A &= F_r \end{aligned}\right\}$$

Then $dP_1 = dx.(a^2+x^2)^{m+1} + 2(m+1)x^2dx.(a^2+x^2)$

$= dx.(a^2+x^2)^{m+1} + 2(m+1)dx.(a^2+x^2)^m.(x^2+a^2-a^2)$

$= \overline{2(m+1)+1} \,.\times dx.(a^2+x^2)^{m+1} - 2(m+1).a^2dx.\times$ $(a^2+x^2)^m$

$\therefore P_1 = \overline{2(m+1)+1} \times F_1 - 2.(m+1).a^2.F_0$

Similarly $P_2 = \overline{2.(m+2)+1} \times F_2 - 2.(m+2).a^2 \times F_1$

Let $2(m+1) = c$

$$\text{Then } F_0 = \frac{c+1}{ca^2} \times F_1 - \frac{P_1}{ca^2}$$

$$F_1 = \frac{c+3}{(c+2).a^2} \times F_2 - \frac{P_2}{(c+2).a^2}$$

$$\text{Similarly } F_2 = \frac{c+5}{(c+4).a^2} \times F_3 - \frac{P_3}{(c+4).a^2}$$

&c. = &c.

$$F_{r-1} = \frac{c+2r-1}{(c+2r-2).a^2} \times F_r - \frac{P_r}{(c+2r-2)\,a^2}$$

Hence, by substitution, we get

$$F_0 = -\frac{1}{ca^2}\Big\{P_1 + \frac{c+1}{c+2} \times \frac{1}{a^2} \times P_2 + \frac{(c+1).(c+3)}{(c+2).(c+4)} \times \frac{1}{a^4} \times P_3 + \&c.$$

$$+ \frac{(c+1).(c+3).(c+5).....(c+2r-3)}{(c+2).(c+4).(c+6).....(c+2r-2)} \times \frac{1}{a^{2.(r-1)}} \times P_r\Big\}$$

$$+ \frac{(c+1).(c+3).(c+5).....(c+2r-1)}{(c+2).(c+4).....(c+2r-2)} \times \frac{1}{a^{2r}} \times F_r,$$ an expression which contains known quantities $F_r = A$, P_1 &c. only.

$\therefore F_0$ or $\int (a^2+x^2)^m dx$ is known.

To integrate $z^n dy$, we have the form $\int udv = uv - \int vdu$, by which we get

$$\int z^n dy = z^n y - \int yd.(z^n)$$

$$\text{But } d.(z^n) = nz^{n-1}dz = nz^{n-1}.\frac{dy}{\sqrt{1-y^2}}$$

$$\therefore \int yd.(z^n) = \int nz^{n-1} \times \frac{ydy}{\sqrt{1-y^2}}$$

$= \sqrt{1-y^2} \times (-nz^{n-1}) - \int \sqrt{1-y^2} \times (-n) \times$ $(n-1)z^{n-2}dz$

$$= -nz^{n-1}\sqrt{1-y^2} + n.(n-1)\int z^{n-2}dy$$

$$\therefore \int z^n dy = z^n y + nz^{n-1}\sqrt{1-y^2} - n.(n-1)\int z^{n-2}dy$$

Similarly $\int z^{n-2}dy = z^{n-2}y + (n-2)z^{n-3}\sqrt{1-y^2} - (n-2).(n-3)\times \int z^{n-4}dy$

&c. = &c.

$\int z^{n-2m}dy = z^{n-2m}y + (n-2m)z^{n-2m-1}.\sqrt{1-y^2} - (n-2m)\times (n-2m-1)\int z^{n-2m-2}dy$

Hence, by substitution, we get

$\int z^n dy = z^n y - n.(n-1).z^{n-2}y + n.(n-1).(n-2).(n-3)z^{n-4}y - \ldots \pm n.(n-1)\ldots(n-2m+2).(n-2m+1 \times z^{n-2m}y + \sqrt{1-y^2} \times \{nz^{n-1} - n.(n-1).(n-2)z^{n-3} + n.(n-1).(n-2).(n-3)\times (n-4)z^{n-5} - \ldots \pm n.(n-1)\ldots(n-2m)z^{n-2m-1}\} \mp n.(n-1)\ldots (n-2m).(n-2m-1)\times\int z^{n-2m-2}dy$, the signs $+$ and $-$ being used according as m is even or odd.

Let now n be an even number, and let the operation be repeated $\frac{n-2}{2}$ times, then $m = \frac{n-2}{2}$, and $n-2m-2=0$, $n-2m+1=3$, $n-2m=2$, $n-2m-1=1$, whence we have

$$\left.\begin{array}{l} \int z^n dy = z^n y - n.(n-1)z^{n-2}y + \ldots \pm n.(n-1).(n-2)\ldots 4\times 3\times z^2 y \\ \quad + \sqrt{1-y^2}\times\{nz^{n-1} - n.(n-1).(n-2)z^{n-3} + \ldots \pm n\times (n-1)\ldots\times 3\times 2\times z\} \\ \quad \mp n.(n-1).(n-2)\ldots\times 3\times 2\times 1\times y \end{array}\right\}$$

Again, let n be odd, and the operation be repeated $\frac{n-3}{2}$ times, then $m = \frac{n-3}{2}$, $n-2m-2=1$, $n-2m+1=4$, $n-2m=3$, $n-2m-1=2$, whence we have, in this case,

$$\left.\begin{array}{l} \int z^n dy = z^n y - n.(n-1)z^{n-2}y + \ldots \pm n.(n-1)\ldots 5\times 4\times z^2 y \\ \quad + \sqrt{1-y^2}\times\{nz^{n-1} - n.(n-1).(n-2)z^{n-3} + \ldots \pm n\times (n-1)\ldots 4\times 3\times z\} \\ \quad \mp n.(n-1).(n-2)\ldots 3\times 2\times\int z dy \end{array}\right\}$$

But $\int z dy = zy - \int y dz = zy - \int\frac{y dy}{\sqrt{1-y^2}}$

$$= zy + \sqrt{1-y^2}$$

Hence, then we finally get, when n is odd,

$$\int z^n dy = z^n y - n.(n-1)\,z^{n-2}y + \ldots. \pm n.(n-1)\ldots. \times 5\times 4\times z^2 y$$
$$\mp\, n.(n-1).(n-2)\ldots. 3\times 2\times zy + \sqrt{1-y^2}\,\{n.z^{n-1} - n.(n-1)\times (n-2)\,z^{n-3} + \ldots. \pm n.(n-1)\ldots. 4\times 3z \mp n.(n-1).(n-2)\ldots. 3\times 2\}$$

A similar process will lead to the integral of $z^n d.\varphi z$, where φz is any function whatever of the arc z, as cosine, versine, tangent, cotangent, secant, cosecant, &c., log. z, log. (cos. z) &c. e^z, $e^{\cos. z}$, &c. &c. &c.

535. To integrate $\dfrac{d\times z^{pn+\frac{n}{2}-1}\,dz}{a+bz^n}$, put $z^{\frac{n}{2}} = u$

$$\therefore\ z^{pn} = u^{2p}$$

$$\text{And } dz^{\frac{n}{2}} = \frac{n}{2}\,z^{\frac{n}{2}-1}\,dz = du$$

$$\therefore\ \frac{d\times z^{pn+\frac{n}{2}-1}\,dz}{a+bz^n} = \frac{2}{n}\,d\times\frac{u^{2p}du}{a+bu^2} = \frac{2}{nb}\,d\times\frac{u^{2p}du}{u^2+\frac{a}{b}} = \frac{2d}{nb}$$

$$\times\left\{u^{2(p-1)}du - \frac{a}{b}.u^{2(p-2)}\,du + \frac{a^2}{b^2}.\,u^{2.(p-3)}\,du - \ldots. \pm\frac{a^{p-1}}{b^{p-1}}\,du \mp\frac{a^p}{b^p}\times\frac{du}{u^2+\frac{a}{b}}\right\}$$ according as p is odd or even.

$$\text{Hence } \int\frac{d\times z^{pn+\frac{n}{2}-1}\,dz}{a+bz^n} = \frac{2d}{nb}\times\left\{\frac{u^{2p-1}}{2p-1} - \frac{a}{b}\times\frac{u^{2p-3}}{2p-3}\right.$$
$$\left. +\,\frac{a^2}{b^2}\times\frac{u^{2p-5}}{2p-5} - \&\text{c.} \pm\frac{a^{p-1}}{b^{p-1}}\,u \mp\frac{a^{p-1}}{b^{p-1}}\times\tan.^{-}\sqrt{\tfrac{a}{b}}\,u\right\}$$ in

which, if for u we substitute its assumed value $z^{\frac{n}{2}}$, we shall have the integral required.

To integrate $d\times\dfrac{z^{\frac{5}{2}n-1}\,dz}{\sqrt{a+bz^n}}$, put $z^{\frac{n}{2}} = u$

$$\text{Then } z^{\frac{5n}{2}} = u^5$$

$$\text{And } z^{\frac{5n}{2}-1}dz = \frac{2}{n}\times u^4\,du$$

Also $\sqrt{a+bz^n} = \sqrt{a+bu^2}$

Hence $\dfrac{d\times z^{\frac{5}{2}n-1}dz}{\sqrt{u+bz^n}} = \dfrac{2d}{n} \times \dfrac{u^4du}{\sqrt{a+bu^2}}$, which is a reduced form. For a further reduction, assume

$$\left.\begin{array}{l} P_1 = u^3\sqrt{a+bu^2} \\ P_2 = u\sqrt{a+bu^2} \end{array}\right\} \quad \left.\begin{array}{l} \int\dfrac{u^4du}{\sqrt{a+bu^2}} = F_1 \\ \int\dfrac{u^2du}{\sqrt{a+bu^2}} = F_2 \\ \int\dfrac{du}{\sqrt{a+bu^2}} = F_3 \end{array}\right\}$$

$$\therefore dP_1 = 3u^2du\sqrt{a+bu^2} + \frac{bu^4du}{\sqrt{a+bu^2}}$$

$$= \frac{3au^2du}{\sqrt{a+bu^2}} + \frac{4bu^4du}{\sqrt{a+bu^2}}$$

$$\therefore P_1 = 3a\,F_2 + 4b.\,F_1$$

Similarly $P_2 = a\,F_3 + 2b.\,F_2$

$$\therefore F_1 = \frac{P_1}{4b} - \frac{3a}{4b}\times F_2$$

$$= \frac{P_1}{4b} - \frac{3a}{4b}\times\frac{P_2}{2b} + \frac{3a^2}{8b^2}\times F_3$$

But $F_3 = \int\dfrac{du}{\sqrt{a+bu^2}} = \dfrac{1}{\sqrt{b}}.\int\dfrac{du}{\sqrt{\dfrac{a}{b}+u^2}} = \dfrac{1}{\sqrt{b}}$

$l.\left(u + \sqrt{\dfrac{a}{b}+u^2}\right)$

Hence $\int\dfrac{d\times z^{\frac{5}{2}n-1}dz}{\sqrt{a+bz^n}} = \dfrac{2d}{n}\times F_1$

$$= \frac{d}{2nb}\times\left\{P_1 - \frac{3a}{2b}\,P_2 + \frac{3a}{2b^{\frac{3}{2}}}\times\right.$$

$\left.l.\left(u + \sqrt{\dfrac{a}{b}+u^2}\right)\right\}$ in which P_1 P_2 and u may be expressed in terms of.

536. To integrate $\frac{dz}{(a+bz)^{\frac{m}{n}}}$, put $(a+bz)^{\frac{m}{n}} = u$

Then $a + bz = u^{\frac{n}{m}}$

And $bdz = \frac{n}{m} u^{\frac{n}{m}-1} du$

$$\therefore \int \frac{dz}{(a+bz)^{\frac{m}{n}}} = \frac{n}{mb} \int u^{\frac{n}{m}-2} du = \frac{n}{mb} \times \frac{m}{n-m} u^{\frac{n}{m}-1}$$

$$= \frac{n}{b.(n-m)} \times (a+bz)^{\frac{n-m}{n}}$$

To integrate $\frac{x^5dx}{a^2-x^2}$, we have

$$\frac{x^5dx}{a^2-x^2} = -\frac{x^5dx}{x^2-a^2} = -x^3dx - a^2xdx - \frac{a^4xdx}{x^2-a^2}$$

by actual division.

$$\therefore \int \frac{x^4dx}{a^2-x^2} = -\frac{x^4}{4} - \frac{a^2x^2}{2} - \frac{a^4}{2} l.(x^2-a^2)$$

To integrate $(a^2+x^2)^m \times x^2dx$, we have

$$(a^2+x^2)^m = a^{2m}\left(1+\frac{x^2}{a^2}\right)^m = a^{2m}\left\{1 + m\frac{x^2}{a^2} + m.\frac{m-1}{2} \times \frac{x^4}{a^4} + \&c.\right\}$$

$$\therefore \int (a^2+x^2)^m x^2dx = a^{2m}\int \left\{x^2dx + \frac{m}{a^2} x^4dx + \frac{m.(m-1)}{2a^4} \times x^6dx + \frac{m.(m-1)(m-2)}{2.3a^3} x^8dx + \&c.\right\} = a^{2m}.\left\{\frac{x^3}{3} + \frac{m}{5a^2}.x^5 + \frac{m.(m-1)}{2.7.a^4} x^7 + \&c.\right\}$$

which will terminate when m is a positive integer, the number of terms being $(m+1)$

Otherwise.

Assume

$$\left.\begin{aligned} P_1 &= x^3.(a^2+x^2)^m \\ P_2 &= x^3.(a^2+x^2)^{m-1} \\ P^3 &= x^3.(a^2+x^2)^{m-2} \end{aligned}\right\} \begin{aligned} F_0 &= \int x^2dx.(a^2+x^2)^m \\ F_1 &= \int x^2dx.(a^2+x^2)^{m-1} \\ \&c. &= \&c. \end{aligned}$$

$$\&c. = \&c.$$

Then $dP_1 = 3x^2dx\,(a^2+x^2)^m + 2m\,x^4dx.\,(a^2+x^2)^{m-1}$

But $x^4dx\,(a^2+x^2)^{m-1} = x^2dx\,(a^2+x^2)^{m-1} \times (a^2+x^2-a^2)$

$= x^2dx.(a^2+x^2)^m - a^2x^2dx.(a^2+x^2)^{m-1}$

Hence $dP_1 = (3+2m)x^2dx.(a^2+x^2)^m - 2ma^2.x^2dx \times (a^2+x^2)^{m-1}$

$\therefore\ P_1 = (3+2m).\,F_1 - 2ma^2.\,F_2$

Similarly $P_2 = 3 + 2(m-1)\,F_2 - 2.\,(m-1).\,a^2.\,F_3$

&c. $=$ &c.

Let $3 + 2m = p$, $2m = q$

Then $F_1 = \dfrac{P_1}{p} + \dfrac{q}{p}\,a^2\,F_2$

$F_2 = \dfrac{P_2}{p-2} + \dfrac{q-2}{p-2}.\,a^2\,F_3$

$F_3 = \dfrac{P_3}{p-4} + \dfrac{q-4}{p-4}.\,a^2\,F_4$

&c. $=$ &c.

from which equations, by elimination, we get $F_1 = \dfrac{P_1}{p} + \dfrac{q}{p.(p-2)}\,a^2.\,P_2 + \dfrac{q.(q-2)}{p(p-2).(p-4)}\,a^4\,P_3$ + &c., which will terminate when m = a positive integer.

When m is negative, a different assumption must be made; which case as well as that in which m is fractional we leave to be investigated by the Reader.

537. First we have $v^m = \left(l.\dfrac{1}{x^2}\right)^m = \left(2l.\dfrac{1}{x}\right)^m$

$= 2^m.\left(l.\dfrac{1}{x}\right)^m = 2^m\,(-lx)^m = (-2)^m.\,(lx)^m$

$\therefore \int v^m x^n dx = (-2)^m \int x^n dx \times (l.x)^m$

Now $\int x^n dx.\,(lx)^m = \dfrac{x^{n+1}}{n+1}.(l.x)^m - \int \dfrac{x^{n+1}}{n+1}\,d.\,(lx)^m$ from the form $\int udv = uv - \int vdu$

But $d.\,(lx)^m = \dfrac{m(l.x)^{m-1}.dx}{x}$

$\therefore \int \dfrac{x^{n+1}}{n+1}.\,d.(lx.)^m = \dfrac{m}{n+1}\int x^n dx.\,(l.x)^{m-1}$

Hence $\therefore \int x^n dx.(l.x)^m = \frac{x^{n+1}}{n+1}.(l.x)^m - \frac{m}{n+1}.\int x^n dx.(l.x)^{m-1}$

Similarly $\int x^n dx.(l.x)^{m-1} = \frac{x^{n+1}}{n+1}.(l.x)^{m-1} - \frac{m-1}{n+1}.\int x^n dx.(l.x)^{m-2}$.

&c. $=$ &c. the law of continuation being evident. By substitution we get

$$\int x^n dx.(lx).^m = \frac{x^{n+1}}{n+1}.(l.x)^m - \frac{m}{(n+1)^2}.(l.x)^{m-1} +$$

$$\frac{m.(m-1)x^{n+1}}{(n+1)^3}.(l.x)^{m-2} - \text{\&c.} = \frac{x^{n+1}}{n+1} \times \Big\{(lx)^m - \frac{m}{n+1}(lx.)^{m-1}$$

$$+ \frac{m.(m-1).}{(n+1)^2}.(l.x)^{m-2} \text{ \&c.} \pm \frac{m.(m-1).\text{\&c.} \times (m-\overline{p-2})}{(n+1)^{p-1}}(lx)^{m-p+1}$$

$\mp$ &c.$\}$ p being a positive integer.

Hence, then, we get $\int v^m x^n dx = (-2)^m \int x^n dx\ (l.x)^m = (-2)^m \frac{x^{n+1}}{n+1} \times$

$$\Big\{(l.x)^m - \frac{m}{n+1}(l.x)^{m-1} + \frac{m.(m-1)}{(n+1)^2}.(l.x)^{m-2} - \text{\&c.}$$

$\pm \frac{m.(m-1)....(m-p+2)}{(n+1)^{p-1}}.(l.x)^{m-p+1} \mp$ &c.$\Big\}$ which being continued until $p-1$ is the nearest integer in m, will give us the required integral in its most simple form. If m be an integer, the series will terminate in the $(p-1)^{th}$ term, and in this case we shall have the complete integral.

Otherwise.

We learn from the above result, that we are at liberty to assume $\int x^n dx.(l.x)^m$

$= \frac{x^{n+1}}{n+1}.\{(l.x)^m - \text{A}.\ (l.x)^{m-1} + \text{B}.\ (l.x)^{m-2} - \text{\&c.}\}$ whence

by differentiation, &c. and by the comparison of homologous terms, we get the values of A, B, C, &c. the same as before.

For this kind of general assumptions, see Vol. I. *Simpson's Fluxions*, (last edition); which, on this subject as well as for a collection of excellent problems, is exceedingly valuable. The writings of this very ingenious and original Author, appearto be little known,

although their great perspicuity in theory, elucidated by numerous and elegant applications, entitles them to the very first rank.

To integrate, $\frac{x^2dx}{x^2-3x+2}$ we must take away the second term of the denominator by assuming

$$x - \frac{3}{2} = u$$

$$\therefore x^2 - 3x + 2 = u^2 - \frac{9}{4} + 2 = u^2 - \frac{1}{4}$$

$$\text{And } x^2 = 3x - 2 + u^2 - \frac{1}{4} = 3u + \frac{9}{2} - 2 - \frac{1}{4} + u^2$$

$$= u^2 + 3u + \frac{9}{4}$$

$$\therefore \int \frac{x^2dx}{x^2-3x+2} = \int \frac{(u^2+3u+\frac{9}{4})du}{u^2-\frac{1}{4}} = \int \frac{(u^2-\frac{1}{4}+3u+\frac{5}{2})\,du}{u^2-\frac{1}{4}}$$

$$= \int du + \int \frac{3udu}{u^2-\frac{1}{4}} + \frac{5}{2}\int \frac{du}{u^2-\frac{1}{4}}$$

$$= u + \frac{3}{2}.\, l.\, \left(u^2 - \frac{1}{4}\right) + \frac{5}{2}.\ l.\ \frac{u-\frac{1}{2}}{u+\frac{1}{2}}$$

where $u = x - \frac{3}{2}$

Otherwise.

Assume $\frac{1}{x^2-3x+2} = \frac{A}{x-1} + \frac{B}{x-2}$, 1 and 2 being the roots of the denominator. By reducing to a common denominator and equating homologous terms with regard to x of the numerator, we shall find A and B; the rest of the investigation will be easy.

538. To integrate $\frac{d \times dz}{z\sqrt{a+bz^n}}$ assume $\sqrt{a+bz^n} = u$

Then $a + bz^n = u^2$

And $z = \dfrac{(u^2 - a)^{\frac{1}{n}}}{b^{\frac{1}{n}}}$

$$\therefore dz = \frac{2}{nb^{\frac{1}{n}}} . udu \times (u^2 - a)^{\frac{1}{n}-1}$$

Hence $\dfrac{d \times dz}{z\sqrt{a+bz^n}} = \dfrac{2d}{nb^{\frac{1}{n}}} \times b^{\frac{1}{n}} \times \dfrac{(u^2 - a)^{\frac{1}{n}-1}}{(u^2 - a)^{\frac{1}{n}}} \times udu$

$$= \frac{2d}{n} \times \frac{udu}{u^2 - a}$$

$$\therefore \int \frac{d \times dz}{z.\sqrt{a + bz^n}} = \frac{d}{n\sqrt{a}} \times \int \frac{2\sqrt{a}udu}{u^2 - a}$$

$$= \frac{d}{n\sqrt{a}} \times l. \frac{u - \sqrt{a}}{u + \sqrt{a}}$$

$$= \frac{d}{n\sqrt{a}} \times l. \frac{(u - \sqrt{a})^2}{u^2 - a}$$

But $u^2 - a = bz^n = (\sqrt{bz^n})^2$

And $u - \sqrt{a} = \sqrt{a + bz^n} - \sqrt{a}$

Hence $\displaystyle\int \frac{d \times dz}{z\sqrt{a + bz^n}} = \frac{2d}{n\sqrt{a}} \times l. \frac{\sqrt{a + bz^n} - \sqrt{a}}{\sqrt{bz^n}}$

To integrate $\dfrac{z^{\frac{n}{2}-1}dz}{z^{pn} \times (a + bz^n)}$, let $z^{\frac{n}{2}} = u$

$$\left.\begin{aligned} \text{Then } z^{\frac{n}{2}}dz &= \frac{2}{n} du \\ z^{pn} &= u^{2p} \\ \text{And } a + bz^n &= a + bu^2 \end{aligned}\right\}$$

$$\therefore \frac{z^{\frac{n}{2}-1}dz}{z^{pn}.(a + bz^n)} = \frac{2}{n} \times \frac{du}{u^{2p} \times (a + bu^2)}$$

Again, let $\dfrac{1}{u} = w$

Then $\dfrac{1}{u^{2p-1}} = w^{2p-1}$

And $-\dfrac{(2p-1)du}{u^{2p}} = (2p-1)\,w^{2p-2}dw$

$\therefore\ \dfrac{du}{u^{2p}} = -\,w^{2p-2}\,dw$

Also $a + bu^2 = a + b.\,\dfrac{1}{w^2} = \dfrac{aw^2+b}{w^2}$

$$\therefore \int \frac{z^{\frac{n}{2}-1}\,dz}{z^{pn}.(a+bz^n)} = -\frac{2}{na} \times \int \frac{w^{2p}dw}{w^2+\frac{b}{a}}$$

But $\dfrac{w^{2p}}{w^2+\frac{b}{a}} = w^{2p-2} - \dfrac{b}{a}\,w^{2p-4} + \dfrac{b^2}{a^2}.\,w^{2p-6} - \&c.$

$$\therefore \int \frac{z^{\frac{n}{2}-1}\,dz}{z^{pn}.(a+bz^n)} = -\frac{2}{na} \times \left\{\frac{w^{2p-1}}{2p-1} - \frac{b}{a}.\,\frac{w^{2p-3}}{2p-3} + \frac{b^2}{a^2}.\,\frac{w^{2p-5}}{2p-5} - \&c.\right\}$$ which will terminate when p is a positive integer, and the last term will be $\pm\dfrac{b^p}{a^p}\displaystyle\int\frac{dw}{w^2+\frac{b}{a}}$ or $\pm\dfrac{b^p}{a^p}$ $\times\dfrac{a}{b}\times\displaystyle\int\frac{\frac{b}{a}dw}{w^2+\frac{b}{a}} = \pm\dfrac{b^{p-1}}{a^{p-1}}\times\tan.^{-}\sqrt{\tfrac{b}{a}}\,w$. If p be positive and not an integer, let the series be continued until we arrive at the lowest positive value of the index of w.

Then substitute for w its value in terms of z, and we have the integral in more simple terms.

To integrate $\dfrac{dz}{z^{\frac{n}{2}+1}\times\sqrt{a+bz^n}} = d.\,\mathrm{F}.$

Assume $z^{-\frac{n}{2}} = \dfrac{\sqrt{b}u}{\sqrt{a}}$

Then $-\dfrac{n}{2}\times z^{-\frac{n}{2}-1}dz = \dfrac{\sqrt{b}du}{\sqrt{a}}$

$\therefore\ \dfrac{dz}{z^{\frac{n}{2}+1}} = -\dfrac{2}{n}\,\dfrac{\sqrt{b}du}{\sqrt{a}}$

Also $\sqrt{a+bz^n} = \sqrt{a+\dfrac{a}{u^2}} = \dfrac{\sqrt{a}.\sqrt{1+u^2}}{u}$

$\therefore$ F $= \frac{-2\sqrt{b}}{na} \cdot \int \frac{udu}{\sqrt{1+u^2}} = - \frac{2\sqrt{b}}{na} \sqrt{1+u^2}$ which is $\therefore$ known.

539. $$\int \frac{d \times dz}{\sqrt{a+bz}} = \frac{2d}{b} \times \int \frac{\frac{1}{2}bdz}{\sqrt{a+bz}} = \frac{2d}{b} \sqrt{a+bz}$$

Again, $\int \frac{d \times dz}{\sqrt{a+bz^2}} = \frac{d}{\sqrt{b}} \int \frac{dz}{\sqrt{\frac{a}{b}+z^2}} = \frac{d}{\sqrt{b}} l.(z+\sqrt{\frac{a}{b}+z^2})$

Again, $\int \frac{xdx}{\sqrt{ax-x^2}} = \int \frac{\frac{adx}{2} - \left(\frac{adx}{2} - xdx\right)}{\sqrt{ax-x^2}} = \int \frac{\frac{a}{2}dx}{\sqrt{ax-x^2}}$

$- \int \frac{1}{2} \frac{adx-2xdx}{\sqrt{ax-x^2}} =$ vers.$^{-\frac{a}{2}} x - \sqrt{ax-x^2}$ by common forms.

540. To integrate $\frac{dx}{x^3\sqrt{a^4+x^4}} = d.$F, assume $\frac{1}{x^2} = u$

Then $\frac{dx}{x^3} = -\frac{1}{2} du$

And $\sqrt{a^4+x^4} = \sqrt{a^4+\frac{1}{u^2}} = \frac{\sqrt{u^2a^4+1}}{u}$

$\therefore d.$F $= \frac{-\frac{1}{2}udu}{\sqrt{a^4u^2+1}}$

And F $= -\frac{1}{2a^4} \times \sqrt{a^4u^2+1} = -\frac{1}{2a^4} \times \frac{\sqrt{a^4+x^4}}{x^2}$

To integrate $dx\sqrt{bx-cx^2} = d.$F, we have

$d.$F $= \sqrt{c}.\, dx \sqrt{\frac{b}{c}x - x^2}$

Assume $x - \frac{b}{2c} = u$

Then $x^2 - \frac{b}{c}x = u^2 - \frac{b^2}{4c^2}$

And $dx = du$

$$\therefore \; d.\mathrm{F} = \sqrt{c}\, du. \sqrt{\frac{b^2}{4c^2} - u^2}$$

Now, if $u =$ the abcissa measured from the centre of a circle whose radius is $\frac{b}{2c}$, the corresponding ordinate will be $\sqrt{\frac{b^2}{4c^2} - u^2}$; whence by the common expression for the differential of an area, we have $\mathrm{F} = \sqrt{c} \times$ area between the ordinates, the ordinates at the extremities of u, or of $x - \frac{b}{2c}$ and is therefore known approximately, because the circular area cannot accurately be found.

541. To integrate $\dfrac{dx}{(a^n + x^n)^{\frac{n+1}{n}}} = d.\mathrm{F}$, assume

$$\mathrm{P} = \frac{x}{(a^n + x^n)^{\frac{1}{n}}}$$

$$\text{Then } d\mathrm{P} = \frac{dx}{(a^n + x^n)^{\frac{1}{n}}} - \frac{x^n dx}{(a^n + x^n)^{\frac{1}{n}+1}}$$

$$\text{But } \frac{x^n dx}{(a^n + x^n)^{\frac{1}{n}+1}} = \frac{dx}{(a^n + x^n)^{\frac{1}{n}+1}} \times (a^n + x^n - a^n)$$

$$= \frac{dx}{(a^n + x^n)^{\frac{1}{n}}} - \frac{a^n dx}{(a^n + x^n)^{\frac{1}{n}+1}}$$

$$\text{Hence } d\mathrm{P} = \frac{a^n dx}{(a^n + x^n)^{\frac{1}{n}+1}} = a^n \times d\mathrm{F}$$

$$\therefore \; \mathrm{F} = \frac{\mathrm{P}}{a^n} = \frac{1}{a^n} \times \frac{x}{(a^n + x^n)^{\frac{1}{n}}}$$

To integrate $\dfrac{dx}{(x-1)^{\frac{3}{2}}.(x+1)^{\frac{1}{2}}} = d.\mathrm{F}$, assume

$$\sqrt{\frac{x+1}{x-1}} = \mathrm{P}$$

$$\text{Then } d.\text{P} = \frac{\frac{1}{2}dx}{\sqrt{x+1}\times\sqrt{x-1}} - \frac{\frac{1}{2}dx.\sqrt{x+1}}{(x-1)^{\frac{3}{2}}}$$

$$= \frac{1}{2}\,dx.\frac{x-1-x-1}{(x-1)^{\frac{3}{2}}.(x+1)^{\frac{1}{2}}}$$

$$= -\frac{dx}{(x-1)^{\frac{3}{2}}.(x+1)^{\frac{1}{2}}} = -\,d.\text{F}$$

$$\therefore \text{F} = -\text{P} = -\sqrt{\frac{x+1}{x-1}}$$

This kind of assumption is useful in the integration of all functions of the form $\dfrac{dx}{(x-a)^{\frac{n}{m}+1}\times(x+a)^{-\frac{n}{m}+1}}$, and of many other forms which may be found in *Euler*, and in a very useful work lately published, entitled, *Examples of the Applications of the Differential and Integral Calculus.* By G. Peacock, A.M. F.R.S., &c. The reader will find this work a great treasure.

To integrate $dx\int dx\int\dfrac{dx}{x} = d\,\text{F}$, we have

$$\int\frac{dx}{x} = l.x$$

$$\therefore \int dx\int\frac{dx}{x} = \int dx.\,l.x = xl.x - \int x.\frac{dx}{x}$$

$$= xl.x - x \text{ by the form} \int udv = uv - \int vdu$$

$$\text{Hence F} = \int du\int dx\int\frac{dx}{x} = \int xdx.\,l.x - \int xdx$$

$$= \frac{x^2}{2}\,l.x - \int\frac{x^2}{2}.\frac{dx}{x} - \frac{x^2}{2}$$

$$= \frac{x^2}{2}\,l.x - \frac{x^2}{4} - \frac{x^2}{2}$$

$$= \frac{x^2}{4}\times(2l\,x - 3)$$

To investigate the value of $e^{\int \frac{d\theta}{\sin.\theta}} = \mathrm{F}$, we have

$$l.\,\mathrm{F} = \int \frac{d\,\theta}{\sin.\theta}$$

$$\therefore \frac{d\,\mathrm{F}}{\mathrm{F}} = \frac{d\,\theta}{\sin.\theta} = \frac{d.\frac{\theta}{2}}{\sin.\frac{\theta}{2}.\cos.\frac{\theta}{2}} = \frac{\dfrac{d\frac{\theta}{2}}{(\cos.\frac{\theta}{2})^2}}{\dfrac{\sin.\frac{\theta}{2}}{\cos.\frac{\theta}{2}}}$$

$$= \frac{d.\tan.\frac{\theta}{2}}{\tan.\frac{\theta}{2}}$$

$$\therefore l.\,\mathrm{F} = l.\tan.\frac{\theta}{2}$$

And $\mathrm{F} = \tan.\dfrac{\theta}{2}$

$\therefore e^{\int \frac{d\theta}{\sin.\theta}} = \tan.\dfrac{\theta}{2}$ the value required.

542. To integrate $\dfrac{x^2dx}{\sqrt{2ax-x^2}} = d\,\mathrm{F}$, assume

$$x\sqrt{2ax-x^2} = \mathrm{P}$$

Then $d\mathrm{P} = dx\sqrt{2ax-x^2} + \dfrac{axdx - x^2dx}{\sqrt{2ax-x^2}}$

$$= \frac{3axdx}{\sqrt{2ax-x^2}} - \frac{2x^2dx}{\sqrt{2ax-x^2}}$$

$$\therefore d\mathrm{F} = \frac{3a}{2} \times \frac{xdx}{\sqrt{2ax-x^2}} - \frac{d\mathrm{P}}{2}$$

$$= \frac{3a}{2} \times \frac{adx - (adx - xdx)}{\sqrt{2ax-x^2}} - \frac{d\mathrm{P}}{2}$$

$$= \frac{3a}{2} \times \frac{adx}{\sqrt{2ax-x^2}} - \frac{3a}{2} \times \frac{adx - xdx}{\sqrt{2ax-x^2}} - \frac{d\mathrm{P}}{2}$$

$$\therefore \mathrm{F} = \frac{3a}{2} \times \text{vers.}^{-1}x - \frac{3a}{2}\sqrt{2ax-x^2} - \frac{x\sqrt{2ax-x^2}}{2}$$

To integrate $\frac{dx}{x}\,(2ax - x^2)^{\frac{3}{2}} = d.\mathrm{F}$,

Assume $x - a = u$

Then $x^2 - 2ax = u^2 - a^2$

And $2ax - x^2 = a^2 - u^2$

Also $dx = du$

And $x = a + u$

$$\therefore\ \mathrm{F} = \frac{du}{a+u} \times (a^2 - u^2)^{\frac{3}{2}} = du\,.\,(a^2 - u^2)^{\frac{1}{2}} \times (a - u)$$

$$= adu\,(a^2 - u^2)^{\frac{1}{2}} - udu\,(a^2 - u^2)^{\frac{1}{2}}$$

Let now u be the abscissa of a circle (whose radius is (a)) measured from its centre, then the corresponding ordinate is $(a^2 - u^2)^{\frac{1}{2}}$, and $\int du\,.\,(a^2 - u^2)^{\frac{1}{2}} =$ (by the differential expression of an area) the area comprised between the ordinates at the extremities of the abscissa u, which area we will call A.

$$\text{Hence } \mathrm{F} = a\,\mathrm{A} + \frac{1}{3}\,(a^2 - u^2)^{\frac{3}{2}}$$

$$= a\,\mathrm{A} + \frac{1}{3}\,(2ax - x^2)^{\frac{3}{2}}$$

543. $$\int \frac{a^2dx}{b^4 + c^2x^2} = \int \frac{a^2dx}{c^2.\left(\frac{b^4}{c^2} + x^2\right)} = \frac{a^2}{b^4}.\int \frac{\frac{b^4}{c^2}\,dx}{\frac{b^4}{c^2} + x^2}$$

Let $\frac{b^4}{c^2} = r^2$

$$\text{Then } \int \frac{a^2dx}{b^4 + c^2x^2} = \frac{a^2}{b^4}\int \frac{r^2dx}{r^2 + x^2} = \frac{a^2}{b^4} \times \tan.^{-r}x$$

544. To integrate $dx \int \frac{dx}{1 - x^3} = d\,\mathrm{F}$, we have

$$1 - x^3 = (1 - x).(x^2 + x + 1) = (1 - x).\left(x + \frac{1 - \sqrt{-3}}{2}\right)\left(x + \frac{1 + \sqrt{-3}}{2}\right)$$

Hence, assuming $\dfrac{1}{1-x^3} = \dfrac{A}{1-x} + \dfrac{Bx+C}{x^2+x+1}$ and reducing to a common denominator, we get

$$A.(x^2+x+1) + B.x.(1-x) + C \times (1-x) = 1$$

Let $x = 1$

Then $A \times 3 = 1$, and $A = \dfrac{1}{3}$

Again, let $x = -\dfrac{1+\sqrt{-3}}{2}$

Then $B \times -\dfrac{1+\sqrt{-3}}{2} \times \dfrac{3+\sqrt{-3}}{2} + C \times \dfrac{3+\sqrt{-3}}{2} = 1$

Or $-\sqrt{-3} \times B + C \times \dfrac{3}{2} + \dfrac{\sqrt{-3}}{2} \times C = 1$. Hence, equating real and imaginary quantities, we have

$$\left.\begin{array}{r} C \times \dfrac{3}{2} = 1 \\ \text{And } \dfrac{C}{2} - B = 0 \end{array}\right\} \quad \begin{array}{l} \therefore C = \dfrac{2}{3} \\ \text{and } B = \dfrac{1}{3} \end{array}$$

$$\therefore \int \frac{dx}{1-x^3} = \frac{1}{3}\int\frac{dx}{1-x} + \frac{1}{3}\int\frac{xdx+2dx}{x^2+x+1}$$

$$= -\frac{1}{3}l.(1-x) + \frac{1}{6}\int\frac{2xdx+dx}{x^2+x+1} + \frac{3}{6}\int\frac{dx}{x^2+x+1}$$

$$= -\frac{1}{3}l.(1-x) + \frac{1}{6}l.(x^2+x+1) + \frac{1}{2}\int\frac{dx}{x^2+x+1}$$

Now, put $x + \dfrac{1}{2} = u$

Then $x^2 + x + 1 = u^3 - \dfrac{1}{4} + 1 = u^3 + \dfrac{3}{4}$

And $dx = du$

$$\therefore \int\frac{dx}{x^2+x+1} = \int\frac{du}{u^2+\frac{3}{4}} = \frac{4}{3}\int\frac{\frac{3}{4}du}{u^2+\frac{3}{4}} = \frac{4}{3} \times \tan.^{-\sqrt{\frac{4}{3}}} u$$

$$\therefore \int\frac{dx}{1-x^3} = -\frac{1}{3} \times l.(1-x) + \frac{1}{6}l.(x^2+x+1) + \frac{2}{3} \times \tan.^{-\frac{\sqrt{3}}{2}}\left(x + \frac{1}{2}\right).$$

Hence $d.\mathrm{F} = -\frac{1}{3}\,dx.\,l.\,(1-x) + \frac{1}{6}.dxl.\,(x^2+x+1) + \frac{2dx}{3} \times$

$\tan.^{-\frac{\sqrt{3}}{2}}\left(x+\frac{1}{2}\right)$ which three differentials we must integrate separately.

1. We have $\int dx.\,l.\,(1-x) = x.\,l.\,(1-x) - \int\frac{-xdx}{1-x} =$

$x.l.(1-x) - \int\frac{xdx}{x-1} = x.\,l.\,(1-x) - \int\left(dx + \frac{dx}{x-1}\right)$ by division.

$$\therefore \int dx.\,l.\,(1-x) = x.\,l.\,(1-x) - l.\,(x-1) - x.$$

2. $\int dx.\,l.\,(x^2+x+1) = \int du.\,l.\,\left(u^2+\frac{3}{4}\right)$

$$= u.\,l.\,\left(u^2+\frac{3}{4}\right) - \int\frac{2u^2du}{u^2+\frac{3}{4}}$$

But $\frac{2u^2du}{u^2+\frac{3}{4}} = 2du - \frac{\frac{3}{2}\,du}{u^2+\frac{3}{4}} = 2du - \frac{\frac{3}{4}\,du}{2\left(u^2+\frac{3}{4}\right)}$

$\therefore \int dx.\,l.\,(x^2+x+1) = u.\,l.\,\left(u^2+\frac{3}{4}\right) - 2u + \frac{1}{2}\tan.^{-\frac{\sqrt{3}}{2}}u$

3. $\int dx \times \tan.^{-\frac{\sqrt{3}}{2}}\left(x+\frac{1}{2}\right) = \int du \times \tan.^{-\frac{\sqrt{3}}{2}}u$; let θ be the circular arc whose radius is $\frac{\sqrt{3}}{2}$ and $\tan. = u$.

Then $u = \tan.\,\theta$

And $\int du \times \tan.\,u^{-\frac{\sqrt{3}}{2}} = \int d.\tan.\,\theta \times \theta = \theta \times \tan.\theta - \int d\theta.\tan.\theta$

$$= \theta.\,\tan.\,\theta - \int\frac{d\,\theta.\,\sin.\,\theta}{\cos.\,\theta}$$

But $\int\frac{d\theta.\,\sin.\,\theta}{\cos.\,\theta} = -\int\frac{d.\cos.\,\theta}{\cos.\,\theta} = -\,l.\,(\cos.\,\theta).$

$\therefore \int du \times \tan.^{-\frac{\sqrt{3}}{2}}u = \theta.\,\tan.\,\theta - l.\,(\cos.\,\theta)$ which is $\therefore$ known.

Hence, by substitution, we arrive at the value of

$$\left.\begin{aligned} F = & -\frac{1}{3}x\,l.(1-x)+\frac{1}{3}l.(x-1)+\frac{1}{3}x \\ & +\frac{1}{6}.ul.\left(u^2+\frac{3}{4}\right)+\frac{1}{12}\times\theta-\frac{1}{8}u \\ & +\frac{2}{3}\times\theta.\tan.\theta+\frac{2}{3}\times(l.\cos.\theta) \end{aligned}\right\}$$ which, by substituting for u and θ their assumed values, may be expressed in terms of x and constants.

To integrate $\dfrac{z^{2n-1}dz}{(g+hz^n)\times\sqrt{e+fz^n}} = \dfrac{z^{2n-1}dz}{\left(\frac{g}{h}+z^n\right)\times\sqrt{\frac{e}{f}+z^n}}$ $\times\dfrac{1}{h\sqrt{f}} = d.\text{F}$, let $\dfrac{g}{h}=a^2$, $\dfrac{e}{f}=b^2$, and assume $\sqrt{b^2+z^n}=u$

$$\text{Then } z^n = u^2 - b^2$$

$$\text{And } z^{n-1}dz = \frac{2}{n}.udu$$

$$\therefore\ z^{2n-1}dz = \frac{2}{n}.udu\times(u^2-b^2)$$

Hence, also $a^2+z^n = u^2-b^2+a^2 = u^2+c^2$ $(c^2=a^2-b^2)$

$$\therefore\ d\text{F} = \frac{2}{n.h\sqrt{f}}\times\frac{du\times(u^2-b^2)}{u^2+c^2}.$$

$$= \frac{2}{nh\sqrt{f}}\times\left(\frac{u^2du}{u^2+c^2}-\frac{b^2du}{u^2+c^2}\right)$$

$$= \frac{2}{nh\sqrt{f}}\times\left(du-\frac{c^2du}{u^2+c^2}-\frac{b^2du}{u^2+c^2}\right)$$

Let c^2 be positive, or $a > b$

$$\text{Then F} = \frac{2}{nh\sqrt{f}}\times\left(u-\tan.^{-1}\frac{u}{c}-\frac{b^2}{c^2}\times\tan.^{-1}\frac{u}{c}\right)$$

$$= \frac{2}{c^2nh.\sqrt{f}}\times\left(c^2u-(c^2+b^2).\tan.^{-1}\frac{u}{c}\right)$$

Let c^2 be negative, or $a < b$

Then $F = \frac{2}{n h \sqrt{f}} \times \left\{ u + \frac{c}{2} l. \frac{u-c}{u+c} - \frac{b^2}{2c} . l. \frac{u-c}{u+c} \right\} =$

$\frac{2}{n h \sqrt{f}} \times \left\{ u + \frac{c^2 - b^2}{2c} . l. \frac{u-c}{u+c} \right\}$ in which two forms, if for a, b, c, and u, we substitute their assumed values, the integral in both cases will be expressed in terms of given quantities.

545.
$$\int \frac{dz}{\sec. z. \operatorname{cosec}. z} = \int dz \times \cos. z. \sin. z$$
$$= \int d. \sin. z \times \sin. z$$
$$= \frac{\sin.^2 z}{2}$$

Now $\cos. 2z = 1 - 2 \sin.^2 z$

$$\therefore \frac{\sin.^2 z}{2} = \frac{1 - \cos. 2z}{2} = \frac{\text{vers. sin. } 2z}{2}$$

$$\therefore \int \frac{dz}{\sec. z. \operatorname{cosec}. z} = \frac{\text{vers. sin. } 2z}{2}$$

546.
$$\int \frac{xdx + dx}{x\sqrt{a^2 + x^2}} = \int \frac{dx}{\sqrt{a^2 + x^2}} + \int \frac{dx}{x\sqrt{a^2 + x^2}}$$
$$= l. (x + \sqrt{a^2 + x^2}) + \frac{1}{2a} \times l. \frac{\sqrt{a^2 + x^2} - a}{\sqrt{a^2 + x^2} + a}$$
$$= l. (x + \sqrt{a^2 + x^2}) + \frac{1}{a} . l. \frac{x}{\sqrt{a^2 + x^2} + a}$$

547. To integrate $\frac{x^5 dx}{\sqrt{a^2 - x^2}} = d.F_1$, assume

$$\left. \begin{array}{l} x^4 . \sqrt{a^2 - x^2} = P_1 \\ x^2 . \sqrt{a^2 - x^2} = P_2 \end{array} \right\} \int \frac{x^3 dx}{\sqrt{a^2 - x^2}} = F_2$$
$$\int \frac{xdx}{\sqrt{a^2 - x^2}} = F_3 = -\sqrt{a^2 - x^2}$$

Then $dP_1 = 4x^3dx\sqrt{a^2-x^2} - \dfrac{x^5dx}{\sqrt{a^2-x^2}}$

$$= \frac{4a^2x^3\,dx}{\sqrt{a^2-x^2}} - \frac{5x^5dx}{\sqrt{a^2-x^2}}$$

$$\left.\begin{array}{l} \therefore\ P_1 = 4a^2 \times F_2 - 5 \times F_1 \\ \text{Similarly } P_2 = 2a^2 \times F_3 - 3 \times F_2 \end{array}\right\}$$

Hence $F_1 = \dfrac{4a^2}{5} \times F_2 - \dfrac{1}{5} \times P_1 = \dfrac{8a^4}{15} \times F_3 - \dfrac{4a^2}{15}$

$\times P_2 - \dfrac{1}{5} P_1 = - \dfrac{\sqrt{a^2-x^2}}{15} \times \{8a^4 + 4a^2x^2 + 3x^4\}$

To integrate $\dfrac{dx}{x^4\sqrt{a^2+x^2}} = dF_1$, assume

$$\left.\begin{array}{l} x^{-3} \times \sqrt{a^2+x^2} = P_1 \\ x^{-1} \times \sqrt{a^2+x^2} = P_2 \end{array}\right\} \left.\begin{array}{l} \displaystyle\int \frac{dx}{x^2\sqrt{a^2+x^2}} = F_2 \\ \displaystyle\int \frac{dx}{\sqrt{a^2+x^2}} = F_3 \end{array}\right\}$$

Then $dP_1 = -3x^{-4}dx \times \sqrt{a^2+x^2} + \dfrac{x^{-2}dx}{\sqrt{a^2+x^2}}$

$$= - \frac{3a^2dx}{x^4\sqrt{a^2+x^2}} - \frac{2dx}{x^2\sqrt{a^2+x^2}}$$

$$\therefore\ P_1 = -3a^2 \times F_1 - 2 \times F_2$$

$$\left.\begin{array}{l} \therefore\ F_1 = -\dfrac{P_1}{3a^2} - \dfrac{2}{3a^2} \times F_2 \\ \text{Similarly } F_2 = -\dfrac{P_2}{a^2} \end{array}\right\}$$

$$\therefore\ F_1 = -\frac{P_1}{3a^2} + \frac{2}{3a^4} \times P_2$$

$$= \frac{\sqrt{a^2+x^2}}{3a^4 \times x^3} \times (2x^2 - a^2) \text{ the integral required.}$$

548. Assume $\displaystyle\int \frac{x^{2n+1}dx}{\sqrt{1-x^2}} = \sqrt{1-x^2}\{Ax^{2n} + Bx^{2n-2} +$

$Cx^{2n-4} + \&c.\ Qx^2 + L\}$

$$\text{Then } \frac{x^{2n+1}dx}{\sqrt{1-x^2}} = -\frac{xdx}{\sqrt{1-x^2}} \times (Ax^{2n} + Bx^{2n-2} + \&c.) \left.\begin{matrix} \\ + \sqrt{1-x^2} \times 2\{Ax^{2n-1} + (2n-)2Bx^{2n-3} + \&c.\} \times dx \end{matrix}\right\}$$

$$\text{Hence } x^{2n+1} = \left.\begin{matrix}(1-x^2) \times \{2n.Ax^{2n-1} + (2n-2)Bx^{2n-3} + \&c.\} \\ - Ax^{2n+1} - Bx^{2n-1} - Cx^{2n-3} - \&c.\end{matrix}\right\} =$$

$$\left.\begin{matrix} -2n\,Ax^{2n+1} - (2n-2).\ B \\ -\ \ Ax^{2n+1} -\qquad\quad B \\ +\ 2n \times A \end{matrix}\right\} x^{2n-1} - \left.\begin{matrix} -(2n-4)\ C \\ C \\ +(2n-2)\ B \end{matrix}\right\} x^{2n-3}\ \&c.\ \&c.$$

$$\left.\begin{matrix} \text{Hence, we get } -2nA - A = 1 \\ 2nA - (2n-1) \times B = 0 \\ (2n-2)B - (2n-3) \times C = 0 \\ \&c. = \&c. \\ \{2n-(n-1)\} \times Q - (2n-n) \times L = 0 \end{matrix}\right\} \text{from which equations}$$

we have $A = -\dfrac{1}{2n+1}$

$$B = -\frac{2n}{(2n+1).(2n-1)}$$

$$C = -\frac{2n\,(2n-2)}{(2n.+1).(2n-1).(2n-3)}$$

$\&c. = \&c.$

$$Q = -\frac{2n.(2n-2).(2n-4)\ldots\times 6\times 4}{(2n+1).(2n-1)(2n-3)..7\times 5\times 3}$$

$$\text{And } L = -\frac{2n.(2n-2)\ldots..6\times 4\times 2}{(2n+1).(2n-1).(2n-3)\ldots.5\times 3\times 1}$$

Now, all the terms in the assumed series except the last (L) have x as a factor.

$\therefore$ when $x = 0$, we have $\sqrt{1-x^2} = 1$ and the integral $= L$

Now, let $x = 1$ then $\sqrt{1-x^2} = 0$

And the integral (between $x = 0$ and 1) $= \sqrt{1-x^2} \times (Ax^{2n} + Bx^{2n-2} + \&c. + L) - L$

$$= -L = \frac{2\times 4\times 6\times \ldots.(2n-2)\times 2n}{1\times 3\times 5\times \ldots..(2n-1)\times(2n+1)}$$

$$= 2^n \times \frac{1\times 2\times 3\times \ldots.(n-1)\times n}{1\times 3\times 5\times \ldots.(2n-1)\times(2n+1)}$$

A similar process will conduct us to the required value of the other integral. For the sake of variety, however, we will use the following method:

$$\left.\begin{aligned}\text{Assume } \int\frac{x^{2n}dx}{\sqrt{1-x^2}} &= F_0 \\ \int\frac{x^{2n-2}dx}{\sqrt{1-x^2}} &= F_1 \\ \&c. &= \&c. \\ \int\frac{x^2dx}{\sqrt{1-x^2}} &= F_{n-1} \\ \int\frac{x^0dx}{\sqrt{1-x^2}} &= F_n\end{aligned}\right\} \qquad \left.\begin{aligned}\text{And } x^{2n-1}\sqrt{1-x^2} &= P_1 \\ x^{2n-3}\sqrt{1-x^2} &= P_2 \\ x^{2n-5}\sqrt{1-x^2} &= P_3 \\ \&c. &= \&c. \\ x^3.\ \sqrt{1-x^2} &= P_{n-1} \\ x.\ \sqrt{1-x^2} &= P_n\end{aligned}\right\}$$

$$\text{Then } dP_1 = (2n-1)\,x^{2n-2}dx\sqrt{1-x^2} - \frac{x^{2n}dx}{\sqrt{1-x^2}}$$

$$= (2n-1)\,\frac{x^{2n-2}dx}{\sqrt{1-x^2}} - 2n.\,\frac{x^{2n}dx}{\sqrt{1-x^2}}$$

$$\left.\begin{aligned}\therefore\ P_1 &= (2n-1)\,F_1 - 2n \times F_0 \\ \text{Similarly } P_2 &= (2n-3)\,F_2 - (2n-2)\,F_1 \\ \&c. &= \&c. \\ P_{n-1} &= 3 \times F_{n-1} - 4 \times F_{n-2} \\ P_n &= F_n - 2 \times F_{n-1}\end{aligned}\right\}$$

$$\left.\begin{aligned}\text{Hence } F_0 &= -\frac{1}{2n}.\,P_1 + \frac{2n-1}{2n}.\,F_1 \\ F_1 &= -\frac{1}{2n-2}\times P_2 + \frac{2n-3}{2n-2}\times F_2 \\ \&c. &= \&c. \\ F_{n-1} &= -\frac{P_n}{2} + \frac{1}{2}\times F_n\end{aligned}\right\}$$

$$\therefore\ F_0 = -\frac{1}{2n}.\,P_1 - \frac{2n-1}{2n\times(2n-2)}\times P_2 - \frac{(2n-1)(2n-3)}{2n.(2n-2)\,(2n-4)}$$

$$\times P_3 - \&c.$$

$$-\frac{(2n-1).(2n-3).\,(2n-5)\ldots5\times3}{2n.\,(2n-2).\,(2n-4)\ldots.4\times2}\times P_n$$

$$+\frac{(2n-1).\,(2n-3)\ldots.5\times3\times1}{2n.\,(2n-2)\ldots.4\times2}\times F_n$$

Now, since P_1, P_2....P_n have each x for a multiplier, when $x = 0$, all the terms involving them must vanish.

But $F_n = \int \frac{dx}{\sqrt{1-x^2}}$ = arc, whose radius is unity and sin. $= x$.

$\therefore$ When $x = 0$, $F_n = \pm m\pi$, where m may be 0, 1, 2, &c.

Hence, when $x = 0$, we have

$$\int \frac{x^{2n}dx}{\sqrt{1-x^2}} = \frac{(2n-1).(2n-3)....3 \times 1}{2n.(2n-2)....4 \times 2} \times (\pm m\pi)$$

Again, when $x = 1$, P_1, P_2, &c. P_n vanish, because $\sqrt{1-x^2}$ $(= 0)$ is a multiplier of each of them.

But $F_n = \sin.^{-1}x = \frac{\pi}{2} + 2p\pi$ in this case, p being any integer whatever.

Hence, when $x = 1$, we have

$$\int \frac{x^{2n}dx}{\sqrt{1-x^2}} = \frac{(2n-1).(2n-3)....3 \times 1}{2n.(2n-2)....4 \times 2} \times \left(\frac{\pi}{2} + 2p\pi\right)$$

$\therefore$ the integral of $\frac{x^{2n}dx}{\sqrt{1-x^2}}$ (between $x = 0$ and 1)

$$= \frac{(2n-1).(2n-3).....3 \times 1}{2n.(2n-2)....4 \times 2} \times \left(\frac{\pi}{2} + 2p\pi \mp m\pi\right)$$

$$= \frac{1 \times 3 \times ....(2n-3).(2n-1)}{2 \times 4....(2n-2).2n} \times \frac{\pi}{2} \times (4p \mp 2m + 1)$$

of which the value stated in the enunciation of the problem is a particular case, *viz.* that when $p = 0$, and $m = 0$.

By supposing n infinite, and m and $p = 0$, we shall get Wallis's expression; for then we have $2n + 1 = 2n$

$$\text{And } \frac{\pi}{2} \times \frac{1.3.5.7.....\text{ad inf.}}{2.4.6.8.....\text{ad inf.}} = \int \frac{x^{2n}dx}{\sqrt{1-2^2}} \div \int \frac{x^{2n+1}dx}{\sqrt{1-x^2}}$$

$$= \frac{2.4.6.....\text{ad inf.}}{1.3.5.....\text{ad inf.}}, \text{ whence } 2\pi = 4 \times \frac{2^2.4^2.6^2......\text{ad inf.}}{1\ .3^2.5^2......\text{ad inf.}}$$

549. To integrate $d \times \frac{dz}{z}.(a+bz^n)^{\frac{5}{2}}$, assume

$$z^{\frac{n}{2}} = u$$

Then $z = u^{\frac{2}{n}}$

$$\text{And } \frac{dz}{z} = \frac{\frac{2}{n}u^{\frac{2}{n}-1}du}{u^{\frac{2}{n}}} = \frac{2}{n}.\frac{du}{u}$$

$$\therefore d\times\frac{dz}{z}.(a+bz^n)^{\frac{5}{2}} = \frac{2d}{n}\times\frac{du}{u}.(a+bu^2)^{\frac{5}{2}}$$

$$\text{But } \frac{du}{u}\times(a+bu^2)^{\frac{5}{2}} = a\frac{du}{u}.(a+bu^2)^{\frac{3}{2}} + b.udu.(a+bu^2)^{\frac{3}{2}}$$

$$\text{Also } \frac{du}{u}\times(a+bu^2)^{\frac{3}{2}} = a.\frac{du}{u}.(a+bu^2)^{\frac{1}{2}} + b.udu.(a+bu^2)^{\frac{1}{2}}$$

$$\text{And } \frac{du}{u}\times(a+bu^2)^{\frac{1}{2}} = \frac{adu}{u(a+bu^2)^{\frac{1}{2}}} + \frac{b.udu}{(a+bu^2)^{\frac{1}{2}}}$$

$$\therefore \frac{2d}{n}\times\frac{du}{u}.(a+bu^2)^{\frac{5}{2}} = \frac{2d}{n}\times\Big\{budu\times(a+bu^2)^{\frac{3}{2}} + ab.udu\times$$

$$(a+bu^2)^{\frac{1}{2}} + \frac{a^2budu}{\sqrt{a+bu^2}} + \frac{a^3du}{u\sqrt{a+bu^2}}\Big\}$$ which are common forms, giving

$$\int\frac{2d}{n}.\frac{du}{u}.(a+bu^2)^{\frac{5}{2}} = \frac{2d}{n}.\Big\{\frac{1}{5}(a+bu^2)^{\frac{5}{2}} + \frac{a}{3}\times$$

$$(a+bu^2)^{\frac{3}{2}} + a^2.(a+bu^2)^{\frac{1}{2}} + \frac{a^3}{\sqrt{b}}\times\frac{1}{2\sqrt{\frac{a}{b}}}\times$$

$$l.\frac{\sqrt{\frac{a}{b}+u^2}-\sqrt{\frac{a}{b}}}{\sqrt{\frac{a}{b}+u^2}+\sqrt{\frac{a}{b}}}\Big\}$$

$$= \frac{2d}{n}\times\Big\{\frac{1}{5}.(a+bz^n)^{\frac{5}{2}} + \frac{a}{3}.(a+bz^n)^{\frac{3}{2}} + a^2.(a+bz^n)^{\frac{1}{2}} + a^{\frac{5}{2}}\times$$

$$l.\frac{\sqrt{\frac{a}{b}+z^n}-\sqrt{\frac{a}{b}}}{z^{\frac{n}{2}}}\Big\}.$$

To integrate $\dfrac{d\times dz}{z\times(a+bz^n)^{\frac{5}{2}}} = d.\mathrm{F}$

Assume $z^{\frac{n}{2}} = u$

Then $z = u^{\frac{2}{n}}$

nd $\dfrac{dz}{z} = \dfrac{2}{n}\,\dfrac{u^{\frac{2}{n}-1}\,du}{u^{\frac{2}{n}}} = \dfrac{2}{n}\cdot\dfrac{du}{u}$

$$\therefore\ d.\mathrm{F} = \frac{2d}{n}\times\frac{du}{u.(a+bu^2)^{\frac{5}{2}}} = \frac{2d}{n(b)^{\frac{5}{2}}}\times\frac{du}{u(c^2+u^2)^{\frac{5}{2}}}\left(c^2=\frac{a}{b}\right)$$

Now, make $\left.\begin{aligned}\mathrm{P}_1 &= \frac{1}{(c^2+u^2)^{\frac{3}{2}}}\\ \mathrm{P}_2 &= \frac{1}{(c^2+u^2)^{\frac{1}{2}}}\end{aligned}\right\}\ \left.\begin{aligned}\int\frac{du}{u.(c^2+u^2)^{\frac{5}{2}}} &= \mathrm{F}_0\\ \int\frac{du}{u.(c^2+u^2)^{\frac{3}{2}}} &= \mathrm{F}_1\\ \int\frac{du}{u.(c^2+u^2)^{\frac{1}{2}}} &= \mathrm{F}_2\end{aligned}\right\}$

$$\text{Then } d\mathrm{P}_1 = \frac{-3udu}{(c^2+u^2)^{\frac{5}{2}}} = \frac{-3du}{u(c^2+u^2)^{\frac{5}{2}}}\times(u^2+c^2-c^2)$$

$$= \frac{-3du}{u.(c^2+u^2)^{\frac{3}{2}}} + \frac{3c^2du}{u.(c^2+u^2)^{\frac{5}{2}}}$$

$$\left.\begin{aligned}\therefore\ \mathrm{P}_1 &= -\ 3\,\mathrm{F}_1 + 3c^2\,\mathrm{F}_0\\ \text{Similarly } \mathrm{P}_2 &= -\ \mathrm{F}_2 + c^2\,\mathrm{F}_1\end{aligned}\right\}$$

$$\therefore\ \mathrm{F}_0 = \frac{\mathrm{P}_1}{3c^2} + \frac{\mathrm{F}_1}{c^2} = \frac{1}{3c^2}\times\mathrm{P}_1 + \frac{1}{c^4}\times\mathrm{P}_2 + \frac{1}{c^4}\times\mathrm{F}_2$$

$$\text{Hence, } \int\frac{d\times dz}{(a+bz^n)^{\frac{5}{2}}} = \frac{2d}{nb^{\frac{5}{2}}}\times\mathrm{F}_0 = \frac{2d}{3nc^4b^{\frac{5}{2}}}\times\ (c^2\,\mathrm{P}_1 + 3\,\mathrm{P}_2$$
$+\ 3\,\mathrm{F}_2)$

$$\text{But } \mathrm{F}_2 = \int\frac{du}{u\sqrt{c^2+u^2}} = \frac{1}{2c}.l.\frac{\sqrt{c^2+u^2}-c}{\sqrt{c^2+u^2}+c}$$

$$= \frac{1}{c}.l.\frac{\sqrt{c^2+u^2}-c}{u}$$

$$\therefore \int \frac{d \times dz}{(a+bz^n)^{\frac{5}{2}}} = \frac{2d}{3nc^4 b^{\frac{5}{2}}} \times \left\{ c^2 P_1 + 3P_2 + \frac{3}{c} l. \frac{\sqrt{c^2+u^2} - c}{u} \right\}$$

in which, substituting for P_1, P_2, and u, we shall have the integral in terms of z.

550. $$\int \frac{bxdx}{(x-a).(x+a)} = \int \frac{bxdx}{x^2 - a^2} = \frac{b}{2} \int \frac{2xdx}{x^2 - a^2}$$

$= \frac{b}{2}. l. (x^2 - a^2)$ by a common form.

To integrate $\frac{x^2dx}{\sqrt{a^2+x^2}}$, we will assume

$$x\sqrt{a^2+x^2} = P$$

Then $dP = dx . \sqrt{a^2+x^2} + \frac{x^2dx}{\sqrt{a^2+x^2}}$

$$= \frac{a^2dx}{\sqrt{a^2+x^2}} + \frac{2x^2dx}{\sqrt{a^2+x^2}}$$

$$\therefore \int \frac{x^2dx}{\sqrt{a^2+x^2}} = \frac{P}{2} - \frac{a^2}{2} \int \frac{dx}{\sqrt{a^2+x^2}}$$

$$= \frac{x\sqrt{a^2+x^2}}{2} - \frac{a^2}{2}.l.(x+\sqrt{a^2+x^2})$$

551. $$\frac{x^3dx}{x^2+1} = xdx - \frac{xdx}{x^2+1}, \text{ by division.}$$

$$\therefore \int \frac{x^3dx}{1+x^2} = \frac{x^2}{2} - \frac{1}{2} l. (x^2+1)$$

To integrate $x^6dx \sqrt{a^2+x^2} = dF_0$ assume

$$\left.\begin{aligned} x^5 \times (a^2+x^2)^{\frac{3}{2}} &= P_1 \\ x^3 \times (a^2+x^2)^{\frac{3}{2}} &= P_2 \\ x \times (a^2+x^2)^{\frac{3}{2}} &= P_3 \end{aligned}\right\} \left\{\begin{aligned} \int x^4dx\sqrt{a^2+x^2} &= F_1 \\ \int x^2dx\sqrt{a^2+x^2} &= F_2 \\ \int dx\sqrt{a^2+x^2} &= F_3 \end{aligned}\right\}$$

Then, $dP_1 = 5x^4dx. (a^2+x^2)^{\frac{3}{2}} + 3.x^6dx\sqrt{a^2+x^2}$

$$= 5a^2\ x^4dx\ (a^2+x^2)^{\frac{1}{2}} + 8\ x^6dx\sqrt{a^2+x^2}$$

$$\left.\begin{aligned} \therefore\ P_1 &= 5a^2 \times F_1 + 8 \times F_0 \\ \text{Similarly } P_2 &= 3a^2 \times F_2 + 6 \times F_1 \\ \text{And } P_3 &= a^2 \times F_3 + 4 \times F_2 \end{aligned}\right\}$$

$$\left.\begin{aligned} \therefore\ F_0 &= \frac{1}{8}. P_1 - \frac{5a^2}{8}. F_1 \\ F_1 &= \frac{1}{6}\times P_2 - \frac{a^2}{2} \times F_2 \\ F_2 &= \frac{1}{4}\times P_3 - \frac{a^2}{4} \times F_3 \end{aligned}\right\}$$

$$\text{Hence } F_0 = \frac{1}{8} P_1 - \frac{5a^2}{6\times 8} P_2 + \frac{5a^4}{64}. P_3 - \frac{5a^6}{256} \times F_3.$$

But $F_3 = \int dx \sqrt{a^2+x^2}$ which may be found thus; let $x.\sqrt{a^2+x^2} = Q$

$$\text{Then } dQ = dx.\sqrt{a^2+x^2} + \frac{x^2dx}{\sqrt{a^2+x^2}}$$

$$= dx.\sqrt{a^2+x^2} + \frac{dx}{\sqrt{a^2+x^2}} \times (x^2 + a^2 - a^2)$$

$$= 2dx\sqrt{a^2+x^2} - \frac{a^2dx}{\sqrt{a^2+x^2}}$$

$$\therefore\ F_2 = \int dx\sqrt{a^2+x^2} = \frac{Q}{2} + \frac{a^2}{2}\int\frac{dx}{\sqrt{a^2+x^2}} = \frac{Q}{2} + \frac{a^2}{2}\times$$

$l.\ (x + \sqrt{a^2+x^2})$ which being substituted, in the above equation, we shall have attained the integral required.

Again, to integrate $\dfrac{z^{\frac{n}{2}-1}dz}{a^n+z^n} = d.F$, assume

$$\left(\frac{z}{a}\right)^{\frac{n}{2}} = u$$

$$\text{Then } \frac{n}{2a^{\frac{n}{2}}} z^{\frac{n}{2}-1}\,dz = du$$

$$\text{And } a^n + z^n = a^n + a^n u^2 = a^n.(1 + u^2)$$

$$\therefore\ dF = \frac{2}{na^{\frac{n}{2}}} \times \frac{du}{1+u^2}$$

$$\text{And } F = \frac{2}{na^{\frac{n}{2}}} . \tan.^{-1} u$$

$$= \frac{2}{na^{\frac{n}{2}}} . \tan.^{-1} \left(\frac{z}{a}\right)^{\frac{n}{2}}$$

552. To integrate $\frac{dx}{1 + x^n}$, n being an odd number.

To find the factors of the denominator, we must assume $x^n + 1 = 0$, then since n is odd, we have (-1) for a root, and all the rest are imaginary, and of the form $a \pm b . \sqrt{-1}$

To find these, we have

$$x^n = -1 = \cos. (2p + 1). \pi \pm \sqrt{-1} \sin. (2p + 1) \pi$$

$$\therefore x = \cos. \frac{2p+1}{n} \pi \pm \sqrt{-1} . \sin. \frac{2p+1}{n} \pi,$$ where p may have any value from 0 to $\frac{n-1}{2}$ (afterwards the values of x recur.)

Hence the roots of $x^n + 1$ are $\cos. \frac{\pi}{n} \pm \sqrt{-1} \sin. \frac{\pi}{n}$, $\cos. \frac{3\pi}{n} \pm \sqrt{-1} . \sin. \frac{3\pi}{n}$, $\cos. \frac{5\pi}{n} \pm \sqrt{-1} \sin. \frac{5\pi}{n}$, &c. ..... $\cos. \frac{n-2}{n} \pi \pm \sqrt{-1} . \sin. \frac{n-2}{n} \pi$, and -1. Now $\cos. \theta - \sqrt{-1} \times$

$$\sin. \theta = \frac{\cos.^2 \theta + \sin.^2 \theta}{\cos. \theta + \sqrt{-1} \sin. \theta} = \frac{1}{\cos. \theta + \sqrt{-1} \sin. \theta}$$

$$\left.\begin{aligned}
\text{Let } \therefore \cos. \frac{\pi}{n} + \sqrt{-1} \sin. \frac{\pi}{n} &= r_1 \\
\cos. \frac{3\pi}{2} + \sqrt{-1} \sin. \frac{\pi}{n} &= r \\
\&c. &= \&c. \\
\cos. \frac{n-2}{n} \pi + \sqrt{-1} . \sin. \frac{n-2}{n} \pi &= r_{\frac{n-1}{2}}
\end{aligned}\right\}$$

$$\left.\begin{aligned}\text{Then}\ \cos.\frac{\pi}{n} - \sqrt{-1}.\sin.\frac{\pi}{n} &= \frac{1}{r_1}\\ \cos.\frac{3\pi}{n} - \sqrt{-1}.\sin.\frac{3\pi}{n} &= \frac{1}{r_2}\\ \&c. &= \&c.\\ \cos.\frac{n-2}{n}\pi - \sqrt{-1}.\sin.\frac{n-2}{n}.\pi &= \frac{1}{r_{\frac{n-1}{2}}}\end{aligned}\right\}$$

Let $\frac{n-1}{2} = m$

Hence, the factors of x^n+1, are $(x+1)$, $(x-r_1)$, $\left(x-\frac{1}{r_1}\right)$, $(x-r_2)$, $\left(x-\frac{1}{r^2}, \&c.\right)$, $(x-r_m)$, $\left(x-\frac{1}{r_m}\right)$

$\therefore x^n+1 = (x+1).(x-r_1)\times\left(x+\frac{1}{r^1}\right) \times (x-r_2)\left(x-\frac{1}{r_2}\right)$ $\times \&c. \times (x-r_m)\times\left(x-\frac{1}{r_m}\right) = (x+1) \times \left\{x^2-\left(r_1+\frac{1}{r_1}\right)x+1\right\}$ $\times \left\{x^2-\left(r_2+\frac{1}{r_2}\right)+1\right\} \times \&c. \times \left\{x^2-\left(r_m+\frac{1}{r_m}\right)x+1\right\}$. Having now found the quadratic divisors of x^n+1, let us assume

$$\frac{1}{1+x^n} = \frac{A}{x+1} + \frac{A_1x+B_1}{x^2-\left(r_1+\frac{1}{r_1}\right)x+1} + \frac{A_2x+B_2}{x^2-\left(r_2+\frac{1}{r_2}\right)x+1} + \&c.$$

$+\ \dfrac{A_mx+B_m}{x^2-\left(r_m+\frac{1}{r_m}\right)x+1}$, which fractions being reduced to a common denominator, we have $A\times D + (A_1x+B_1)D_1 + (A_2x+B_2) \times D_2 + \&c.. + (A_mx+B_m)\times D_m = 1$ (A_1, D_1, D_2 .... D_m, denoting the product of all the denominators, except $(x+1)$; except $\left\{x^2-2\left(r_1+\frac{1}{r_1}\right)x+1\right\}$&c....

Now, let $x = -1$; then D_1, D_2 &c. D_m become zero, and $\therefore$ we get $A\times R = 1$ (R being the value of D when $x = -1$)

$$\therefore A = \frac{1}{R} = \frac{1}{n}, \text{ for } R = \frac{1+x^n}{1+x} = n, \text{ when } x = -1.$$

To find the p^{th} pair of indeterminates at once, (A_p, B_p) we have

$$\frac{1}{1+x^n} = \frac{A}{1+x} + \frac{A_p x + B_p}{x^2 - \left(r_p + \frac{1}{r^p}\right)x + 1} + \frac{N}{M}$$

(N and M being real quantities.)

$$\therefore A \times \left\{x^2 - \left(r_p + \frac{1}{r_p}\right)x + 1\right\} \times M + (1+x) \times M \times (A_p x + B_p)$$

$$+ N \times (1+x) . (x^2 - \overline{r_p + \frac{1}{r_p}} \times x + 1) = 1; \text{ let } x = r_p$$

Then $(1+r_p)\ M_p \times (A_p r_p + B_p) = 1$ (M_p being the value of M on that supposition)

But $M = \dfrac{1+x^n}{(1+x) \times \left\{x^2 - \left(r_p + \frac{1}{r_p}\right)x + 1\right\}}$; to find whose value, when both numerator and denominator become zero, when $x = r_p$,

Assume $x - r_p = y \quad \therefore x = y + r_p$

$$\text{And } M = \frac{1 + y^n + n . y^{n-1} r_p + \ldots\ldots n y r_p^{n-1} + r_p^n}{(1+x) \times y \times \left(x - \frac{1}{r_p}\right)}$$

$$= \frac{y^{n-1} + n . y^{n-2} r_p + \ldots\ldots n r_p^{n-1}}{(1+x) \times \left(x - \frac{1}{r_p}\right)} \text{ because}$$

$r_p^n = \left(\cos. \frac{(2p-1)\pi}{u} + \sqrt{-1} \sin. \frac{(2p-1)}{u}\pi\right)^n = \cos. (2p-1)\pi$
$+ \sqrt{-1} . \sin. (2p-1)\pi = -1$

Let now $x = r_p$, then $y = 0$

$$\text{And } M_p = \frac{n r_p^{n-1}}{(1+r_p) \times \left(r_p - \frac{1}{r_p}\right)}$$

$$\text{Hence } A_p r_p + B_p = \left(r_p - \frac{1}{r_p}\right) \times \frac{1}{n r_p^{n-1}}$$

$$\text{But } r_p - \frac{1}{r_p} = 2\sqrt{-1} \sin. \frac{2p-1}{n} . \pi$$

And $r_p^{n-1} = (\cos. \frac{2p-1}{n}\pi + \sqrt{-1}\sin. \frac{2p-1}{n}\pi)^{n-1} = \cos. \left\{\frac{n-1}{n} \times (2p-1)\pi\right\} + \sqrt{-1}\sin. \left\{\frac{n-1}{n} \times (2p-1)\pi\right\} = (\cos. \overline{2p-1}.\pi - \frac{2p-1}{n}.\pi) + \sqrt{-1}.\sin. (\overline{2p-1}.\pi - \frac{2p-1}{n}\pi)$

$$= -(\cos. \frac{2p-1}{n}\pi - \sqrt{-1}\sin. \frac{2p-1}{n}\pi) = -\frac{1}{r_p}$$

$$\therefore A_p r_p + B_p = -\frac{2}{n}\sqrt{-1}\sin. \frac{2p-1}{n}\pi \times r_p$$

$\therefore A_p(\cos. \frac{2p-1}{n}\pi + \sqrt{-1}\sin. \frac{2p-1}{n}\pi) + B_p = -\frac{2}{n}\sqrt{-1} \times \sin. \frac{2p-1}{n}\pi \times \cos. \frac{2p-1}{n}\pi + \frac{2}{n} \times (\sin. \frac{2p-1}{n}\pi)^2$ whence, equating real and imaginary quantities, we get

$$A_p = -\frac{2}{n}.\cos. \frac{2p-1}{n}\pi$$

$$B_p = \frac{2}{n}\left\{(\cos. \frac{2p-1}{n}\pi)^2 + (\sin. \frac{2p-1}{n}\pi)^2\right\} = \frac{2}{n}$$

Now, if we give to p the values 1, 2, 3 ..... $\frac{n-1}{2}$ successively, we shall get each of the pairs of indeterminates, A_1, B_1; A_2, B_2 ..... $A_{\frac{n-1}{2}}$, $B_{\frac{n-1}{2}}$, and A we have already found $= \frac{1}{n}$; $\therefore$ we may proceed to integrate $\frac{dx}{1+x^n}$ by taking the integral of each of the differentials $\frac{Adx}{x+1}$, $\frac{A_1xdx+B_1dx}{x^2-\left(r_1+\frac{1}{r_1}\right)x+1}$, &c. $\frac{A_mxdx+B_mdx}{x^2-\left(r_m+\frac{1}{r_m}\right)x+1}$

But $\int\frac{Adx}{x+1} = A.\, l.\,(x+1)$ and generally, we may integrate $dx\frac{A_px+B_p}{x^2-\left(r_p+\frac{1}{r_p}\right)x+1}\left(= dx\frac{A_px+B_p}{x^2-2x\cos. \frac{2p-1}{u}\pi+1}\right)$ by assuming $x - \cos. \frac{2p-1}{n}\pi = u$

For then we have $x^2 - 2x.\cos.\frac{2p-1}{n}\pi + 1 = u^2 + 1 - (\cos.\frac{2p-1}{n}\pi)^2 = u^2 + (\sin.\frac{2p-1}{n}\pi)^2$; and $B_p + A\ x = \frac{2}{n}\times$

$$(1-x\cos.\frac{2p-1}{n}\pi) = \frac{2}{n}.\left\{1-u\cos.\frac{2p-1}{n}\pi - (\cos.\frac{2p-1}{n}\pi)^2\right\}$$

$$= \frac{2}{n}\left\{(\sin.\frac{2p-1}{n}\pi)^2 - u\ \cos.\frac{2p-1}{n}\pi\right\}$$

And $dx = du$

$$\therefore \int dx.\frac{A_p x + B_p}{x^2 - \left(r_p + \frac{1}{r_p}\right)x + 1} = \frac{2}{\pi}\int\frac{(\sin.\frac{2p-1}{n}\pi)^2 du}{u^2 + (\sin.\frac{2p-1}{n}\pi)^2}$$

$$-\frac{2}{n}\int\frac{(\cos.\frac{2p-1}{n}\pi)\ udu}{u^2+(\sin.\frac{2p-1}{n}\pi)^2};\ \text{put}\ \sin.\frac{2p-1}{n}\pi = a$$

$$\text{Then}\int dx\ \frac{A_p x + B_p}{x^2 - \left(r_p+\frac{1}{r_p}\right)x+1} = \frac{2}{n}\int\frac{a^2 du}{u^2+a} - \frac{\cos.\frac{2p-1}{n}\pi}{n}\times$$

$$\int\frac{2udu}{u^2+a^2} = \frac{2}{n}.\tan.^{-1}u - \frac{\cos.\frac{2p-1}{n}.\pi}{n}.\ l.\ (u^2+a^2)$$

$$= \frac{2}{n}.\tan.^{-1}u - \frac{\cos.\frac{2p-1}{n}.\pi}{n}.\ l.\ (x^2-2x.\cos.\frac{2p-1}{n}.\pi+1)$$

Hence, by substituting for p the numbers $1, 2, 3 \ldots\ldots \frac{n-1}{2}$, we shall find each of the pairs of integrals in succession. The sum of all the integrals will be the integral required.

The preceding solution affords a striking instance of the utility of the theory of *Vanishing Fractions*. We performed the operation algebraically; but the Reader may apply, if he pleases, the principles of the *Differential Calculus*.

That the roots of $\frac{x^n+1}{x+1}=0$ are reciprocal, will be perceived by the Theory of Reciprocal Equations, as well as by our method.

To integrate $\frac{dx.\sqrt{a^2-x^2}}{x^6}=d\,F_0$, assume

$$\left.\begin{array}{l}\frac{(a^2-x^2)^{\frac{3}{2}}}{x^5}=P_1\\[2ex] \frac{(a^2-x^2)^{\frac{3}{2}}}{x^3}=P_2\end{array}\right\}\quad\text{and}\int\frac{dx\,\sqrt{a^2-x^2}}{x^4}=F_1$$

$$\text{Then } dP_1=-\frac{5dx.(a^2-x^2)^{\frac{3}{2}}}{x^6}-\frac{3dx\,(a^2-x^2)^{\frac{1}{2}}}{x^4}$$

$$=-\frac{5a^2dx\,\sqrt{a^2-x^2}}{x^6}+\frac{2dx\sqrt{a^2-x^2}}{x^4}$$

$$\therefore\ P_1=-5a^2\,F_0+2\,F_1$$

$$\therefore\ F_0=\frac{2}{5a^2}F_1-\frac{1}{5a^2}.P_1$$

$$\text{Similarly } F_1=-\frac{1}{3a^2}.P_2$$

$$\therefore\ F_0=-\frac{1}{5a^2}P_1-\frac{2}{15a^4}P_2$$

$$=-\frac{(a^2-x^2)^{\frac{3}{2}}}{15a^4\,x^5}\times(3a^2+2x^2)$$ the integral required.

To integrate $\frac{-dx}{\sqrt{l.\frac{a}{x}}}$

Let $l.\frac{a}{x}=u$; then if e be the hyperbolic base, we have

$$\frac{a}{x}=e^u$$

Also $x=ae^{-u}$

$$\therefore\ dx=-ae^{-u}\,du$$

Hence $\int \frac{-dx}{\sqrt{l.\frac{a}{x}}} = \int \frac{+ae^{-u}\,du}{u^{\frac{1}{2}}} = \int \frac{adu}{u^{\frac{1}{2}}}\left\{1 - u + \frac{u^2}{2} - \frac{u^3}{2.3} + \&c.\right\}$

$$= \int \frac{adu}{u^{\frac{1}{2}}} - \int au^{\frac{1}{2}}du + \int \frac{a}{2}u^{\frac{3}{2}}du - \int \frac{a}{2.3}u^{\frac{5}{2}}du + \&c.$$

$$= 2au^{\frac{1}{2}} - \frac{2a}{3}u^{\frac{3}{2}} + \frac{2a}{5.2}u^{\frac{5}{2}} - \frac{2a}{7.2.3}u^{\frac{7}{2}} + \&c.$$

$$= al.\frac{a}{x}\left\{1 - u + \frac{u^2}{2.5} - \frac{u^3}{2.3.7} + \&c.\right\}$$

553. To integrate $\frac{dx}{\sqrt{1+x^2} - \sqrt{1-x^2}}$ we have

$$\frac{dx}{\sqrt{1+x^2} - \sqrt{1-x^2}} = \frac{dx.(\sqrt{1-x^2} + \sqrt{1-x^2})}{2x^2} = \frac{dx\,\sqrt{1+x^2}}{2x^2}$$

$+ \frac{dx.\sqrt{1-x^2}}{2x^2}$ which may be integrated separately.

Assume $\frac{\sqrt{1+x^2}}{x} = \text{P}$

Then $d\text{P} = \frac{-dx}{x^2}\sqrt{1+x^2} + \frac{dx}{\sqrt{1+x^2}}$

$$\therefore \int \frac{dx}{2x^2}\sqrt{1+x^2} = \int \frac{dx}{2\sqrt{1+x^2}} - \frac{\text{P}}{2}$$

$$= \frac{1}{2}\,l.\,(x + \sqrt{1+x^2}) - \frac{\sqrt{1+x^2}}{2x}.$$

Similarly $\int \frac{dx}{2x^2}\sqrt{1-x^2} = -\int \frac{dx}{2\sqrt{1-x^2}} - \frac{\sqrt{1-x^2}}{2x}$

$$= -\frac{1}{2}.\sin.^{-1}x - \frac{\sqrt{1-x^2}}{2x}$$

$$\therefore \int \frac{dx}{\sqrt{1+x^2} - \sqrt{1-x^2}} = \frac{1}{2}\left\{l.\,(x + \sqrt{1+x^2}) - \sin.^{-1}x\right.$$

$$\left. - \frac{1}{x}.\sqrt{1+x^2} - \frac{1}{x}\sqrt{1-x^2}\right\}$$

To integrate $x^x dx$, we have

$$\int x^x dx = \int dx \times \left\{ 1 + x l.x + \frac{x^2}{1.2}.(l.x)^2 + \frac{x^3}{1.2.3}.(l.x)^3 + ..\infty \right\}$$

$$= \int dx + \int x dx.l.x + \int \frac{x^2 dx}{1.2}(l.x)^2 + \int \frac{x^3 dx}{1.2.3}(l.x)^3 + ...\infty$$

But, by the form $\int u dv = uv - \int v du$, we have (T) the $(n)^{th}$ term $\left(= \int \frac{x^{n-1}dx}{1.2.3....n-1} \times (l.x)^{n-1}\right) = \frac{x^n}{1.2..(n-1).n} \times (lx)^{n-1}$ $- \int \frac{x^n}{1.2..(n-1)n} \times d.(lx.)^{n-1} = \frac{x^n}{1.2.3..(n-1).n} \times (l.x)^{n-1} -$ $\int \frac{x^{n-1}dx}{1.2.....(n-2).n} \times (lx)^{n-2}$

Or, $\frac{1}{1.2...n-1} \int x^{n-1}dx \times (lx)^{n-1} = \frac{x^n}{1.2.3....(n-1)n} \times (lx)^{n-1} -$ $\int \frac{x^{n-1}dx}{1.2..(n-2)n} \times (lx)^{n-2}$

Similarly $\frac{1}{1.2.....(n-2)n} \times \int x^{n-1}dx\,(lx)^{n-2} = \frac{x^n}{1.2...(n-2).n^2} \times$ $(lx)^{n-2} - \int \frac{x^{n-1}dx}{1.2....(n-3)n^2} \times (l.x)^{n-3}$

&c. = &c.

$$\frac{1}{1.2.n^{n-2}} \int x^{n-1}dx.lx = \frac{x^n}{1.2.n^{n-1}}.\,lx - \int \frac{x^{n-1}dx}{1.n^{n-1}}.\,(lx)^0$$

And $\frac{1}{1.n^{n-1}} \int x^{n-1}dx = \frac{x^n}{n^n}$

Hence $T = \frac{x^n(lx)^{n-1}}{1.2....(n-1)n} - \frac{x^n(lx)^{n-2}}{1.2...(n-2)n^2} +$ $\frac{x^n(lx)^{n-3}}{1.2....(n-3.n^3} - \&c. \pm \frac{x^n.lx}{1.2.n^{n-1}} \mp \frac{x^n}{n^n}$. This being the n^{th}, or general term in the above series of integrals, we shall obtain each term separately by substituting for n the numbers 1,2,3....∞ successively; and their sum will be the integral of $x^x dx$.

Let $x = 0$. Then since every term of T is multiplied by x, it vanishes on this supposition. Again, let $x = 1$. Then, every

term of T but the last being multiplied by lx, or 0, vanishes, and T becomes $\mp \frac{1}{n^n}$.

Hence, then, $\int x^x dx$, between the values of $x = 0$ and $x = 1$, is $\frac{1}{1} - \frac{1}{2^2} + \frac{1}{3^3} - \frac{1}{4^4} + \ldots \infty$.

554. To integrate $\frac{dx}{\sqrt{x^2+2ax}}$ we have (*Vince, Simpson, &c.*)

$$\int \frac{dx}{\sqrt{x^2+2ax}} = l.(x + a + .\sqrt{x^2+2ax}) =$$

$$= l.(\sqrt{x+a+\sqrt{x^2+2ax}})^2 =$$

$$2l.\sqrt{x+a+\sqrt{x^2+2ax}}$$

Now, supposing $\sqrt{2ax+x^2}$ a surd, assume

$$\left.\begin{array}{l} \sqrt{x+a+\sqrt{2ax+x^2}} = u + v \\ \text{Then } \sqrt{x+a-\sqrt{2ax+x^2}} = u - v \end{array}\right\}$$

And $\sqrt{(x+a)^2-2ax-x^2} = a = u^2 - v^2$

Also $x + a + \sqrt{2ax+x^2} = u^2 + 2uv + v^2$

$\therefore u^2 + v^2 = x + a$

And $u^2 - v^2 = a$

$$\therefore u = \frac{\sqrt{x+2a}}{\sqrt{2}}$$

$$\text{And } v = \frac{\sqrt{x}}{\sqrt{2}}$$

$$\therefore u + v = \frac{\sqrt{x} + \sqrt{x+2a}}{\sqrt{2}}$$

Hence, then we have

$$\int \frac{dx}{\sqrt{x^2+2ax}} = 2.l.\left\{\frac{\sqrt{x} + \sqrt{x+2a}}{\sqrt{2}}\right\}$$

The enunciation is wrong.

To integrate $\frac{mdz}{a+bz^2}$ we have

$$\frac{mdz}{a+bz^2} = \frac{mdz}{a.(1+\frac{b}{a}z^2)}. \text{ Let } \sqrt{\frac{b}{a}}.z = u$$

$$\therefore dz = \sqrt{\frac{a}{b}}.du$$

And we have $\frac{m.dz}{a+bz^2} = \frac{m}{\sqrt{ab}}.\frac{du}{1+u^2}$

$$\therefore \int\frac{mdz}{a+bz^2} = \frac{m}{\sqrt{ab}}\int\frac{du}{1+u^2} = \frac{m}{\sqrt{ab}}.\tan.^{-1}u$$

$$= \frac{m}{\sqrt{ab}}.\tan.^{-1}\sqrt{\frac{b}{a}}.z.$$

555. To integrate $\frac{dx}{x.(a^2+x^2)^{\frac{5}{2}}} = F_0$

$$\text{Assume } \left.\begin{array}{l}\frac{1}{(a^2+x^2)^{\frac{3}{2}}} = P_1\\ \frac{1}{(a^2+x^2)^{\frac{1}{2}}} = P_2\end{array}\right|\left.\begin{array}{l}\int\frac{dx}{x.(a^2+x^2)^{\frac{3}{2}}} = F_1\\ \int\frac{dx}{x.(a^2+x^2)^{\frac{1}{2}}} = F_2\end{array}\right\}$$

Then $dP_1 = \frac{-3xdx}{(a^2+x^2)^{\frac{5}{2}}} = \frac{-3dx}{x.(a^2+x^2)^{\frac{5}{2}}} \times (x^2+a^2-a^2)$

$$= \frac{-3dx}{x.(a^2+x^2)^{\frac{3}{2}}} + \frac{3a^2dx}{x.(a^2+x^2)^{\frac{5}{2}}}$$

$$\therefore P_1 = -3F_1 + 3a^2F_0$$

$$\therefore F_0 = \frac{1}{3a^2}.P_1 + \frac{1}{a^2}.F_1$$

Similarly $F_1 = \frac{1}{a^2}.P_2 + \frac{1}{a^2}.F_2$

But $F_2 = \int\frac{dx}{x.\sqrt{a^2+x^2}} = \frac{1}{2a}.l.\frac{\sqrt{a^2+x^2}-a}{\sqrt{a^2+x^2}+a}$

$$= \frac{1}{a}.l.\frac{\sqrt{a^2+x^2}-a}{x}$$

$\therefore F_0 = \frac{1}{3a^2}.P_1 + \frac{1}{a^4}.P_2 + \frac{1}{a^5}.l.\frac{\sqrt{a^2+x^2}-a}{x}$ which is $\therefore$ known.

To integrate $(a + cz^n)^{m+p}. z^{vn-1}dz = d\,F_p$, having given $\int(a+cz^n)^m z^{vn-1}dz = F_0$

$$\left.\begin{array}{r}\text{Assume } (a+cz^n)^{m+1}z^{vn} = P_1 \\ (a+cz^n)^{m+2}z^{vn} = P_2 \\ \&c. = \&c. \\ (a+cz^n)^{m+p}z^{vn} = P_p\end{array}\right\}\text{and}\left\{\begin{array}{r}\int(a+cz^n)^{m+1}z^{vn-1}dz = F_1 \\ \int(a+cz^n)^{m+2}z^{vn-1}dz = F_2 \\ \&c. = \&c.\end{array}\right\}$$

Then $dP_1 = vn.z^{vn-1}dz.\ (a + cz^n)^{m+1} + cn.(m+1) \times z^{vn+n-1}dz.(a+cz^n)^m$

But $z^{vn+n-1}dz.(a+cz^n)^m = \frac{z^{vn-1}dz}{c}.(a+cz^n)^m(cz^n + a - a)$

$= \frac{z^{vn-1}dz.}{c}.(a+cz^n)^{m+1} - \frac{a}{c}.z^{vn-1}dz.(a+cz^n)^m.$

$\therefore dP_1 = \{vn + n.(m+1)\}\ z^{vn-1}dz.\ (a + cz^n)^{m+1} - an.(m+1)\ z^{vn-1}dz\ (a + cz^n)^m$

$\therefore P_1 = (vn + n + nm)\,F_1 - (anm + an)\,F_0$

imilarly $P_2 = (vn + nm + 2n)\,F_2 - (anm + an + an).\,F_1$

&c. = &c.

$P_p = (vn+nm+pn) \times F_p - \{anm+an+an.(p-1)\}F_{p-1}$

$$\text{Let}\left.\begin{array}{r} vn + nm = s \\ anm + an = t\end{array}\right\}$$

Then, we have $F_p = \frac{1}{s+pn}.\,P_p + \frac{t+an(p-1)}{s+pn}\,F_{p-1}$

Similarly $F_{p-1} = \frac{1}{s+(p-1).n}.\,P_{p-1} + \frac{t+an.(p-2)}{s+n.(p-1)}.F_{p-2}$

&c. = &c.

$F_1 = \frac{1}{s+n}.\,P_1 + \frac{t}{s+n}\,F_0$

Hence may be perceived the facility of expressing F_p in terms of F_0; i. e., of finding the integral as required by the Problem.

556. $$\int \frac{dx}{a-mx.} = -\frac{1}{m}\int \frac{dx}{x-\frac{a}{m}} = -m.\,l.\left(x-\frac{a}{m}\right)$$

To integrate $\dfrac{z^{\frac{n}{2}-1}dz}{a^n+z^n}$, assume $z^{\frac{n}{2}} = u$ and $a^n = c^2$

Then $\dfrac{n}{2}.\, z^{\frac{n}{2}-1}dz = du$

And $\dfrac{z^{\frac{n}{2}-1}dz}{a^n+z^n} = \dfrac{2}{n}.\dfrac{du}{c^2+a^2} = \dfrac{2}{nc^2}.\dfrac{c^2du}{c^2+u^2}$

$\therefore \displaystyle\int \frac{z^{\frac{n}{2}}dz}{a^n+z^n} = \frac{z}{nc^2}\tan.^{-c} u.$

557. To integrate $\dfrac{x^3dx}{\sqrt{2ax-x^2}} = d.\,F_0$

Assume $\left.\begin{aligned} x^2.\sqrt{2ax-x^2} &= P_1 \\ x.\sqrt{2ax-x^2} &= P_2 \end{aligned}\right\}$ and $\left.\begin{aligned}\int \frac{x^2dx}{\sqrt{2ax-x^2}} &= F_1 \\ \frac{xdx}{\sqrt{2ax-x^2}} &= F^2\end{aligned}\right\}$

Then $dP_1 = 2xdx.\sqrt{2ax-x^2} + \dfrac{(adx-xdx)x^2}{\sqrt{2ax-x^2}}$

$$= \frac{4ax^2dx}{\sqrt{2ax-x^2}} - \frac{2x^3dx}{\sqrt{2ax-x^2}} + \frac{ax^2dx}{\sqrt{2ax-x^2}} - \frac{x^3dx}{\sqrt{2ax-x^2}}$$

$$= \frac{5ax^2dx}{\sqrt{2ax-x^2}} - \frac{3x^3dx}{\sqrt{2ax-x^2}}$$

$$\therefore F_0 = \frac{5a}{3}\int \frac{x^2dx}{\sqrt{2ax-x^2}} - \frac{P_1}{3} = \frac{5a}{3}.F_1 - \frac{P_1}{3}$$

Similarly $F_1 = \dfrac{3a}{2}F_2 - \dfrac{P_2}{2}$

But $F_2 = \displaystyle\int \frac{xdx}{\sqrt{2ax-x^2}} = -\int \frac{adx-xdx}{\sqrt{2ax-x^2}} + \int \frac{adx}{\sqrt{2ax-x^2}}$

$$= -\sqrt{2ax-x^2} + \text{vers.}^{-1}x$$

$$\therefore F_0 = -\frac{P_1}{3} + \frac{5a}{6} P_2 - \frac{5a^2}{z}\sqrt{2ax - x^2} + \frac{5a^2}{2}\text{vers.}^{-1} x$$

which is the integral required.

To integrate $dz. \sin.^2 z$, we have

$$\sin.^2 z = \frac{1 - \cos. 2z}{2}$$

$$\therefore \int dz \sin.^2 z = \int \frac{dz}{2} - \frac{1}{4}\int d.(2z) \times \cos. 2z$$

$$= \frac{z}{2} - \frac{1}{4} \sin. 2z.$$

558. To integrate $\frac{1+\sqrt{-x}.\sqrt[3]{x^2}}{1+\sqrt[3]{x}} \times dx = d.F$ we have

$$1 + \sqrt{-x}.\sqrt[3]{x^2} = 1 + \sqrt{-1}.x^{\frac{1}{2}+\frac{2}{3}} = 1 + \sqrt{-1}.x^{\frac{7}{6}}$$

Assume $\therefore x^{\frac{1}{6}} = u.$

$$\text{Then } dF = \frac{1+\sqrt{-1}.u^7}{1+u^2} \times 6u^5 du$$

$$= \frac{6u^5 du}{u^2+1} + 6\sqrt{-1}\,\frac{u^{12}\,du}{u^2+1}$$

$$= 6\left\{u^3 du - u du + \frac{u du}{u^2+1}\right\} + 6\sqrt{-1}.\left\{u^{10} du - u^8 du + u^6 du\right.$$

$$\left. - u^4 du + u^2 du - du + \frac{du}{u^2+1}\right\} \text{ by common division.}$$

$$\therefore F = 6.\left\{\frac{u^4}{4} - \frac{u^2}{2} + \frac{1}{2} l.(u^2+1)\right\} + 6\sqrt{-1}.\left\{\frac{u^{11}}{11} - \right.$$

$$\left.\frac{u^9}{9} + \frac{u^7}{7} - \frac{u^5}{5} + \frac{u^3}{3} - u + \tan.^{-1} u\right\} \text{ the integral required.}$$

To integrate $\frac{dx}{(1+x^2)^n} = d.F_0$

$$\text{Assume } \left.\begin{aligned} \frac{x}{(1+x^2)^{n-1}} &= P_1 \\ \frac{x}{(1+x^2)^{n-2}} &= P_2 \\ \&c. &= \&c. \\ \frac{x}{1+x^2} &= P_{n-1} \end{aligned}\right\} \text{ and } \int \left\{\begin{aligned} \frac{dx}{(1+x^2)^{n-1}} &= F_1 \\ \frac{dx}{(1+x^2)^{n-2}} &= F_2 \\ \&c. &= \&c. \\ \frac{dx}{1+x^2} &= F_{n-1} \end{aligned}\right\}$$

$$\text{Then } d.P_1 = \frac{dx}{(1+x^2)^{n-1}} - \frac{2(n-1)\,x^2dx}{(1+x^2)^{n-1}}$$

$$= \frac{dx}{(1+x^2)^{n-1}} - \frac{2(n-1)\,dx}{(1+x^2)^n}.(x^2+1-1)$$

$$= (1-2.\overline{n-1})\frac{dx}{(1+x^2)^{n-1}} + \frac{2(n-1)dx}{(1+x^2)^n}$$

$$\therefore\ P_1 = (3-2n).F_1 + (2n-2).\ F_0$$

$$\text{Let } \left.\begin{aligned} 3-2n &= a \\ 2n-2 &= b \end{aligned}\right\}$$

$$\text{Then } F_0 = \frac{1}{b}.\ P_1 - \frac{a}{b}.F_1$$

$$\text{Similarly } F_1 = \frac{1}{b-2}.\ P_2 - \frac{a+2}{b-2}.\ F_2$$

$$F_2 = \frac{1}{b-4}.\ P_3 - \frac{a+4}{b-4}.\ F_3$$

$$\&c. = \&c.$$

$$F_{n-2} = \frac{1}{b-2.(n-2)}.\ P_{n-1} - \frac{a+2.(n-2)}{b-2.(n-2)}.\ F_{n-1};$$

whence it is easy, by substitution, to find F_0 in terms of F_{n-1} and other known ($F_{n-1} = \tan.^{-1}x$) integrals.

Let $n=3$, and let (in this case) F_1, F_2 &c. become F'_1, F'_2 &c.

$$\text{Then } F'_0 = \frac{1}{4}.\frac{x}{(1+x^2)^2} + \frac{3}{4}.\ F'_1$$

$$F'_1 = \frac{1}{2}.\frac{x}{1+x^2} + \frac{1}{2}F'_2 = \frac{1}{2}\,\frac{x}{1+x^2} + \frac{1}{2}.\ \tan.^{-1}x$$

$$\therefore\ F'_0 = \frac{1}{4}.\frac{x}{(1+x^2)^2} + \frac{3}{8}\frac{x}{(1+x^2)}. + \frac{3}{8}.\ \tan.^{-1}x$$

Let $x = 0$

Then $\tan.^{-1}x = m\pi$ ($\pi = 180°$ and m = any whole number), and the other terms vanish.

Let $x = 1$.

Then $\tan.^{-1}x = \frac{\pi}{4} + 2p\pi$ (p being any integer whatever) and

F_0 becomes $\frac{1}{16} + \frac{3}{16} + \frac{3}{8}\left(\frac{\pi}{4} + 2p\pi\right) = \frac{1}{4} + \frac{3}{8}\left(\frac{\pi}{4} + 2p\pi\right)$

$\therefore$ F_0 (between $x = 0$ and 1) $= \frac{1}{4} + \frac{3}{8}.\left(\overline{2p - m}.\pi + \frac{\pi}{4}\right)$

of which the value, stated in the problem, is a particular case; viz., when $m = 0$, and $p = 0$.

559. To integrate $\frac{d \times dz}{z^3.(a+cz^2)^2}$, assume

$$\frac{1}{z^2.(a+cz^2)} = P$$

$$\therefore dP = \frac{-2dz}{z^3.(a+cz^2)} - \frac{2cdz}{z.(a+cz^2)^2}$$

$$= \frac{-2dz \times (a+cz^2)}{z^3.(a+cz^2)^2} - \frac{2cdz}{z.(a+cz^2)^2}$$

$$= \frac{-2adz}{z^3.(a+cz^2)^2} - \frac{4cdz}{z.(a+cz^2)^2}$$

$$\therefore \int \frac{d \times dz}{z^3.(a+cz^2)^2} = -\frac{d}{2a} \times P - \frac{2c}{a}. \int \frac{d \times dz}{z.(a+cz^2)^2}$$

$$= -\frac{d}{2a} \times P - \frac{2c}{a} \times A$$

560. To integrate $\frac{dx}{x.\sqrt{x^2-a^2}}$

Put $\sqrt{x^2-a^2} = u$

$\therefore x = \sqrt{u^2+a^2}$

And $dx = \frac{udu}{\sqrt{u^2+a^2}}$

$\therefore \int \frac{dx}{x.\sqrt{x^2-a^2}} = \int \frac{du}{u^2+a^2} = \frac{1}{a^2}. \tan.^{-1}u$, which is $\therefore$ known.

To integrate $\frac{x^3 dx}{a^2 - x^2}$, we have

$$\frac{x^3}{x^2 - a^2} = x + \frac{a^2 x}{x^2 - a^2} \text{ by division}$$

$$\therefore \int \frac{x^3 dx}{a^2 - a^2} = -\int x dx - \int \frac{a^2 x dx}{x^2 - a^2} = \frac{-x^2}{2} - \frac{a^2}{2}. \, l. \, (x^2 - a^2)$$

To integrate $x^2 dx. \sqrt{a^2+x^2} = dF$.

Assume $x\,(a^2+x^2)^{\frac{3}{2}} = P_1$

Then $dP_1 = dx. \, (a^2+x^2)^{\frac{3}{2}} + 3 \,.\, x^2 dx \,.\, \sqrt{a^2+x^2}$

$$= a^2 dx \,.\, \sqrt{a^2+x^2} + 4\, x^2 dx \sqrt{a^2+x^2}$$

$$\therefore F = \frac{P_1}{4} - \frac{a^2}{4} \int dx. \sqrt{a^2+x^2}$$

Again, put $x \sqrt{a^2 + x^2} = P_2$

Then $dP_2 = dx. \sqrt{a^2+x^2} + \frac{x^2 dx}{\sqrt{a^2+x^2}}$

$$= dx. \sqrt{a^2+x^2} + \frac{dx}{\sqrt{a^2+x^2}}. \, (x^2+a^2-a^2)$$

$$= 2dx. \sqrt{a^2+x^2} - \frac{a^2.dx}{\sqrt{a^2+x^2}}$$

$$\therefore \int dx \sqrt{a^2+x^2} = \frac{1}{2} P_2 + \frac{a^2}{2}. \int \frac{dx}{\sqrt{a^2+x^2}}$$

$$= \frac{1}{2} P_2 + \frac{a^2}{2}. \, l. \, (x + \sqrt{a^2+x^2})$$

Hence $\int x^2 dx \sqrt{a^2+x^2} = \frac{1}{4} P_1 - \frac{a^2}{8}. P_2 - \frac{a^4}{8}. \, l. \, (x + \sqrt{a^2+x^2})$

which is $\therefore$ known.

561. To integrate $\frac{dz}{\sin. z. \cos. z}$, we have $\int \frac{dz}{\sin. z. \cos. z}$

$$\int \frac{\frac{dz}{\cos.^2 z}}{\frac{\sin. z}{\cos. z}} = \int \frac{d \tan. z}{\tan. z} = l. \tan.$$

To integrate $\dfrac{dx}{\sqrt{A+Bx-x^2}}$, take away the second term of the denominator, by assuming $x - \dfrac{B}{2} = u$.

Then $dx = du$

$$x^2 - Bx + \frac{B^2}{4} = u^2$$

And we have $\sqrt{A+Bx-x^2} = \sqrt{A+\dfrac{B^2}{4}-u^2} = \sqrt{a^2-u^2}$

(a being put $= A + \dfrac{B^2}{4}$)

$\therefore \displaystyle\int \frac{dx}{\sqrt{A+Bx-x^2}} = \int \frac{du}{\sqrt{a^2-u^2}} = \frac{1}{a}.\ \sin.^{-1}u$ which is $\therefore$ known.

562. Let $\int (e+fx^n)^m \times x^p dx = F$, and assume

$$(e+fx^n)^{m+1} . x^{p+1} = P$$

Then $dP = (p+1) . x^p dx\,(e+fx^n)^{m+1} + fn\,(m+1) . (e+fx^n)^m x^{p+n} dx$

$= e . (p+1) . x^p dx . (e+fx^n)^m + f(\overline{p+1} + n . \overline{m+1}) \times (e+fx^n)^m x^{p+n} dx$

$\therefore \displaystyle\int (e+fx^n)^m . x^{p+n} dx = \frac{1}{f(\overline{p+1}+n.\overline{m+1})} \times P - \frac{e . (p+1)}{f . (\overline{p+1}+n . \overline{m+1})} \times F$, which is $\therefore$ known.

$d.P$ also $= (p+1) . x^p dx . (e+fx^n)^{m+1} + n . (m+1) . (e+fx^n)^m x^p dx \times (fx^n + e - e)$

$= (\overline{p+1} + n . \overline{m+1})\, x^p dx . (e+fx^n)^{m+1} - en . (m+1) \times (e+fx^n)^m x^p dx$.

$\therefore \displaystyle\int (e+fx^n)^{m+1} x^p dx = \frac{1}{\overline{p+1}+n.\overline{m+1}} \times P + \frac{en . (m+1)}{\overline{p+1}+n.\overline{m+1}} \times F$ which is $\therefore$ known.

To integrate $\dfrac{a^{\frac{1}{2}}+y^{\frac{1}{2}}}{y^{\frac{1}{3}}+y^{\frac{2}{3}}} \times dy = d.F$, assume $y^{\frac{1}{6}} = u$

Then $dF = \frac{a^{\frac{1}{2}}+u^3}{u^3+u^4} \times 6u^5\,du = \frac{6a^{\frac{1}{2}}u^2du}{u+1} + \frac{6u^5}{u+1} = 6a^{\frac{1}{2}} \times$ $\left(udu - du + \frac{du}{u+1}\right) + 6\,\left(u^4du - u^3du + u^2du - udu + du\right.$ $\left.- \frac{du}{u+1}\right)$

$\therefore\ F = 6\,a^{\frac{1}{2}}.\left(\frac{u^2}{2} - u + l.\,\overline{u+1}\right) + 6\,\left(\frac{u^5}{5} - \frac{u^4}{4} + \frac{u^3}{3}\right.$ $\left.- \frac{u^2}{2} + u - l.\,\overline{u+1}\right) = 6u^3 \times \left(\frac{u^2}{5} - \frac{u}{4} + \frac{1}{3}\right) + 3\,(a^{\frac{1}{2}} - 1) \times$ $(u^2 - 2u + 2\,l.\,\overline{u+1})$ } which is $\therefore$ known.

563. $\int \frac{adx}{a^2 - x^2} = \frac{1}{2}\int \frac{2adx}{a^2 - x^2} = \frac{1}{2}.\,l.\frac{x+a}{x-a}$ (*Vince, Lacroix, Simpson,* &c.)

To integrate $\frac{x^2dx}{\sqrt{a^2+x^2}}$, assume $x\sqrt{a^2+x^2} = u$

$$\therefore\ du = dx.\sqrt{a^2+x^2} + \frac{x^2dx}{\sqrt{a^2+x^2}} = \frac{a^2dx}{\sqrt{a^2+x^2}} + \frac{2x^2\,dx}{\sqrt{a^2+x^2}}$$

$$\therefore \int \frac{x^2dx}{\sqrt{a^2+x^2}} = \int \frac{du}{2} - \int \frac{a^2dx}{2\sqrt{a^2+x^2}}$$

$$= \frac{u}{2} - \frac{a^2}{2}.\,l.\,(x+\sqrt{a^2+x^2})$$

$$= x.\sqrt{a^2+x^2} - \frac{a^2}{2}.\,l.\,(x+\sqrt{a^2+x^2})$$

564. $\int \frac{(x-1)dx}{(x^2+1)^2} = \int \frac{xdx}{(x^2+1)^2} - \int \frac{dx}{(x^2+1)^2}$

$$= -\frac{1}{2} \times \frac{1}{x^2+1} - \int \frac{dx}{(x^2+1)^2}$$

Assume $\frac{x}{x^2+1} = P$

Then $dP = \frac{dx}{x^2+1} - \frac{x^2 dx}{(x^2+1)^2} = \frac{dx}{1+x^2} - \frac{dx}{(x^2+1)^2}$
$\times (x^2+1-1) = \frac{dx}{1+x^2} - \frac{dx}{1+x^2} + \frac{dx}{(1+x^2)^2} = \frac{dx}{(1+x^2)^2}$

$$\therefore \int \frac{(x-1)dx}{(x^2+1)^2} = -\frac{1}{2} \times \frac{1}{x^2+1} - P = -\frac{1}{2} \times \frac{1}{1+x^2} - \frac{x}{1+x^2}$$
$$= -\frac{1+2x}{2.(1+x^2)}$$

To integrate $\frac{d \times dz}{\sqrt{a^2 - bz^2}}$, we have

$$\int \frac{d \times dz}{\sqrt{a^2 - bz^2}} = \int \frac{d}{a} \times \frac{dz}{\sqrt{1 + \frac{b}{a^2}.z^2}} = \frac{d}{\sqrt{b}} \times \int \frac{\frac{\sqrt{b}}{a}dz}{\sqrt{1 - \frac{b}{a^2}.z^2}}$$
$$= \frac{d}{\sqrt{b}}.\sin.^{-1}\frac{\sqrt{b}.z}{a} \quad (\textit{Vince, Lacroix, Simpson.})$$

565. To integrate $\frac{e^z z dz}{(1+z)^2}$

Assume $1 + z = u$

Then $z = u - 1$

$dz = du$

$e^z = e^{u-1} = \frac{e^u}{e}$

$$\therefore \int \frac{e^z z dz}{(1+z)^2} = \frac{1}{e} \times \int \frac{e^u \times (u-1).du}{u^2}$$
$$= \frac{1}{e} \times \left(\int \frac{e^u du}{u} - \int \frac{e^u du}{u^2} \right)$$

Now $\int \frac{e^u du}{u} = \int \left(e^u du \times \frac{1}{u} \right) = \frac{e^u}{u} - \int e^u \times d.\frac{1}{u}$
$= \frac{e^u}{u} + \int \frac{e^u du}{u^2}$, by the form $\int xdy = xy - \int ydx$.

$$\therefore \int \frac{e^z z dz}{(1+z)^2} = \frac{1}{e}.\frac{e^u}{u} = \frac{e^{u-1}}{u} = \frac{e^z}{z+1}.$$

By this method we may integrate all differentials of the form $e^x dx \times \left(\phi x + \frac{d\phi x}{dx}\right)$, ϕx being any function whatever of x.

For $e^x dx \times \phi x + e^x \times d.\,\phi x = d.\,(e^x \times \phi x)$

To integrate $\frac{d \times y^2 dy}{(ar^2+by^2).\sqrt{r^2-y^2}} = d\text{F}$, we have

$$d\text{F} = \frac{d}{b} \times \frac{dy}{(ar^2+by^2).\sqrt{r^2-y^2}} \times (by^2+ar^2-ar^2)$$

$$= \frac{d}{b} \times \frac{dy}{\sqrt{r^2-y^2}} - \frac{r^2 d}{b} \times \frac{dy}{(r^2+\frac{b}{a}y^2).\sqrt{r^2-y^2}}$$

$$\therefore \text{F} = \frac{d}{br}\sin.^{-r} y - \frac{r^2 d}{b}\int\frac{dy}{(r^2+\frac{b}{a}y^2).\sqrt{r^2-y^2}}$$

Let now $u = \sqrt{\frac{r^2-y^2}{r^2+cy^2}}$ (c being $= \frac{b}{a}$)

Then $y^2 = \frac{r^2.(1-u^2)}{1+cu^2}$

And $y = \frac{r.\sqrt{1-u^2}}{\sqrt{1+cu^2}}$

$$\therefore dy = \frac{-rudu}{\sqrt{1-u^2} \times \sqrt{1+cu^2}} - \frac{cudu \times r\sqrt{1-u^2}}{(1+cu^2)^{\frac{3}{2}}}$$

$$= \frac{-r.(c+1)\,udu}{\sqrt{1-u^2} \times (\sqrt{1+cu^2})^{\frac{3}{2}}}$$

Also $\sqrt{r^2-y^2} = \sqrt{r^2 \times \left(1 - \frac{1-u^2}{1+cu^2}\right)} = \frac{r.\sqrt{c+1}.u}{\sqrt{1+cu^2}}$

And $r^2+cy^2 = r^2\left(1 + \frac{c-cu^2}{1+cu^2}\right) = \frac{r^2.(c+1)}{1+cu^2}$

$$\therefore \int\frac{dy}{(r^2+\frac{b}{a}y^2)\sqrt{r^2-y^2}} = \int\frac{-du}{r^2.\sqrt{c+1} \times \sqrt{1-u^2}}$$

$$= \frac{1}{r^2.\sqrt{c+1}} \times \cos.^{-1} u$$

$$\therefore \text{F} = \frac{d}{br}\sin.^{-r}y - \frac{d}{b\sqrt{c+1}} \times \cos.^{-1}\sqrt{\frac{r^2-y^2}{r^2+cy^2}}$$

To integrate $\dfrac{dx}{\sin.^2 x \times \cos.^4 x} = d\text{F}$

Assume $\text{P} = \dfrac{1}{\sin. x \times \cos.^3 x}$

$$\text{Then } d\text{P} = \frac{-dx \times \cos.^4 x + 3dx \cos.^2 x \times \sin.^2 x}{\sin.^2 x \times \cos.^6 x}$$

$$= \frac{-dx \times \cos.^2 x + 3dx \times \sin.^2 x}{\sin.^2 x \times \cos.^4 x}$$

$$= \frac{-dx + 4dx \times \sin.^2 x}{\sin.^2 x \times \cos.^4 x}$$

$$= \frac{-dx}{\sin.^2 x \times \cos.^4 x} + \frac{4dx}{\cos.^4 x}$$

$$\therefore \text{ F} = \int \frac{4dx}{\cos.^4 x} - \text{P}$$

But $\dfrac{dx}{\cos.^2 x} = d\tan. x$ and $\dfrac{1}{\cos.^2 x} = \sec.^2 x = 1 + \tan.^2 x$

$$\therefore \int \frac{4dx}{\cos.^4 x} = \int 4d.(\tan. x) \times (1 + \tan.^2 x)$$

$$= \int 4d. \tan. x + \int 4 \tan.^2 x \times d. \tan. x$$

$$= 4. \tan. x + \frac{4}{3}\tan.^3 x$$

$$\therefore \text{ F} = 4\tan. x + \frac{4}{3}\tan.^3 x - \frac{1}{\sin. x \times \cos.^3 x}$$

$$= 4\frac{\sin. x}{\cos. x} + \frac{4}{3} \times \frac{\sin.^3 x}{\cos.^3 x} - \frac{1}{\sin. x \times \cos.^3 x}$$

$$= \frac{4}{3}\sin.^2 x \times \frac{3\cos.^2 x + \sin.^2 x}{\sin. x \times \cos.^3 x} - \frac{1}{\sin. x \times \cos.^3 x}$$

$$= \frac{4}{3}.(1-\cos.^2 x) \times \frac{2\cos.^2 x + 1}{\sin. x.\cos.^3 x} - \frac{1}{\sin. x.\cos.^3 x}$$

$$= \frac{4}{3}.\frac{1 + \cos.^2 x - 2\cos.^4 x}{\sin. x.\cos.^3 x} - \frac{1}{\sin. x.\cos.^3 x}$$

$$= \frac{1}{3}.\frac{1}{\sin. x.\cos.^3 x} - \frac{4}{3}.\frac{2\cos.^2 x - 1}{\sin. x.\cos. x}$$

$$= \frac{1}{3} \cdot \frac{1}{\sin. x. \cos.^3 x} - \frac{8}{3} \times \frac{\cos. 2x}{\sin. 2x}$$

$$= \frac{1}{3} \cdot \frac{1}{\sin. x. \cos.^3 x} - \frac{8}{3} \cot. 2x$$

Otherwise.

$$\frac{dx}{\sin.^2 x. \cos.^4 x} = \frac{dx}{\cos.^2 x} \times \frac{1}{\sin.^2 x} \times \frac{1}{\cos.^2 x} = d. \tan. x$$
$$\times \operatorname{cosec.}^2 x \times \sec.^2 x = d. \tan. x \times (1 + \cot.^2 x) \times (1 + \tan.^2 x)$$
$$= d. \tan. x \times \left(2 + \tan.^2 x + \frac{1}{\tan.^2 x}\right) = 2d. \tan. x + d. \tan. x \times$$
$$\tan.^2 x + \frac{d. \tan. x}{\tan.^2 x}$$

$$\therefore \int \frac{dx}{\sin.^2 x. \cos.^4 x} = 2 \tan. x + \frac{\tan.^3 x}{3} - \frac{1}{\tan. x}$$ which may be reduced.

This latter method may be applied to all differentials of the form $\frac{dx}{\sin.^{2m} x \times \cos.^{2n} x}$.

To integrate $\frac{bdy}{(a^2 - y^2).(a+y)^{\frac{2}{3}}} = d.F.$

Let $(a+y)^{\frac{1}{3}} = u$

Then $du = \frac{1}{3} \cdot \frac{dy}{(a+y)^{\frac{2}{3}}}$

Also $a + y = u^3$

And $a - y = 2a - u^3$

$\therefore a^2 - y^2 = u^3 \times (2a - u^3)$

Hence $d.F = \frac{3b \times du}{u^3 \times (2a - u^3)}$.

Again, put $\frac{1}{u^2} = v^2$

$\therefore \frac{du}{u^3} = -vdv$

$$\text{And } d.\mathrm{F} = 3b \times \frac{-v^4 dv}{2av^3 - 1} = -\frac{3b}{2a} \cdot \frac{v^4 dv}{v^3 - \frac{1}{2a}}$$

$$= -\frac{3b}{2a} vdv - \frac{3b}{4a^2} \times \frac{vdv}{v^3 - \frac{1}{2a}}$$

$$\therefore \mathrm{F} = -\frac{3b}{4a} \times v^2 - \frac{3b}{4a^2} \int \frac{vdv}{v^3 - \frac{1}{2a}}$$

$$\text{Again, let } \frac{1}{v^3 - \frac{1}{2a}} \left(= \frac{1}{v^3 - c^3}\right) = \frac{\mathrm{A}}{v - c} + \frac{\mathrm{B}v + \mathrm{C}}{v^2 + c + v}$$

$$\left.\begin{array}{r} \text{Then } \mathrm{A}v^2 + \mathrm{A}cv + \mathrm{A}c^2 \\ + \mathrm{B}v^2 + \mathrm{C}v - \mathrm{C}c \\ - \mathrm{B}cv \end{array}\right\} = 1 \quad \left.\begin{array}{c} \therefore \mathrm{A} + \mathrm{B} = 0 \\ \mathrm{A}c - \mathrm{B}c + \mathrm{C} = 0 = 2\mathrm{A}c + \mathrm{C} \\ \text{and } \mathrm{A}c^2 - \mathrm{C}c = 1 \end{array}\right\}$$

$$\text{Hence } 2\mathrm{A}c^2 + \mathrm{C}c = 0$$

$$\text{And } 2\mathrm{A}c^2 - 2\mathrm{C}c = 2$$

$$\therefore 3\,\mathrm{C}c = -2$$

$$\text{And } \mathrm{C} = -\frac{2}{c^3}$$

$$\text{Also } \mathrm{A} = -\frac{\mathrm{C}c}{2c^2} = \frac{1}{3c^2}$$

$$\text{And } \mathrm{B} = -\mathrm{A} = -\frac{1}{3c^2}$$

$$\text{Hence } \mathrm{F} = -\frac{3b}{4a} \times v^2 - \frac{b}{4a^2c^2} \times \int \frac{vdv}{v - c} + \frac{b}{4a^2c^2} \times$$

$$\int \frac{v^2 dv}{v^2 + cv + c^2} + \frac{b}{2a^2c} \cdot \int \frac{vdv}{v^2 + cv + c^2}$$

$$\text{But } \int \frac{vdv}{v - c} = \int dv + c \int \frac{dv}{v - c} \text{ by division.}$$

$$= v + l.(v - c)$$

$$\therefore \mathrm{F} = -\frac{3b}{4a} \times v^2 - \frac{b}{4a^2c^2}(v + c\; l.\overline{v - c}) + \frac{b}{4a^2c^2}$$

$$\int \frac{v^2 dv}{v^2 + cv + c^2} + \frac{b}{2a^2c} \cdot \int \frac{vdv}{v^2 + cv + c^2}$$

Again, put $v + \frac{c}{2} = z$

Then $dv = dz$

And $v^2 + cv + c^2 = z^2 + c^2 - \frac{c^2}{4} = z^2 + \frac{3}{4}c^2$

Hence $\int \frac{v^2 dv}{v^2 + cv + c^2} = \int \frac{\left(z^2 - cz + \frac{c^2}{4}\right)dz}{z^2 + \frac{3}{4}c^2} = \int dz -$

$\int \frac{czdz}{z^2 + \frac{3}{4}c^2} - \int \frac{\frac{c^2}{2}dz}{z^2 + \frac{3}{4}.c^2} = z - \frac{c}{2}.l.(z^2 + \frac{3}{4}c^2) - \frac{c^2\sqrt{3}}{4} \times$

$\int \frac{\frac{2}{c\sqrt{3}}dz}{1 + \frac{4z^2}{3c^2}}$

$$= z - \frac{c}{2}.l.(z^2 + \frac{3}{4}c^2) - \frac{c^3\sqrt{3}}{4}.\tan.^{-1}\frac{2z}{c\sqrt{3}}$$

Also $\int \frac{vdv}{v^2 + cv + c} = \int \frac{\left(z - \frac{c}{2}\right)dz}{z^2 + \frac{3}{4}c^2} = \int \frac{zdz}{z^2 + \frac{3}{4}c^2} -$

$\int \frac{\frac{c}{2}.dz}{z^2 + \frac{3}{4}c^2}$

$$= \frac{1}{2}.l.(z^2 + \frac{3}{4}c^2) - \frac{c^2\sqrt{3}}{4} \times \int \frac{2}{c\sqrt{3}}.\frac{dz}{1 + \frac{4z^2}{3c^2}}$$

$$= \frac{1}{2}.l.(z^2 + \frac{3}{4}c^2) - \frac{c^2\sqrt{3}}{4} \times \tan.^{-1}\frac{2z}{c\sqrt{3}}$$

Hence and by substitution we may find the integral required. The integration will be more easily effected by assuming $a + y = \frac{1}{u^3}$; for the differential then becomes $dF = \frac{-u^4 du}{2au^3 - 1}$

566. To integrate $\frac{dx}{x^3.(a^2 - x^2)}$, assume $\frac{a^2}{x^2} = u$

Then $\frac{dx}{x^3} = -\frac{du}{a^2}$

$$\text{And } a^2 - x^2 = a^2 - \frac{a^2}{u} = \frac{a^2}{u} \cdot (u-1)$$

$$\therefore \int \frac{dx}{x^3 \cdot (a^2-x^2)} = \int \frac{-udu}{a^4 \cdot (u-1)} = -\frac{1}{a^4} \int \left(du + \frac{du}{u-1}\right)$$

$$= -\frac{1}{a^4} \times \left\{u + l.(u-1)\right\}$$

$$= -\frac{1}{a^4} \times \left\{\frac{a^2}{x^2} + l.\frac{a^2-x^2}{x^2}\right\}$$

To integrate $x^3dx \times (a^2+x^2)^{\frac{1}{3}}$

$$\text{Let } x^2 \cdot (a^2+x^2)^{\frac{4}{3}} = \text{P}$$

$$\text{Then } d\text{P} = 2xdx\,(a^2+x^2)^{\frac{4}{3}} + \frac{8}{3} x^3\,dx \cdot (a^2+x^2)^{\frac{1}{3}}$$

$$\therefore \int x^3dx \cdot (a^2+x^2)^{\frac{1}{3}} = \frac{3\text{P}}{8} - \frac{3}{8}\int 2xdx \cdot (a^2+x^2)^{\frac{4}{3}}$$

$$= \frac{3\text{P}}{8} - \frac{9}{56} \times (a^2+x^2)^{\frac{7}{3}}$$

$$= \frac{3}{8}(a^2+x^2)^{\frac{4}{3}} \times \left\{x^2 - \frac{3}{7} \cdot (a^2+x^2)\right\}$$

$$= \frac{3}{56}(a^2+x^2)^{\frac{4}{3}} \times (4x^2 - 3a^2)$$

567.
$$\int \frac{dx \cdot \sqrt{a^2-x^2}}{x} = \int \frac{a^2\,dx}{x \cdot \sqrt{a^2-x^2}} - \int \frac{xdx}{\sqrt{a^2-x^2}}$$

$$= \frac{a}{2} \int \frac{2adx}{x \cdot \sqrt{a^2-x^2}} + \sqrt{a^2-x^2}$$

$$= \frac{a}{2} \cdot l \cdot \frac{a-\sqrt{a^2-x^2}}{a+\sqrt{a^2-x^2}} + \sqrt{a^2-x^2}$$

$$= a \cdot l \cdot \frac{a-\sqrt{a^2-x^2}}{x} + \sqrt{a^2-x^2}$$

Again $\int \frac{bdx}{\sqrt{1-ax^2}} = \frac{b}{\sqrt{a}} \cdot \int \frac{\sqrt{a}dx}{\sqrt{1-ax^2}} = \frac{b}{\sqrt{a}} \int \frac{d.(\sqrt{a} \, . \, x)}{\sqrt{1-ax^2}}$

$$= \frac{b}{\sqrt{a}} . \sin.^{-1} x\sqrt{a}$$

568. $\int zdz \sqrt{\frac{a+z}{a-z}} = \int \frac{zdz \times (a+z)}{\sqrt{a^2-z^2}} = \int \frac{azdz}{\sqrt{a^2-z^2}} + \int \frac{z^2 dz}{\sqrt{a^2-z^2}}$

$$= -a.\sqrt{a^2-z^2} + \int \frac{-dz}{\sqrt{a^2-z^2}} \times (a^2-z^2-a^2)$$

$$= -a\sqrt{a^2-z^2} - \int dz . \sqrt{a^2-z^2} + a\int \frac{adz}{\sqrt{a^2-z^2}}$$

$$= -a\sqrt{a^2-z^2} - \int dz\sqrt{a^2-z^2} + a . \sin.^{-1} z$$

Let z be the abscissa measured from the centre of a circle whose radius is a. Then $\sqrt{a^2-z^2}$ = the corresponding ordinate.

$\therefore \int \sqrt{a^2-z^2} \times dz = \int d.$ (area between ordinates at extremities of z) = A by supposition.

$$\therefore \int zdz \sqrt{\frac{a+z}{a-z}} = -a\sqrt{a^2-z^2} - \text{A} + a . \sin.^{-1} z.$$

To integrate $\frac{dz}{z} . [(a^2+z^2)^{\frac{3}{2}}$, we have

$$\frac{(a^2+z^2)^{\frac{3}{2}}}{z} = (a^2+z^2)^{\frac{1}{2}} \times \frac{a^2}{z} + z . (a^2+z^2)^{\frac{1}{2}}$$

$$= \frac{a^4}{z . \sqrt{a^2+z^2}} + \frac{a^2 z}{\sqrt{a^2+z^2}} + z . (a^2+z^2)^{\frac{1}{2}}$$

$$\therefore \int \frac{dz}{z} (a^2+z^2)^{\frac{3}{2}} = \frac{a^3}{2} \int \frac{2adz}{z . \sqrt{a^2+z^2}} + a^2 \int \frac{zdz}{\sqrt{a^2+z^2}} + \int zdz\sqrt{a^2+z^2}$$

$$= \frac{a^3}{2} . l . \frac{\sqrt{a^2+z^2}-a}{\sqrt{a^2+z^2}+a} + a^2\sqrt{a^2+z^2} + \frac{1}{3}(a^2+z^2)^{\frac{3}{2}}$$

$$= a^2 \cdot l \cdot \frac{z}{\sqrt{a^2+z^2}+a} + a^2 \sqrt{a^2+z^2}$$

$$+ \frac{1}{3} \cdot (a^2+z^2)^{\frac{3}{2}}$$

To integrate $\dfrac{d \times y^2 dy}{(ar^2+by^2) \cdot \sqrt{r^2-y^2}} = dF$ we have

$$dF = \frac{d}{b} \times \frac{dy}{(ur^2+by^2) \cdot \sqrt{r^2-y^2}} \times (by^2 + ar^2 - ar^2)$$

$$= \frac{d}{b} \times \frac{dy}{\sqrt{r^2-y^2}} - \frac{r^2 d}{b} \times \frac{dy}{\left(r^2 + \frac{by^2}{a}\right) \cdot \sqrt{r^2-y^2}}$$

$$\therefore\ F = \frac{d}{br} \sin.^{-1} y - \frac{r^2 d}{b} \int \frac{dy}{\left(r^2 + \frac{b}{ay^2}\right) \cdot \sqrt{r^2-y^2}}$$

Let now $u = \sqrt{\dfrac{r^2-y^2}{r^2+cy^2}}$ $\left(c \text{ being } = \dfrac{b}{a}\right)$

Then $y^2 = \dfrac{r^2 \cdot (1-u^2)}{(1+cu^2)}$

and $y = \dfrac{r \cdot \sqrt{1-u^2}}{\sqrt{1+cu^2}}$

$$\therefore\ dy = \frac{-rudu}{\sqrt{1-u^2} \times \sqrt{1+cu^2}} - \frac{cudu \times r\sqrt{1-u^2}}{(1+cu^2)^{\frac{3}{2}}}$$

$$= \frac{-r \cdot (c+1)\ udu}{\sqrt{1-u^2} \times (1+cu^2)^{\frac{3}{2}}}$$

Also $\sqrt{r^2-y^2} = \sqrt{r^2 \times \left(1 - \dfrac{1-u^2}{1+cu^2}\right)} = \dfrac{r \cdot \sqrt{c+1} \cdot u}{\sqrt{1+cu^2}}$

and $r^2 + cy^2 = r^2 \left(1 + \dfrac{c-cu^2}{1+cu^2}\right) = \dfrac{r^2 \cdot (c+1)}{1+cu^2}$

$$\therefore \int \frac{dy}{\left(r^2 + \frac{b}{a} y^2\right)\sqrt{r^2-y^2}} = \int \frac{-du}{r^2 \cdot \sqrt{c+1} \times \sqrt{1-u^2}}$$

$$= \frac{1}{r^2 \cdot \sqrt{c+1}} \times \cos.^{-1} u$$

$$\therefore F = \frac{d}{br} \sin.^{-1} y - \frac{d}{b\sqrt{c+1}} \times \cos.^{-1} \sqrt{\frac{r^2 - y^2}{r^2 - cy^2}}.$$

To integrate $\dfrac{dz}{(1 - az^2)\sqrt{1 - z^2}}$, assume

$$u = \sqrt{\frac{1 - z^2}{1 - az^2}}, \therefore u^2 - az^2u^2 = 1 - z^2.$$

$$\therefore z^2 = \frac{u^2 - 1}{au^2 - 1}, \therefore z = \sqrt{\frac{u^2 - 1}{au^2 - 1}}$$

$$\therefore dz = \frac{udu}{\sqrt{u^2 - 1} \times \sqrt{au^2 - 1}} - \frac{audu \times \sqrt{u^2 - 1}}{(au^2 - 1)^{\frac{3}{2}}}$$

$$= udu \times \frac{au^2 - 1 - au^2 + a}{\sqrt{u^2 - 1} \times (au^2 - 1)^{\frac{3}{2}}} = \frac{(a-1) \cdot udu}{\sqrt{u^2 - 1} \times (au^2 - 1)^{\frac{3}{2}}}.$$

$$\text{Also } \sqrt{1 - z^2} = \sqrt{1 - \frac{u^2 - 1}{au^2 - 1}} = \sqrt{\frac{(a-1) \cdot u^2}{au^2 - 1}} = \frac{\sqrt{a-1} \cdot u}{\sqrt{au^2 - 1}}$$

$$1 - az^2 = \frac{a - 1}{au^2 - 1}$$

$$\therefore (1 - az^2) \cdot \sqrt{1 - z^2} = \frac{(a - 1)^{\frac{3}{2}} u}{(au^2 - 1)^{\frac{3}{2}}}$$

$$\text{Hence } \int \frac{dz}{(1 - az^2)\sqrt{1 - z^2}} = \int \frac{du}{\sqrt{1 - a}\sqrt{u^2 - 1}} = \frac{1}{-\sqrt{1 - a}} \times \int \frac{-du}{\sqrt{1 - u^2}}.$$

$$= \frac{1}{-\sqrt{1 - a}} \times \cos.^{-1} u = \frac{1}{-\sqrt{1 - a}} \times \cos.^{-1} \sqrt{\frac{1 - z^2}{1 - az^2}}.$$

569. To integrate $e^x (d P + P dx)$ we have

$$\begin{aligned}\int e^x(dP + P\,dx) &= \int (e^x dP + P e^x dx)\\ &= \int (e^x d.P + P d.e^x)\\ &= \int d.e^x P\\ &= e^x P\end{aligned}$$

To integrate $\dfrac{d \times z^{pn-1}\,dz}{(a+cz^n)^m\,(e+fz^n)^r} = dF$, assume

$a + cz^n = u$

Then $z^n = \dfrac{u-a}{c}$

And $z^{pn} = \dfrac{(u-a)^p}{c^p}$

$\therefore\ z^{pn-1}\,dz = \dfrac{(u-a)^{p-1}\,du}{nc^p}$

Also $e + fz^n = e + \dfrac{fu}{c} - \dfrac{fa}{c} = \left(u - a + \dfrac{ec}{f}\right)\dfrac{f}{c}$

$= (u-b)\dfrac{f}{c}$, by supposition.

Hence, we get, $\dfrac{d \times z^{pn-1}\,dz}{(a+cz^n)^m.\,(e+fz^n)^r} = \dfrac{d}{nf^r\,c^{p-r}} \times \dfrac{(u-a)^{p-1}\,du}{(u-b)^r . u^m}$

$= \dfrac{d}{nf^r\,c^{p-r}} \times \dfrac{du}{u^m.(u-b)^r} \times \Big\{ u^{p-1} - (p-1)a\,u^{p-2} + (p-1) \times$
$\dfrac{p-2}{2}a^2.\,u^{p-3} - \&c. \pm (p-1).\dfrac{p-2}{2}\ldots\ldots\dfrac{p-q+1}{q-1}.\,a^{q-1}\,u^{p-q} \mp$
$\&c. \pm (p-1).\,a^{p-2}\,u \mp a^{p-1}\Big\}$ of which each term must be integrated separately.

Let $m = p \pm s$,

Then $dF = \dfrac{d}{nf^r\,c^{p-r}} \times \dfrac{du}{(u-b)^r} \times \Big\{ \dfrac{1}{u^{s+1}} - \dfrac{(p-1).\,a}{u^{s+2}} + (p-1) \times$
$\dfrac{(p-2)}{2}.\dfrac{a^2}{u^{s+3}} - \&c. \pm (p-1)\dfrac{a^{p-2}}{u^{s+p-1}} \mp \dfrac{a^{p-1}}{u^{s+p}}\Big\}$ (A)

Or $d\,\mathrm{F} = \dfrac{d}{nf^r \cdot c^{p-r}} \times \dfrac{du}{(u-b)^r} \times \Big\{ u^{p-1} - (p-1)\,a u^{p-2} + (p-1) \times$

$\dfrac{p-2}{2} a^2 u^{p-3} \mp \&c. \pm (p-1) . \dfrac{p-2}{2} \ldots . \dfrac{p-s+1}{s-1} a^{s-1} u^{p-s} \mp (p-1) \times$

$\dfrac{p-2}{2} \ldots . \dfrac{p-s}{s} \cdot \dfrac{a^s}{u} \pm (p-1) \ldots , \dfrac{p-s-1}{s+1} \times \dfrac{a^{s+1}}{u^2} \mp \&c. \pm$

$(p-1) \dfrac{a^{p-2}}{u^{p-s-1}} \mp \dfrac{a^{p-1}}{u^{p-s}} \Big\}$ (B)

Now the integration of the form (A) depends on that of the form $\dfrac{du}{u^q . (u-b)^r} = \mathrm{F}_0$; (q being any positive integer whatever) to effect which, assume

$$\left.\begin{array}{l} \mathrm{P}_1 = \dfrac{1}{u^{q-1} . (u-b)^{r-1}} \\ \mathrm{P}_2 = \dfrac{1}{u^{q-2} . (u-b)^{r-1}} \\ \&c. = \quad \&c. \\ \mathrm{P}_q = \dfrac{1}{(u-b)^{r-1}} \end{array}\right| \quad \left.\begin{array}{l} \mathrm{F}_1 = \displaystyle\int \dfrac{du}{u^{q-1} . (u-b)^r} \\ \mathrm{F}_2 = \displaystyle\int \dfrac{du}{u^{q-2} . (u-b)^r} \\ \&c. = \quad \&c. \\ \mathrm{F}_q = \displaystyle\int \dfrac{du}{(u-b)^r} \end{array}\right\}$$

Then $d\,\mathrm{P}_1 = \dfrac{b . (q-1)\, du}{u^q . (u-b)^r} - \dfrac{(r+q+2) du}{u^{q-1} . (u-b)^r}$

$$\left.\begin{array}{rl} \therefore\ \mathrm{P}_1 &= b . (q-1) . \mathrm{F}_0 - (r+q+2) . \mathrm{F}_1 \\ \text{Similarly } \mathrm{P}_2 &= b . (q-2) . \mathrm{F}_1 - (r+q+3) . \mathrm{F}_2 \\ \mathrm{P}_3 &= b . (q-3) . \mathrm{F}_2 - (r+q+4) . \mathrm{F}_3 \\ \&c. &= \quad \&c. \\ \mathrm{P}_q &= b . (q-q) \quad \mathrm{F}_{q-1} - \{r+q+(q+1)\}\mathrm{F}_q \end{array}\right\}$$

Whence we have F_0 in terms of $\mathrm{F}_q \left(= -\dfrac{1}{r-1} \cdot \dfrac{1}{(u-b)^{r-1}} \right)$ & $\mathrm{P}_1 , \mathrm{P}_2 \ldots\ldots$ which $\therefore$ is known. Hence, by giving to q the values of $s+1$, $s+2$, &c., $s+p$ successively, we shall obtain the integral of the form (A).

The integration of the last ($p-s$) terms of the form (B), may be effected by the above process; and the remainder by continuing the assumptions, &c., in that process thus,

$$\left.\begin{aligned} \text{Let } P_{q+1} &= \frac{u}{(u-b)^{r-1}} \\ P_{q+2} &= \frac{u^2}{(u-b)^{r-1}} \\ \&c. &= \&c. \\ P_{q+s-1} &= \frac{u^{s-1}}{(u-b)^{r-1}} \end{aligned}\right\} \quad \left.\begin{aligned} F_{q+1} &= \int \frac{u\,du}{(u-b)^r} \\ F_{q+2} &= \int \frac{u^2\,du}{(u-b)^r} \\ \&c. &= \&c. \\ F_{q+s-1} &= \int \frac{u^{s-1}\,du}{(u-b)^r} \end{aligned}\right\}$$

$$\begin{aligned} \text{Then } P_{q+1} &= -b\,.\,F_q - (r-2)\,.\,F_{q+1} \\ P_{q+2} &= -2b\,F_{q+1} - (r-3)\,.\,F_{q+2} \\ \&c. &= \&c. \\ P_{q+s-1} &= -(s-1)\,bF_{q+s-2} - (r-s)\,.\,F_{q+s-1} \end{aligned}$$

Whence we have F_{q+s-1}, F_{q+s-2}, &c., in terms of F_q, and other known integrals.

Having conducted the reader thus far, we shall leave to him the remainder of the investigation.

Otherwise.

Let $z^n = u$

Then $z^{pn} = u^p$, and $z^{pn-1}\,dz = \dfrac{u^{p-1}\,du}{n}$

Hence $d\,F = \dfrac{d}{n} \times \dfrac{u^{p-1}\,du}{(a+cu)^m.(e+fu)^r} = \dfrac{d}{efn} \times \dfrac{u^{p-1}\,du}{(a'+u)^m.(e'+u)^r}$

$\left(a' = \dfrac{a}{c}, \text{ and } e' = \dfrac{e}{f}\right)$.

Again, assume $\dfrac{1}{(a'+u)^m.(e'+u)^r} = \dfrac{A_1}{(a'+u)^m} + \dfrac{A_2}{(a'+u)^{m-1}} + \ldots$

$+ \dfrac{A_{m-1}}{a'+u} + A_m + \dfrac{B_1}{(e'+u)^r} + \dfrac{B_2}{(e'+u)^{r-1}} + \ldots \dfrac{B_{r-1}}{e'+u} + B_r$

Reduce the fractions to a common denominator, and equate the numerators to unity. Make $\left.\begin{aligned} a'+u &= 0 \\ e'+u &= 0 \end{aligned}\right\}$ successively,

then $\left.\begin{aligned} A_1 &= \frac{1}{(e'-a')^r} \\ B_1 &= \frac{1}{(a'-e')^m} \end{aligned}\right\}$ Take the differential, divide by dx, and

proceed as before; then we have A_2, B_2. Repeat the differentiations, &c., and thence obtain the values of the other indeterminates. The remaining part of the process will be easy.

To integrate $\dfrac{x^{m-1}\,dx}{(1-x^m).(2x^m-1)^{\frac{1}{2m}}} = d\,F$. Assume

$$2x^m - 1 = u^{2m}$$

$$\therefore\ x^{m-1}\,dx = u^{2m-1}du$$

$$\text{Also } 1 - x^m = 1 - \frac{u^{2m}}{2} - \frac{1}{2} = \frac{1-u^{2m}}{2}$$

$$\text{Hence } d\,F = \frac{2u^{2m-1}\,du}{(1-u^{2m})u} = \frac{-2u^{2m-2}du}{u^{2m}-1}$$

Now, in order to find the factors of $u^{2m}-1$, let us assume $u^{2m}-1=0$, or

$$u^{2m} = 1 = \cos.\ 2p\pi \pm \sqrt{-1}\ \sin.\ 2p\pi \text{ (p being any integer.)}$$

$\therefore\ u = \cos.\dfrac{p\pi}{m} \pm \sqrt{-1}\ .\ \sin.\ \dfrac{p\pi}{m}$; hence the roots of $u^{2m}-1=0$, are ± 1, $\cos.\dfrac{\pi}{m} \pm \sqrt{-1}\ .\ \sin.\dfrac{\pi}{m}$, $\cos.\dfrac{2\pi}{m} \pm \sqrt{-1}\ .\ \sin.\ \dfrac{2\pi}{m}$, $\cos.\dfrac{3\pi}{m} \pm \sqrt{-1}\ .\ \sin.\ \dfrac{3\pi}{m}$; ...... $\cos.\dfrac{m-1}{m}\pi \pm \sqrt{-1}\ \sin.\dfrac{m-1}{m}\pi$ (the values of u afterwards recur)

$$\therefore\ u^{2m}-1 = (u^2-1)\left(u^2-2u\cos.\frac{\pi}{m}+1\right)\left(u^2-2u\cos.\frac{2\pi}{m}+1\right)\times \ldots\ldots \times\left(u^2-2u\cos.\frac{m-1}{m}\pi+1\right).$$

$$\text{Again, let } \frac{1}{u^{2m}-1} = \frac{A_1u+B_1}{u^2-1} + \frac{A_2u+B_2}{u^2-2u\cos.\frac{\pi}{m}+1} + \frac{A_3u+B_3}{u^2-2u\cos.\frac{2\pi}{m}+1} + \ldots\ldots \frac{A_mu+B_m}{u^2-2u\cos.\frac{m-1}{m}\pi+1}$$

Hence, reducing to a common denominator, equating the numerators of both sides of the equation, and substituting for u its several values in succession, (of which there are many examples in the previous part of this subject,) we shall determine the values of A_1, B_1; A_2, B_2; &c. A_m, B_m.

We have $\therefore dF = -2u^{2m-2}du \left\{ \frac{A_1 u}{u^2-1} + \frac{B}{u^2-1} + \frac{A_2 u + B_2}{u^2 - 2u\cos.\frac{\pi}{m} + 1} \right.$

$\left. + \text{\&c.} + \frac{A_m u + B_m}{u^2 - 2u\cos.\frac{m-1}{m}\pi + 1} \right\}$ whose terms must be integrated separately.

To integrate the general term $T = -2u^{2m-2}du \times \frac{A_p u + B_p}{u^2 - 2u\cos.\frac{p-1}{m}\pi + 1}$, assume $u - \cos.\frac{p-1}{m}\pi = v$,

Then $du = dv$

$$u^2 - 2u\,.\cos.\frac{p-1}{m}\pi + 1 = v^2 + 1 - \cos.^2\frac{p-1}{m}\pi$$

$$= v^2 + \sin.^2\frac{p-1}{m}.\pi$$

$$\therefore T = -\frac{2A_p.(v+\cos.\frac{p-1}{m}\pi)^{2m-1}dv}{v^2 + \sin.^2\frac{p-1}{m}\pi} - \frac{2B_p.(v+\cos.\frac{p-1}{m}\pi)^{2m-2}dv}{v^2 + \sin.^2\frac{p-1}{m}\pi}$$

which, by expanding the numerators, and dividing them by the denominator, will be reduced to known forms of integration.

Having thus exhibited the process to be pursued in the solution of the Problem, we leave the completion of it to the Reader.

570. To integrate $d \times \frac{z^{\frac{1}{3}}dz}{(c^{\frac{4}{3}} - az^{\frac{2}{3}})^{\frac{1}{2}}} = dF$, assume

$z^{\frac{1}{3}} = u^2$

Then $z^{\frac{1}{3}}dz = \frac{3}{2}\,udu$

And $(c^{\frac{5}{3}} - az^{\frac{2}{3}})^{\frac{3}{5}} = (c^{\frac{5}{3}} - au)^{\frac{3}{5}}$

$$\therefore dF = \frac{3d}{2} \times \frac{udu}{(c^{\frac{5}{3}} - au)^{\frac{3}{5}}}$$

$$= -\frac{3d}{2a} \cdot \frac{du}{(c^{\frac{5}{3}} - au)^{\frac{3}{5}}} \times (c^{\frac{5}{3}} - au - c^{\frac{5}{3}})$$

$$= -\frac{3d}{2a} \times du \,.\, (c^{\frac{5}{3}} - au)^{\frac{2}{5}} + \frac{3dc^{\frac{5}{3}}}{2a} \times \frac{du}{(c^{\frac{5}{3}} - au)^{\frac{3}{5}}}$$

$$\therefore F = \frac{15d}{14a^2} \,.\, (c^{\frac{5}{3}} - au)^{\frac{7}{5}} - \frac{15dc^{\frac{5}{3}}}{4a^2} \,.\, (c^{\frac{5}{3}} - au)^{\frac{2}{5}}.$$

To integrate $dx \int dx \int dx \ldots\ldots \infty = dF$, we have $\int dx = x + c_1$

$$\therefore \int (dx \int dx) = \int xdx + c_1 \int dx = \frac{x^2}{2} + c_1\, x + c_2$$

$$\therefore \int (dx \int dx \int dx) = \int \frac{x^2}{2}\, dx + c_1 \int xdx + c_2 \int dx$$

$$= \frac{x^3}{2\,.\,3} + c_1 \,.\, \frac{x^2}{2} + c_2 \,.\, x + c_3$$

Similarly $\int (dx \int dx \int dx \int dx) = \frac{x^4}{2.3.4} + c_1 \,.\, \frac{x^3}{2\,.\,3} + c_2 \,.\, \frac{x^2}{2}$

$+ c_3\, x + c_4$

&c. = &c.

And $F = \frac{x^{\infty}}{2.3.4\ldots.\infty} + c_1 \,.\, \frac{x^{\infty-1}}{2.3\ldots.\infty-1} + c_2 \,.\, \frac{x^{\infty-2}}{2.3\ldots.\infty-2} +$ &c.

$+ c_{\infty-3} \,.\, \frac{x^3}{2\,.\,3} + c_{\infty-2} \,.\, \frac{x^2}{2} + c_{\infty-1} \,.\, x + c_{\infty}$, c_1, c_2, c_3, &c. being constants.

571. To integrate $\frac{dx}{x^2 \sqrt{x^2 - a^2}} = dF$, assume

$$\frac{a}{x} = u$$

$$\therefore \frac{dx}{u^2} = \frac{-du}{a}$$

$$\text{But } F_2 = \int \frac{dx}{x.\sqrt{a^2-x^2}} = \frac{1}{a}.\ l.\frac{a-\sqrt{a^2-x^2}}{x}$$

$$\therefore\ F_0 = \frac{3}{8a^5}.\,l.\frac{a-\sqrt{a^2-x^2}}{x} - \frac{1}{8a^4}.\frac{\sqrt{a^2-x^2}}{x^4}.(3x^2+2a^2)$$

To integrate $\dfrac{dx.\sqrt{1+x}}{(1-x)^{\frac{3}{2}}} = d.\mathrm{F}$, assume

$$\frac{1+x}{1-x} = u^2$$

$$\therefore\ x = \frac{u^2-1}{u^2+1}$$

$$\text{And } dx = \frac{2udu}{u^2+1} - \frac{2udu \times (u^2-1)}{(u^2+1)^2} = \frac{4udu}{(u^2+1)^2}$$

$$\text{Also } 1-x = 1 - \frac{u^2-1}{u^2+1} = \frac{2}{u^2+1}$$

$$\therefore\ d\mathrm{F} = \frac{4udu}{(u^2+1)^2} \times u \times \frac{u^2+1}{2} = \frac{2u^2du}{u^2+1}$$

$$= \frac{2du}{u^2+1} \times (u^2+1-1)$$

$$= 2du - \frac{2du}{u^2+1}$$

$$\therefore\ \mathrm{F} = 2u - 2.\tan.^{-1}u$$

To integrate $\dfrac{dx.(1-x^2)}{(1+x^2)\sqrt{1+ax^2+x^4}} = d\mathrm{F}$, assume

$$u = \frac{x\sqrt{2-a}}{1+x^2}$$

$$\text{Then } \frac{du}{\sqrt{2-a}} = \frac{dx}{1+x^2} - \frac{2x^2dx}{(1+x^2)^2} = \frac{dx.(1-x^2)}{(1+x^2)^2}$$

$$\text{Hence } dx.\frac{1-x^2}{1+x^2} = \frac{1+x^2}{\sqrt{2-a}} \times du = \frac{du}{u} \times x$$

$$\text{Again, } \frac{x^2.(2-a)}{u^2} = (1+x^2)^2 = 1 + 2x^2 + x^4$$

$$= 1 + ax^2 + x^4 + (2-a)x^2$$

$$\therefore (1 + ax^2 + x^4)^{\frac{1}{2}} = \frac{x\sqrt{2-a}}{u} \times \sqrt{1-u^2}$$

Hence $\dfrac{dx.(1-x^2)}{(1+x^2)\sqrt{1+ax^2+x^4}} = \dfrac{du}{u} \times x \times \dfrac{u}{x\sqrt{2-a}.\sqrt{1-u^2}}$

$$= \frac{du}{\sqrt{2-a}.\sqrt{1-u^2}}$$

$$\therefore F = \frac{1}{\sqrt{2-a}}.\sin.^{-1} u = \frac{1}{\sqrt{2-a}}.\sin.^{-1}\frac{x\sqrt{2-a}}{1+x^2}$$

This method will serve to integrate the following general forms.

$$\frac{dx.(a^2 \mp x^2)}{(a^2 \pm x^2).\sqrt{a^4 \pm bx^2 + x^4}} \left(u = \frac{\sqrt{\pm 2a^2 \mp b}}{a^2 \pm x^2}\right)$$

$$\frac{x^{\frac{n}{2}-1}dx\,(a^n \mp x^n)}{(a^n \pm x^n)\sqrt{a^{2n} \pm bx^n + x^{2n}}} \left(u = \frac{x^{\frac{n}{2}}\sqrt{\pm 2a^n \mp b}}{a^n \pm x^n}\right)$$

$$\frac{x^{\frac{n}{m}-1}\,dx\,(a^n \mp x^n)}{(a^n \pm x^n)\left\{(a^n \pm x^n)^m - ax^n\right\}^{\frac{1}{m}}} \left(u = \frac{a^{\frac{1}{m}}\,x^{\frac{n}{m}}}{a^n \pm x^n}\right)$$

and many others, which will suggest themselves to the Reader.

574. $$\int \frac{mbx^{-n-1}\,dx}{\sqrt{e+fx^{-n}}} = \int \frac{-2mb}{nf} \times \frac{\frac{1}{2}(-nf)dx}{\sqrt{e+fx^{-n}}}$$

$$= -\frac{2mb}{nf}\int d.\sqrt{e+fx^{-n}}$$

$$= -\frac{2mb}{nf}.\sqrt{e+fx^{-n}}$$

To integrate $\dfrac{\sqrt{y}dy}{1+y^{\frac{3}{2}}}$, we have $\sqrt{y}dy = \dfrac{2}{3} \times \dfrac{3}{2}y^{\frac{3}{2}-1}dy$

$$= \frac{2}{3}.d.(1 + y^{\frac{3}{2}})$$

$$\therefore \int \frac{\sqrt{y}dy}{1+y^{\frac{3}{2}}} = \frac{2}{3}\int \frac{d.(1+y^{\frac{3}{2}})}{(1+y^{\frac{3}{2}})} = \frac{2}{3}.l.(1+y^{\frac{3}{2}})$$

The integral of $\dfrac{dx}{\sqrt{x^2-a^2}} = l.\,(x+\sqrt{x^2-a^2})$ (*Vince, Simpson, Lacroix.*)

To integrate $\dfrac{x^{\frac{1}{2}}dx}{\sqrt{2a-x}}$ we have

$$\int\frac{x^{\frac{1}{2}}dx}{\sqrt{2a-x}} = \int\frac{xdx}{\sqrt{2ax-x^2}} = \int\frac{adx-(adx-xdx)}{\sqrt{2ax-x^2}}$$

$$= \int\frac{adx}{\sqrt{2ax-x^2}} - \int\frac{adx-xdx}{\sqrt{2ax-x^2}}$$

$$= \text{vers.}^{-1}x - \sqrt{2ax-x^2}$$

To integrate $\dfrac{zdz}{\sqrt{a^4-z^4}}$, put $z^2 = u$, and $a^2 = b$

Then $$\int\frac{zdz}{\sqrt{a^4-z^4}} = \int\frac{\frac{du}{2}}{\sqrt{b^2-u^2}} = \frac{1}{2b}\int\frac{bdu}{\sqrt{b^2-u^2}}$$

$$= \frac{1}{2b}.\ \sin.^{-1}.\ z^2$$

575. $$\int\frac{d\times dx}{a^3-mx^2} = \frac{d}{a^3}\times\int\frac{dx}{1-\frac{mx^2}{a^3}}$$

Put $\dfrac{mx^2}{a^3} = u^2$; then $dx = \dfrac{a^{\frac{3}{2}}}{m^{\frac{1}{2}}}du$

Hence $$\int\frac{d\times dx}{a^3-mx^2} = \frac{d}{2\sqrt{a^3m}}\times\int\frac{2du}{1-u^2}$$

$$= \frac{d}{2\sqrt{a^3m}}\times l.\,\frac{1+u}{1-u}$$ (*Vince, Simpson, Lacroix.*)

$$\int \frac{d\times\sqrt{xdx}}{\sqrt{1-x}} = d \times \int \frac{xdx}{\sqrt{x-x^2}} = d\times \int \frac{\frac{dx}{2} - \left(\frac{dx}{2} - xdx\right)}{\sqrt{x-x^2}}$$

$$= d \times \int \frac{\frac{dx}{2}}{\sqrt{2.\frac{x}{2} - x^2}} - d\times \int \frac{\frac{dx}{2} - xdx}{\sqrt{x-x^2}}$$

$$= d \times \text{vers.}^{-1}\tfrac{1}{2}x - d \times \sqrt{x-x^2}$$

To integrate $\dfrac{x^2dx}{(x-a)^2 \times (x+a)} = d\text{F}$, since two of the factors of the denominator are equal, assume

$$\frac{1}{(x-a)^2 \times (x+a)} = \frac{\text{A}}{(x-a)^2} + \frac{\text{B}}{x-a} + \frac{\text{C}}{x+a}$$

Then $\left\{\text{A} + \text{B}.(x-a)\right\}(x+a) + \text{C} \times (x-a)^2 = 1 \ldots (m)$

Let $x = a$

Then $\text{A} \times 2a = 1, \therefore \text{A} = \dfrac{1}{2a}$

Hence $\dfrac{1 - \frac{1}{2a}\times(x+a)}{x-a} = \text{B}.(x+a) + \text{C} \times (x-a) = \dfrac{-1}{2a}$

Let $x = a$

Then $\text{B} \times 2a = -\dfrac{1}{2a}, \ \therefore \ \text{B} = -\dfrac{1}{4a^2}$

Again, in equation (m) substitute $(-a)$ for (x) and there results.

$\text{C} \times 4a^2 = 1, \therefore \text{C} = \dfrac{1}{4a^2}$

But $\displaystyle\int \frac{x^2dx}{(x-a)^2.(x+a)} = \int \frac{(x^2 - a^2 + a^2)dx}{(x-a)\times(x^2-a^2)} = \int \frac{dx}{x-a} +$
$\displaystyle\int \frac{a^2dx}{(x-a)^2.(x^2+a^2)}$

$$= l.(x-a) + \int \frac{a^2dx}{(x-a)^2 \times (x+a)}$$

And, by the preceding process, we have

$$\int \frac{a^2 dx}{(x-a)^2 \times (x+a)} = \int \frac{a^2}{2a} \times \frac{dx}{(x-a)^2} - \int \frac{a^2}{4a^2} \times \frac{dx}{x-a}$$

$$+ \int \frac{a^2}{4a^2} \times \frac{dx}{x+a}$$

$$= -\frac{a}{2} \times \frac{1}{x-a} - \frac{1}{4}.\ l.\ (x-a) + \frac{1}{4} \times$$

$l.\ (x+a)$.

Hence, then we finally get

$$F = \frac{3}{4}.\ l.\ (x-a) + \frac{1}{4}\ l.\ (x+a) - \frac{a}{2} \times \frac{1}{x-a}$$

576. To integrate $\frac{(a+bx)d \times dx}{x^3-1} \left(= d \times \frac{a+b+b.(x-1)}{x^3-1}dx\right.$

$$\left. = \frac{d \times (a+b)dx}{x^3-1} + \frac{bd \times dx}{x^2+x+1}\right) = d\,F$$

Let us assume $\frac{1}{x^3-1} = \frac{Ax+B}{x^2+x+1} + \frac{C}{x-1}$

Then $(Ax+B) \times (x-1) + C \times (x^2+x+1) = 1$

Let $x = 1$

Then $C = \frac{1}{3}$

Again, let $x = \frac{-1+\sqrt{-3}}{2}$ (one of the values of x in $x^2 + x + 1 = 0$)

Then $\left(A \times \frac{-1+\sqrt{-3}}{2} + B\right) \times \left(-\frac{3}{2} + \frac{\sqrt{-3}}{2}\right) = 1$

Or $-\frac{3}{2}B + (B \times 2A) \times \frac{\sqrt{-3}}{2} = 1$ $\therefore$ equating real imaginary quantities we get

$$\left.\begin{aligned} -\frac{3}{2}B &= 1 \\ \text{And } B - 2A &= 0 \end{aligned}\right\} \therefore \begin{aligned} B &= -\frac{2}{3} \\ A &= \frac{B}{2} = -\frac{1}{3} \end{aligned}$$

Hence $\frac{d\times(a+b)dx}{x^3-1} = -\frac{d\times(a+b)}{3}\left(\frac{xdx+2dx}{x^2+x+1} - \frac{dx}{x-1}\right)$

$$= -\frac{d\times(a+b)}{6}\times\frac{2xdx+dx}{x^2+x+1} - \frac{d\times(a+b)}{2}\times\frac{dx}{x^2+x+1}$$
$$+\frac{d\times(a+b)}{3}\,\frac{dx}{x-1}$$

Hence $F = -\frac{d\times(a+b)}{6}\,l.\,(x^2+x+1)+\frac{d\times(a+b)}{3}\times$
$$l.\,(x-1) + \int\left(bd-\frac{ad}{2}-\frac{bd}{2}\right)\frac{dx}{x^2+x+1}$$

Again, let $x+\frac{1}{2}=u$

Then $x^2+x+1 = u^2+1-\frac{1}{4} = u^2+\frac{3}{4}$

And $dx = du$

$$\therefore \frac{dx}{x^2+x+1} = \frac{du}{u^2+\frac{3}{4}}$$

$$\therefore F = \frac{d\times(a+b)}{6}\times l.\,\frac{(x-1)^2}{x^2+x+1} + \int\frac{d}{2}.(b-a)\,\frac{du}{u^2+\frac{3}{4}}$$

$$= \frac{d\times(a+b)}{6}\times l.\,\frac{(x-1)^2}{x^2+x+1} + \frac{2}{3}.(b-a).\tan.^{-1}\sqrt{\tfrac{4}{3}}\,u$$

To integrate $\frac{dx}{x^n.\sqrt{a^2-x^2}} = dF_0$ assume

$$\left.\begin{array}{rcl} \frac{\sqrt{a^2-x^2}}{x^{n-1}} & = & P_1 \\ \frac{\sqrt{a^2-x^2}}{x^{n-3}} & = & P_2 \\ \&c. & = & \&c. \\ \frac{\sqrt{a^2-x^2}}{x} & = & P_{\frac{n}{2}} \end{array}\right\} \left.\begin{array}{rcl} \int\frac{dx}{x^{n-2}\sqrt{a^2-x^2}} & = & F_1 \\ \int\frac{dx}{x^{n-4}\sqrt{a^2-x^2}} & = & F_2 \\ \&c. & = & \&c. \\ \int\frac{dx}{x^{n-n}\sqrt{a^2-x^2}} & = & F_{\frac{n}{2}} \end{array}\right\}$$

Then $dP_1 = -\frac{(n-1)dx}{x^n}.\sqrt{a^2-x^2} - \frac{xdx}{x^{n-1}\sqrt{a^2-x^2}}$

$$= -\frac{a^2.(n-1)dx}{x^n\sqrt{a^2-x^2}} + \frac{(n-2)dx}{x^{n-2}\sqrt{a^2-x^2}}$$

$$\text{Hence } F_0 = \frac{n-2}{a^2.(n-1)} \times F_1 - \frac{1}{a^2.(n-1)} \times P_1$$

$$\text{Similarly } F_1 = \frac{n-4}{a^2.(n-3)} \times F_2 - \frac{1}{a^2.(n-3)} \times P_2$$

$$\&c. = \&c.$$

$$F_{\frac{n}{2}-1} = \frac{n-n}{a^2} \times F_{\frac{n}{2}} - \frac{1}{a^2} \times P_{\frac{n}{2}}$$

Hence, by substitution, we may express F_0 in terms P_1, P_2, P_3, &c. (n being an even number); i. e., we can find the integral required.

To integrate $\dfrac{d \times dx}{x.(1+x)^2 \times (1+x+x^2)} = d.F$ assume

$$\frac{1}{x.(1+x)^2.(1+x+x^2)} = \frac{A}{x} + \frac{B}{(1+x)^2} + \frac{C}{(1+x)} + \frac{Px+Q}{1+x+x^2}$$

Then $A.(1+x)^2.(1+x+x^2) + B.x.(1+x+x^2) + C \times x.(1+x).(1+x+x^2) + (Px+Q) \times x.(1+x)^2 = 1$

Let $x = 0$

Then $A = 1$

Let $x = -1$

Then $B \times (-1) \times (+1) = 1$ or $B = -1$

Hence $\dfrac{1+x.(1+x+x^2)}{x+1} = C.x.(1+x+x^2) + \left\{A.(1+x+x^2) + \overline{Px+Q}\right\} \times (1+x).$

Let $x = -1$; then $\because$ the value of the Fraction $\dfrac{1+x.(1+x+x^2)}{1+x} = x^2 + 1 = 2$ on that supposition, we have,

$$2 = -1 \times C, \text{ or } C = -2$$

Again, since $x^2 + x + 1 = \left(x + \dfrac{1+\sqrt{-3}}{2}\right) \times \left(x + \dfrac{1-\sqrt{-3}}{2}\right)$

put $x = -\dfrac{1+\sqrt{-3}}{2}$

Then $\left\{P\times\left(-\frac{1+\sqrt{-3}}{2}\right)+Q\right\}\times\left(1-\frac{1+\sqrt{-3}}{2}\right)^2\times$ $\left(-\frac{1+\sqrt{-3}}{2}\right)=1$

Or $\left(Q-\frac{P}{2}-\frac{P}{2}\sqrt{-3}\right)\times(1-\sqrt{-3}).(1+\sqrt{-3})\times$ $(1-\sqrt{-3})=-8$

Or $(Q-\frac{P}{2}-\frac{P}{2}\sqrt{-3})\times(1-\sqrt{-3})=-2$

$\therefore Q-\frac{P}{2}-\frac{3}{2}\times P+\left(\frac{P}{2}-Q-\frac{P}{2}\right)\sqrt{-3}=-2$

$\therefore$ equating real and imaginary quantities, we have

$$\left.\begin{array}{r} Q-2P=-2 \\ \text{and } -Q=0 \end{array}\right\} \therefore P=1$$

Hence $\frac{d\times dx}{x.(1+x)^2\times(1+x+x^2)}=d\times\left\{\frac{dx}{x}-\frac{dx}{(1+x)^2}-\frac{2dx}{1+x}\right.$ $\left.+\frac{dx}{1+x+x^2}\right\}$

$\therefore F=d\times\left\{lx+\frac{1}{1+x}-2l.(1+x)+\int\frac{dx}{1+x+x^2}\right\}$

Again, let $x+\frac{1}{2}=u$

Then $dx=du$, and $x^2+x+1=u^2+\frac{3}{4}$

$\therefore \int\frac{dx}{1+x+x^2}=\int\frac{du}{u^2+\frac{3}{4}}=\frac{4}{3}\tan.^{-\sqrt{\frac{3}{4}}}u$

$\therefore F=d\times\left\{\frac{1}{1+x}+lx-2l.(1+x)+\frac{4}{3}\tan.^{-\sqrt{\frac{3}{4}}}\left(x+\frac{1}{2}\right)\right\}$

577. To integrate $\frac{dz}{z\sqrt{a+cz^n}}$, assume

$z^{\frac{n}{2}}=u$, $\therefore z^{\frac{n}{2}-1}dz=\frac{2}{n}du$

Or $\frac{dz}{z} = \frac{2}{n} du \times \frac{1}{z^{\frac{n}{2}}} = \frac{2}{n} \times \frac{du}{u}$

Also $\sqrt{a+cz^n} = \sqrt{a+cu^2} = \sqrt{1+\frac{cu^2}{a}} \times \sqrt{a}$,

$$\therefore \int \frac{dz}{z\sqrt{a+cz^n}} = \frac{2}{n\sqrt{a}} \times \int \frac{du}{u\,.\sqrt{1+\frac{cu^2}{a}}}$$

$$= \frac{1}{n\sqrt{a}} \int \frac{2d\sqrt{\frac{c}{a}}\,.\,u}{\sqrt{\frac{c}{a}}\,.\,u\sqrt{1+\frac{c}{a}u^2}}$$

$$= \frac{1}{n\sqrt{c}}\,.\,l\,.\,\frac{\sqrt{1+\frac{c}{a}u^2}-1}{\sqrt{1+\frac{c}{a}u^2}+1} = \frac{2}{n\sqrt{c}}\,.\,l\,.\,\frac{\sqrt{1+\frac{c}{a}u^2}-1}{\sqrt{\frac{c}{a}}\,.\,u}$$

$$= \frac{2\sqrt{a}}{nc}\,.\,l\,.\,\frac{\sqrt{1+\frac{c}{a}z^n}-1}{z^n}$$

To integrate $\frac{x^p dx}{(1+x^n)^2}$, assume

$$\frac{x^{p+1}}{1+x^n} = \text{P}$$

Then $d\text{P} = (p+1)\,.\,\frac{x^p dx}{1+x^n} - \frac{nx^{n+p}dx}{(1+x^n)^2}$

$$= (p+1)\frac{x^p dx}{1+x^n} - \frac{nx^p dx}{(1+x^n)^2} \times (x^n+1-1)$$

$$= (p-n+1)\frac{x^p dx}{1+x^n} + \frac{nx^p dx}{(1+x^n)^2}$$

$$\therefore \int \frac{x^p dx}{(1+x^n)^2} = \frac{P}{n} - \frac{p-n+1}{n} \times A$$
$$= \frac{x^{p+1}}{n \cdot (1+x^n)} - \frac{p-n+1}{n} \times A$$

578. $$\int \frac{(a+bx)\,dx}{a^2+x^2} = \frac{1}{a}\int \frac{a^2 dx}{a^2+x^2} + \frac{b}{2}\int \frac{2x dx}{a^2+x^2}$$
$$= \frac{1}{a} \cdot \tan.^{-1} x + \frac{b}{2}\, l.(a^2+x^2)$$

To integrate $\dfrac{x^4 dx}{\sqrt{a^2+x^2}} = dF_0$ assume

$$\left.\begin{array}{l} x^3\sqrt{a^2+x^2} = P_1 \\ x\sqrt{a^2+x^2} = P_2 \end{array}\right\} \quad \left.\begin{array}{l} \displaystyle\int \frac{x^2 dx}{\sqrt{a^2+x^2}} = F_1 \\ \displaystyle\int \frac{dx}{\sqrt{a^2+x^2}} = F_2 \end{array}\right\}$$

Then $dP_1 = 3x^2 dx\,\sqrt{a^2+x^2} + \dfrac{x^4 dx}{\sqrt{a^2+x^2}} = \dfrac{3a^2x^2dx}{\sqrt{a^2+x^2}}$
$+ \dfrac{4x^4dx}{\sqrt{a^2+x^2}}$

$$\therefore P_1 = 3a^2 \times F_1 + 4 \times F_0$$
Similarly $P_2 = a^2 \times F_2 + 2 \times F_1$
$$\therefore F_0 = \frac{P_1}{4} - \frac{3a^2}{8} \times P_2 + \frac{3a^4}{8} \cdot F_2$$
$$= \frac{x^3\sqrt{a^2+x^2}}{4} - \frac{3a^2}{8}\, x\sqrt{a^2+x^2} +$$
$\dfrac{3a^4}{8}\, l.(x + \sqrt{a^2+x^2})$

579. To integrate $\dfrac{x^4 dx}{\sqrt{a^2-x^2}} = dF_0$, assume

$$\left.\begin{array}{l} x^3\sqrt{a^2-x^2} = P_1 \\ x\sqrt{a^2-x^2} = P_2 \end{array}\right\} \quad \left.\begin{array}{l} \displaystyle\int \frac{x^2 dx}{\sqrt{a^2-x^2}} = F_1 \\ \displaystyle\int \frac{dx}{\sqrt{a^2-x^2}} = F_2 \end{array}\right\}$$

Then $dP_1 = 3x^2dx\sqrt{a^2-x^2} - \frac{x^4dx}{\sqrt{a^2-x^2}} = \frac{3a^2x^2dx}{\sqrt{a^2-x^2}}$
$- \frac{4x^4dx}{\sqrt{a^2-x^2}}$

$$\therefore P_1 = 3a^2 \times F_1 - 4 \times F_0$$

Similarly $P_2 = a^2 \times F_2 - 2 \times F_1$

Hence $F_0 = -\frac{1}{4}P_1 + \frac{3a^2}{4} \times F_1 = -\frac{1}{4}P_1 -$
$\frac{3a^2}{8}P_2 + \frac{3a^4}{8} \times F_2$

$$= -\frac{1}{4}.x^3\sqrt{a^2-x^2} - \frac{3a^2}{8} \times x.\sqrt{a^2-x^2}$$
$$+ \frac{3a^3}{8} \times \sin.^{-1}x.$$

To integrate $\frac{2adx}{x\sqrt{a^3+x^3}}$, assume $x^{\frac{3}{2}} = u$

Then $\frac{dx}{x} = \frac{2}{3}\frac{u^{-\frac{1}{3}}du}{u^{\frac{2}{3}}} = \frac{2}{3} \times \frac{du}{u}$

Hence $\int\frac{2adx}{x\sqrt{a^3+x^3}} = \frac{2a}{3}\int\frac{2du}{u \times \sqrt{a^3+u^2}}$

$$= \frac{2a}{3a^{\frac{3}{2}}}.l.\frac{\sqrt{a^3+u^2}-a^{\frac{3}{2}}}{\sqrt{a^3+u^2}+a^{\frac{3}{2}}}$$

$$= \frac{4a^2}{3\,a^{\frac{1}{2}}}.l.\frac{\sqrt{a^3+x^3}-a^{\frac{3}{2}}}{x^{\frac{3}{2}}}.$$

To integrate $d\theta . \cos.^3\theta$, we have

$d\theta . \cos.^3\theta = d\theta . \cos.\theta \times (1 - \sin.^2\theta) =$
$d.\sin.\theta - d.\sin.\theta \times \sin.^2\theta$

$$\therefore \int d\theta.\cos.^3\theta = \sin.\theta - \frac{\sin.^3\theta}{3}.$$

580. To integrate $\frac{xdx}{(1+x)^3\sqrt{1+x+x^2}} = dF$, we have

$$dF = \frac{dx}{(1+x)^3\sqrt{1+x+x^2}} \times (x+1-1) = \frac{dx}{(1+x)^2.\sqrt{1+x+x^2}}$$
$$- \frac{dx}{(1+x)^3.\sqrt{1+x+x^2}}$$

Assume $\frac{1}{1+x} = u$

Then $\frac{dx}{(1+x)^2} = -du, \frac{-dx}{(1+x)^3} = udu$

And $\sqrt{1+x+x^2} = \sqrt{\frac{1}{u^2}-\frac{1}{u}+1} = \frac{\sqrt{u^2-u+1}}{u}$

$$\therefore dF = \frac{-udu}{\sqrt{u^2-u+1}} + \frac{u^2du}{\sqrt{u^2-u+1}}.$$

Again, assume $u - \frac{1}{2} = v$

Then $u^2 - u + 1 = v^2 + 1 - \frac{1}{4} = v^2 + \frac{3}{4}$

$$\therefore dF = \frac{-vdv+\frac{1}{2}dv}{\sqrt{v^2+\frac{3}{4}}} + \frac{v^2dv+vdv+\frac{1}{4}dv}{\sqrt{v^2+\frac{3}{4}}} = \frac{(v^2+\frac{3}{4})\,dv}{\sqrt{v^2+\frac{3}{4}}}$$
$$= dv \times \sqrt{v^2+\tfrac{3}{4}}$$

Again, let $v\sqrt{v^2+\frac{3}{4}} = P$

Then $dP = \frac{v^2dv}{\sqrt{v^2+\frac{3}{4}}} + dv.\sqrt{v^2+\frac{3}{4}}$

$$= 2\,dv\sqrt{v^2+\frac{3}{4}} - \frac{\frac{3}{4}dv}{\sqrt{v^2+\frac{3}{4}}}$$

$$\therefore dv \times \sqrt{v^2+\frac{3}{4}} = \frac{dP}{2} + \frac{3}{8}\frac{dv}{\sqrt{v^2+\frac{3}{4}}}$$

And $\therefore F = \frac{P}{2} + \frac{3}{8}\,l.\left(v + \sqrt{^2+\frac{3}{4}}\right)$

$$\therefore F = \frac{v}{8}\sqrt{v^2+\frac{3}{4}} + \frac{3}{8}.\ l.\ \left(v + \sqrt{v^2+\frac{3}{4}}\right)$$

$$= \frac{1-x}{2.(1+x)^2} \times \sqrt{1+x+x^2} + \frac{3}{8}.\ l.\frac{1-x+2\sqrt{1+x+x^2}}{2.(1+x)}$$

which expresses the integral in terms of x.

To integrate $b \times \dfrac{x^{\frac{3}{2}}dx\sqrt{2a-x}}{(a-x)^2} = dF$, we have

$$dF = \frac{xdx\sqrt{2ax-x^2}}{(a-x)^2}$$

Let $x - a = u$

Then $\sqrt{2ax-x^2} = \sqrt{a^2-u^2}$, $dx = du$, &c.

Hence $dF = \dfrac{(u+a)\,du.\sqrt{a^2-u^2}}{u^2} = \dfrac{du}{u}.\sqrt{a^2-u^2}$

$+ \dfrac{adu}{u^2}\sqrt{a^2-u^2}$

$$= \frac{a^2du}{u\sqrt{a^2-u^2}} - \frac{udu}{\sqrt{a^2-u^2}} + \frac{a^3du}{u^2.\sqrt{a^2-u^2}}$$

$- \dfrac{adu}{\sqrt{a^2-u^2}}$

Again, put $\dfrac{\sqrt{a^2-u^2}}{u} = P$; then $dP = \dfrac{-du}{u^2}.\sqrt{a^2-u^2} -$

$\dfrac{du}{\sqrt{a^2-u^2}} = \dfrac{-a^2du}{u^2.\sqrt{a^2-u^2}}$. Hence $F = \sec.^{-1}u + \sqrt{a^2-u^2}$

$- \sin.^{-1}u - \dfrac{\sqrt{a^2-u^2}}{u}$

581. To integrate $\dfrac{dx}{1+x^3}$, assume

$$\frac{1}{1+x^3} = \frac{A}{1+x} + \frac{Px+Q}{x^2-x+1}$$

Then $A \times (x^2 - x + 1) + (Px + Q)(1 + x) = 1$

Let $x = -1$

Then $A = \frac{1}{3}$

Again, let $x = \frac{1 + \sqrt{-3}}{2}$

Then $\left(\frac{P}{2} + \frac{\sqrt{-3}}{2}P + Q\right) \times \left(\frac{3}{2} + \frac{\sqrt{-3}}{2}\right) = 1$

Hence $\frac{3}{2} \times Q + \left(P + \frac{Q}{2}\right) \times \sqrt{-3} = 1$

$\therefore\ Q = \frac{2}{3}$ and $P + \frac{Q}{2} = 0$

$\therefore\ P = \frac{-Q}{2} = -\frac{1}{3}$

Hence $\frac{dx}{1 + x^3} = \frac{1}{3} \cdot \frac{dx}{1+x} - \frac{1}{3} \times \frac{xdx - 2dx}{x^2 - x + 1}$

$\therefore\ \int \frac{dx}{1 + x^3} = \frac{1}{3} \cdot l \cdot (1 + x) - \frac{1}{6} \cdot l \cdot (x^2 - x + 1) + \frac{1}{2} \times \int \frac{dx}{x^2 - x + 1}.$

Let now $x - \frac{1}{2} = u$

Then $x^2 - x + 1 = u^2 + 1 - \frac{1}{4} = u^2 + \frac{3}{4}$

$\therefore\ \int \frac{dx}{x^2 - x + 1} = \int \frac{du}{u^2 + \frac{3}{4}} = \frac{4}{3} \tan.^{\frac{-\sqrt{3}}{2}} u$

$\therefore\ \int \frac{dx}{1 + x^3} = \frac{1}{3} l \cdot (1 + x) - \frac{1}{6} \cdot l\,(x^2 - x + 1) + \frac{2}{3} \cdot \tan.^{\frac{-\sqrt{3}}{2}} (x - \frac{1}{2})$

582. To integrate $\frac{p \times dx}{\sqrt{x} \cdot \sqrt{a - bx}} = dF$,

we have $dF = \frac{p}{\sqrt{b}} \times \frac{dx}{\sqrt{\frac{a}{b}x - x^2}}$

Put $\frac{a}{b} = 2c$

Then $dF = \frac{p}{c\sqrt{b}} \cdot \frac{c\,dx}{\sqrt{2cx - x^2}}$

$\therefore\ F = \frac{p}{c\sqrt{b}} \times \text{ver. sin.}^{-c}x.$

To integrate $\frac{dx}{x\sqrt{1+\sqrt{x}}} = dF$, assume

$\sqrt{x} = u^2$

Then $dF = \frac{4u^3\,du}{u^4 . \sqrt{1+u^2}} = \frac{4du}{u . \sqrt{1+u^2}}$

$\therefore\ F = 2 . l. \frac{\sqrt{1+u^2}-1}{\sqrt{1+u^2}+1} = 4 . l. \frac{\sqrt{1+u^2}-1}{u}$

$= 4 . l\ \frac{\sqrt{1+\sqrt{x}}-1}{x^{\frac{1}{4}}}$

$= 4 . l. (\sqrt{1+\sqrt{x}}-1) - l. x.$

583. To integrate $\frac{dx}{(a^2+x^2)^{\frac{2r+1}{2}}} = F_0$, assume

$$\left.\begin{array}{l} P_1 = \frac{x}{(a^2+x^2)^{\frac{2r-1}{2}}} \\ P_2 = \frac{x}{(a^2+x^2)^{\frac{2r-3}{2}}} \\ \&c. = \ \&c. \\ P_r = \frac{x}{(a^2+x^2)^{\frac{1}{2}}} \end{array}\right\} \quad \left.\begin{array}{l} F_1 = \int \frac{dx}{(a^2+x^2)^{\frac{2r-1}{2}}} \\ F_2 = \int \frac{dx}{(a^2+x^2)^{\frac{2r-3}{2}}} \\ \&c. = \ \&c. \\ F_r = \int \frac{dx}{(a^2+x^2)^{\frac{1}{2}}} \end{array}\right\}$$

$$\text{Then } dP_1 = \frac{dx}{(a^2+x^2)^{\frac{2r-1}{2}}} - \frac{2r-1}{2} \times \frac{2x^2dx}{(a^2+x^2)^{\frac{2r+1}{2}}}$$

$$= \frac{(2r-1).a^2.dx}{(a^2+x^2)^{\frac{2r+1}{2}}} - (2r-2)\frac{dx}{(a^2+x^2)^{\frac{2r-1}{2}}}$$

$$\text{Or } P_1 = (2r-1).a^2 \times F_0 - (2r-2) \times F_1$$

$$\text{Similarly } P_2 = (2r-3).a^2 \times F_1 - (2r-4) \times F_2$$

&c. = &c.

$$P_{r-1} = 3a^2 \times F_{r-2} - 2 \times F_{r-1}$$

$$P_r = a^2 \times F_{r-1} - 0$$

$$\text{Hence } F_0 = \frac{1}{(2r-1)a^2} \times P_1 + \frac{(2r-2)}{(2r-1)(2r-3) \times a^4} \times P_2 + \&c.$$

$$+ \frac{(2r-2).(2r-4) \ldots\ldots \times 2}{(2r-1).(2r-3) \ldots\ldots \times 3 \times 1} \times \frac{P_r}{a^{2r}}$$

584. To integrate $\frac{pdx}{a - bx^2} = dF$

put $\frac{b}{a} x^2 = u^2$

Then $dx = \sqrt{\frac{a}{b}}\, du$, &c.

$$\therefore\ F = \int \frac{pa^{\frac{1}{2}}}{2b^{\frac{3}{2}}} \times \frac{2du}{1-u^2}$$

$$= \frac{pa^{\frac{1}{2}}}{2b^{\frac{3}{2}}} . l. \frac{1+u}{1-u}$$

To integrate $\frac{x^3dx}{\sqrt{a-x}} = F_0$, assume

$$\sqrt{a-x} = u$$

Then $\frac{dx}{\sqrt{a-x}} = -2du$, and $x^3 = (a-u^2)^3$

$$\text{Hence } dF = -2du \times (a-u^2)^3$$

$$= -2du \times (a^3 - 3a^2u^2 + 3au^4 - u^6)$$

$\therefore F = -2a^3u + 2a^2u^3 - \frac{6a}{5}u^5 + \frac{2}{7}u^7$ which is $\therefore$ known.

585. To integrate $\frac{dx}{x^3 - 7x^2 + 12x} = dF$, assume $\frac{1}{x^3 - 7x^2 + 12x} = \frac{A}{x} + \frac{B}{x-3} + \frac{C}{x-4}$ ($x \times \overline{x-3} \times \overline{x-4}$, being $= x^3 - 7x^2 + 12x$)

Then $A.(x-3).(x-4) + B.x.(x-4) + C.x.(x-3) = 1$

Let $x = 0, 3, 4$, successively, and we get

$$\left.\begin{array}{l} A \times 12 = 1 \\ B \times (-3) = 1 \\ C \times 4 = 1 \end{array}\right\} \therefore \left\{\begin{array}{l} A = \frac{1}{12} \\ B = -\frac{1}{3} \\ C = \frac{1}{4} \end{array}\right.$$

$$\text{Hence } F = \frac{1}{12} \times \int \frac{dx}{x} - \frac{1}{3} \times \int \frac{dx}{x-3} + \frac{1}{4} \times \int \frac{dx}{x-4}$$

$$= \frac{1}{12} \times l.x - \frac{1}{3}.l.(x-3) + \frac{1}{4}.l.(x-4)$$

586. To integrate $\frac{dx}{x^5.\sqrt{1+x^2}} = dF_0$, assume

$$\left.\begin{array}{l} P_1 = \frac{\sqrt{1+x^2}}{x^4} \\ P_2 = \frac{\sqrt{1+x^2}}{x^2} \end{array}\right\} \left.\begin{array}{l} F_1 = \int \frac{dx}{x^3.\sqrt{1+x^2}} \\ F_2 = \int \frac{dx}{x\sqrt{a+x^2}} \end{array}\right\}$$

$$\text{Then } dP_1 = \frac{-4dx.\sqrt{1+x^2}}{x^5} + \frac{dx}{x^3.\sqrt{1+x^2}}$$

$$= \frac{-4dx}{x^5.\sqrt{1+x^2}} - \frac{3.dx}{x^3.\sqrt{1+x^2}}$$

$$\therefore\ P_1 = -4 \times F_0 - 3 \times F_1$$

Similarly $P_2 = -2 \times F_1 - F_2$

Hence $F_0 = -\dfrac{P_1}{4} + \dfrac{3}{2} \times P_2 + \dfrac{3}{2} \times F_2$

$$= -\frac{P_1}{4} + \frac{3}{2} \times P_2 + \frac{3}{4} \times l.\ \frac{\sqrt{1+x^2}-1}{\sqrt{1+x^2}+1}$$

$$= \frac{\sqrt{1+x^2}}{4x^4} \times (6x^2-1) + \frac{3}{2} \cdot l.\ \frac{\sqrt{1+x^2}-1}{x}$$

To integrate $\dfrac{a^x dx}{x^2}$ we have

$$\int \frac{a^x dx}{x^2} = \frac{-a^x}{x} + \int \frac{1}{x} \times d \cdot a^x$$

$$= \frac{-a^x}{x} + \int \frac{l.\ a \times a^x dx}{x}$$

The integral of $\dfrac{a^x dx}{x}$ can only be assigned in the form of a series; thus,

$$\int \frac{a^x dx}{x} = \int \frac{dx}{x} \times \left(1 + x\, l.\ a + \frac{x^2}{1 \cdot 2} \times (l.\ a)^2 + \&c.\right)$$

$$= l.\ x + \frac{x\, l.\ a}{1 \cdot 1} + \frac{x^2 \cdot (l.\ a)^2}{1 \cdot 2 \cdot 2} + \&c.$$

$$\therefore\ \int \frac{a^x dx}{x^2} = \frac{-a^x}{x} + l.\ a\left\{l.\ x + \frac{xla}{1 \cdot 1} + \frac{x^2 \cdot (l.\ a)^2}{1 \cdot 2 \cdot 2} + \&c.\right\}$$

To integrate $dx \times \cos.^2 x \times \sin.^3 x = dF$

we have $dF = dx \times \sin. x \times \cos.^2 x \cdot \sin.^2 x$

$$= -d \cdot \cos. x \times \cos.^2 x \times (1 - \cos.^2 x)$$

$$= -d.\cos. x \times \cos.^2 x + d.\cos. x \times \cos.^4 x)$$

$$\therefore\ F = \frac{-\cos.^3 x}{3} + \frac{\cos.^5 x}{5}$$

587. To integrate $\dfrac{dx \cdot \sqrt{1-x^2}}{(1+x)^2} = dF$,

assume $\dfrac{1}{1+x} = \dfrac{u^2}{2}$

Then $\dfrac{dx}{(1+x)^2} = -udu$

Also $1 + x = \dfrac{2}{u^2}$

And $1 - x = 1 - \dfrac{2}{u^2} + 1 = 2 \times \dfrac{u^2-1}{u^2}$

$\therefore \sqrt{1-x^2} = 2.\dfrac{\sqrt{u^2-1}}{u^2}$

$\therefore F = \int -2u\dfrac{du}{u^2}\sqrt{u^2-1}$

$= -2\int\dfrac{udu}{\sqrt{u^2-1}} + \int\dfrac{2du}{u\sqrt{u^2-1}}$

$= -2\sqrt{u^2-1} + \sec.^{-1}u$

588. $\displaystyle\int\frac{az^2dz}{e-fz} = -\frac{a}{f} \times \int\frac{z^2dz}{z-\frac{e}{f}} = -\frac{a}{f} \times$

$\displaystyle\int\left(zdz + \frac{e}{f}.dz + \frac{e^2}{f^2} \times \frac{dz}{z-\frac{e}{f}}\right)$ by division.

$\displaystyle\therefore \int\frac{az^2dz}{e-fz} = \frac{-a}{2f}z^2 - \frac{ae}{f2}z - \frac{ae^2}{f^3}l.\left(z-\frac{e}{f}\right)$

589. To integrate $\dfrac{x^2dx}{(a^2+x^2)^2} = dF$,

Assume $\dfrac{x}{a^2+x^2} = P$.

Then $dP = \dfrac{dx}{a^2+x^2} - \dfrac{2x^2dx}{(a^2+x^2)^2}$

$\displaystyle\therefore \int\frac{x^2dx}{(a^2+x^2)^2} = \frac{1}{2a^2}\int\frac{a^2dx}{a^2+x^2} - \frac{P}{2}$

$= \dfrac{1}{2a^2} \times \tan.^{-1}x - \dfrac{x}{2(a^2+x^2)}$

To integrate $\dfrac{a^xdx}{\sqrt{1-a^{2x}}} = dF$, we have

$a^xdx = \dfrac{d.(a^x)}{l.a}$

$$\therefore\ dF = \frac{1}{l.a} \times \frac{d.(a^x)}{\sqrt{1-(a^x)^2}}$$

$$\therefore\ F = \frac{1}{l.a} \times \sin.^{-1} a^x$$

To integrate zdx (tan. $z = x$), we have

$$\int zdx = zx - \int xdz$$

$$= zx - \int x \times \frac{dx}{1+x^2}$$

$$= zx - \frac{1}{2}.\ l.\ (1+x^2)$$

590. To integrate $x^2dx \times \sin.^{-1} x$ (rad. $= 1$) $= dF$, we have $\int x^2dx \times \sin.^{-1}x = \frac{x^3}{3} \sin.^{-1} x - \int \frac{x^3}{3} \times d.\sin.^{-1}x = \frac{x^3}{3}. \times \sin.^{-1} x - \int \frac{x^3dx}{3\sqrt{1-x^2}}$

$$\text{Now } -\frac{x^3dx}{\sqrt{1-x^2}} = \frac{xdx}{\sqrt{1-x^2}} \times (1-x^2-1)$$

$$= xdx.\sqrt{1-x^2} - \frac{xdx}{\sqrt{1-x^2}}$$

$$\therefore\ F = \frac{x^3}{3}.\sin.^{-1} x + \frac{1}{3} \int xdx\sqrt{1-x^2} - \frac{1}{3} \times \int \frac{xdx}{\sqrt{1-x^2}} = \frac{x^3}{3}.\sin.^{-1} x - \frac{1}{9}.(1-x^2)^{\frac{3}{2}} + \frac{1}{3}.\sqrt{1-x^2}$$

To integrate $\frac{dx.(1+x^2)}{(1-x^2)\sqrt{1+x^4}} = dF$

$$\text{Assume } \frac{\sqrt{2}.x}{1-x^2} = u$$

$$\text{Then } \frac{du}{\sqrt{2}} = \frac{dx}{1-x^2} + \frac{2x^2dx}{(1-x^2)^2} = \frac{dx.(1+x^2)}{(1-x^2)^2}$$

$$\therefore\ \frac{dx.(1+x^2)}{1-x^2} = du \times \frac{1-x^2}{\sqrt{2}} = x \times \frac{du}{u}$$

Also $\frac{2x^2}{u^2} = 1 - 2x^2 + x^4$

$\therefore \sqrt{1+x^4} = \frac{\sqrt{x^2}}{u} \times (1+u^2)^{\frac{1}{2}}$

$\therefore dF = \frac{1}{\sqrt{2}} \times \frac{du}{\sqrt{1+u^2}}$

$\therefore F = \frac{1}{\sqrt{2}} \times l.(u + \sqrt{1+u^2})$

591. $\int \frac{adx}{b + \frac{c}{x}} = \frac{a}{b} \int \frac{xdx}{x + \frac{c}{b}}$

$= \frac{a}{b} \int \left(dx - \frac{\frac{c}{b}\,dx}{x + \frac{c}{b}}\right)$ by division

$= \frac{a}{b} \times x - \frac{c}{b^2} \times l.\left(x + \frac{c}{b}\right)$

To integrate $\frac{xdx}{(1+x^4)^{\frac{3}{2}}} = dF$, put $\frac{x^2}{(1+x^4)^{\frac{1}{2}}} = P$

Then $dP = \frac{2xdx}{(1+x^4)^{\frac{1}{2}}} - \frac{2x^5dx}{(1+x^4)^{\frac{3}{2}}}$

$= \frac{2xdx}{(1+x^4)^{\frac{1}{2}}} - \frac{2xdx}{(1+x^4)^{\frac{1}{2}}} + \frac{2xdx}{(1+x^4)^{\frac{3}{2}}}$

$= \frac{2xdx}{(1+x^4)^{\frac{3}{2}}}$

Hence $F = \frac{P}{2} = \frac{x^2}{2.(1+x^4)^{\frac{1}{2}}}$

592. $\int \frac{adx}{\sqrt{1-x^4}} = a \times \int \frac{dx}{\sqrt{1-x^4}}$.

For the investigation of $\int \frac{dx}{\sqrt{1-x^4}}$, see the solution of problem 497, beginning at page 297.

To integrate $\frac{(a^2+x^2)^m dx}{(l.x)^n} = d.F_0$, we have

$$dF_0 = (a^2+x^2)^m x \times \frac{dx}{x} \times (l.x)^{-n}$$

Let $l.x = u$

Then $\frac{dx}{x} = du$

$$\therefore \int \frac{dx}{x} \times (l.x)^{-n} = \int u^{-n}\, du = \frac{u^{1-n}}{1-n}$$

Hence $F_0 = (a^2+x^2)^m x \times \frac{u^{1-n}}{1-n} - \int \frac{u^{1-n}}{1-n} \times d(a^2+x^2)^m x$

Suppose n a positive integer, and

Assume $\frac{d.(a^2+x^2)^m x}{dx} = P_1$

$$\frac{d.P_1 x}{dx} = P_2$$

&c. = &c.

$$\frac{d.P_{n-2}x}{dx} = P_{n-1}$$

Then, by continuing the above process, we shall arrive at

$$F_0 = -\frac{(a^2+x^2)^m x}{(n-1).u^{n-1}} - \frac{P_1 \times x}{(n-1).(n-2)\,u^{n-2}}$$

$$-\frac{P_2 \times x}{(n-1).(n-2).(n-3)u^{n-3}} - \&c. +$$

$$\frac{1}{(n-1).(n-2)\ldots\ldots 3.2.1} \times \int \frac{P_{n-1}\,dx}{u}$$

Now $P_{n-1}\frac{dx}{l.x}$ may be integrated approximately in a series, by expanding $(l.x)^{-1}$ according to the powers, multiplying P_{n-1} (which is an algebraic function of x) and integrating each term separately.

If n be not integral, let the process be continued as far as it tends to simplify the differential. Having thus indicated the steps of the process, we shall leave the actual investigation to the patience and assiduity of the Reader.

593. To integrate $dx.(1-x^2)^{\frac{2n-1}{2}} = d\,\mathrm{F}_0$, assume

$$\left.\begin{aligned} \mathrm{P}_1 &= x\,(1-x^2)^{\frac{2n-1}{2}} \\ \mathrm{P}_2 &= x.(1-x^2)^{\frac{2n-3}{2}} \\ \&\text{c.} &= \&\text{c.} \\ \mathrm{P}_n &= x.(1-x^2)^{\frac{1}{2}} \end{aligned}\right\} \qquad \left.\begin{aligned} \mathrm{F}_1 &= \int dx.(1-x^2)^{\frac{2n-3}{2}} \\ \mathrm{F}_2 &= \int dx.(1-x^2)^{\frac{2n-5}{2}} \\ \&\text{c.} &= \&\text{c.} \\ \mathrm{F}_n &= \int dx \times (1-x^2)^{-\frac{1}{2}} \end{aligned}\right\}$$

Then $d\,\mathrm{P}_1 = dx.(1-x^2)^{\frac{2n-1}{2}} - (2n-1)\,x^2dx.(1-x^2)^{\frac{2n-3}{2}}$

$$= dx\,(1-x^2)^{\frac{2n-1}{2}} + (2n-1).dx.(1-x^2)^{\frac{2n-3}{2}} \times$$

$(1-x^2-1)$

$$= 2ndx\,(1-x^2)^{\frac{2n-1}{2}} - (2n-1)\,dx\,(1-x^2)^{\frac{2n-3}{2}}$$

$$\therefore \mathrm{P}_1 = 2n.\,\mathrm{F}_0 - (2n-1)\,\mathrm{F}_1$$

$$\therefore \mathrm{F}_0 = \frac{\mathrm{P}_1}{2n} + \frac{2n-1}{2n} \times \mathrm{F}_1$$

Similarly $\mathrm{F}_1 = \dfrac{\mathrm{P}_2}{2(n-1)} + \dfrac{2n-3}{2(n-1)} \times \mathrm{F}_2$

$$\mathrm{F}_2 = \frac{\mathrm{P}_3}{2.(n-2)} + \frac{2n-5}{2\,(n-2)} \times \mathrm{F}_3$$

$$\&\text{c.} = \&\text{c.}$$

$$\mathrm{F}_{n-2} = \frac{\mathrm{P}_{n-1}}{2\times 2} + \frac{3}{2\times 2} \times \mathrm{F}_{n-1}$$

$$\mathrm{F}_{n-1} = \frac{\mathrm{P}_n}{2} + \frac{1}{2} \times \mathrm{F}_n$$

Hence, by successively substituting for F_1, F_2, &c., we get

$$\mathrm{F}_0 = \frac{1}{2n} \times \mathrm{P}_1 + \frac{2n-1}{2^2 n \times (n-1)} \times \mathrm{P}_2 + \frac{(2n-1).(2n-3)}{2^3 n.(n-1).(n-2)} \times \mathrm{P}_3$$

$$+ \frac{(2n-1).(2n-3).(2n-5)}{2^4.n.(n-1).(n-2).(n-3)} \times \mathrm{P}_4 + \&\text{c.}$$

$$+ \frac{(2n-1).(2n-3)\ \ldots.\ 3.1}{2^{n-1}.n.(n-2).(n-3)\ \ldots.\ 2.1} \times \mathrm{P}_{n-1}$$

$$+ \frac{(2n-1).(2n-3).(2n-5)\ \ldots.\ 5\times 3\times 1}{2^n.n.(n-1).(n-2)\ \ldots.\ 3.2.1} \times \sin.^{-1} x. \ (\mathrm{F}_n \text{ being}$$

$$= \int \frac{dx}{\sqrt{1-x^2}} = \sin.^{-1} x)$$

Let $x = 0$

Then P_1, P_2, P_3, &c. each $= 0$

And $\sin.^{-1} x = p \times \pi$ (p being any integer, and $\pi =$ a semicircle whose radius is unity.)

$$\therefore F_0 \text{ (when } x = 0) = \frac{1.3.5 \ldots. 2n-1}{1.2.3 \ldots. (n-1).n} \times 2^{-n} p \pi$$

Again, let $x = 1$

Then P_1, P_2, &c. each $= 0$, and $\sin.^{-1} x = 2p'\pi + \frac{\pi}{2}$ (p' being any integer)

$$\therefore F_0 \text{ (when } x = 1) = \frac{1.3 \ldots. (2n-1)}{1.2 \ldots. (n-1).n} \times 2^{-n} \times \left(2p'\pi + \frac{\pi}{2}\right)$$

$$\therefore F_0 \text{ (between } x = 0 \text{ and } x = 1) = \frac{1.3.5 \ldots. (2n-1)}{1.2 \ldots. n} \times \frac{\pi}{2^n} \times \left(2p'\pi - p + \frac{1}{2}\right)$$

of which the Problem is a particular case, viz., that when $p = 0$ and $p' = 0$.

594. $$\int \frac{dx}{(bx+cx^2)^2} = \int \frac{dx}{x^2 . (b+cx)^2} = F$$

Assume $\frac{1}{x.(b+cx)} = u$

Then $du = \frac{-dx}{x^2.(b+cx)} - \frac{cdx}{x.(b+cx)^2}$

$$= \frac{-dx}{x^2.(b+cx)} - \frac{dx}{x^2(b+cx)^2} \times (cx + b - b)$$

$$= \frac{-2dx}{x^2.(b+cx)} + \frac{bdx}{x^2.(b+cx)^2}$$

$$\therefore F = \frac{u}{b} + \frac{2}{b} \int \frac{dx}{x^2.(b+cx)}$$

Again, put $\frac{1}{x} = v$

Then $\frac{dx}{x^2} = - dv$ and we get

$$\int \frac{dx}{x^2.(b+cx)} = \int \frac{-vdv}{bv+c}$$

$$= -\frac{1}{b}\int \left(dv - \frac{c}{b} \frac{dv}{v+\frac{c}{b}}\right)$$

$$= -\frac{1}{b} v + \frac{c}{b}.l.\left(v+\frac{c}{b}\right)$$

$$\therefore F = \frac{1}{b.x.(b+cx)} - \frac{2}{b^2 x} + \frac{2c}{b^2}.l.\frac{b+cx}{bx}$$

Otherwise,

Assume $x + \dfrac{b}{2c} = u$

Then $F = \displaystyle\int \frac{du}{c^2 \times \left(u^2 - \frac{b^2}{4c^2}\right)^2}$

Assume $\dfrac{u}{u^2 - \frac{b^2}{4c^2}} = v$ differentiate, &c.

To integrate $e^x dx \times \cos.^2 x = d\,F$, we have $\cos.^2 x = \dfrac{1+\cos. 2x}{2}$

$$\therefore F = \int \frac{e^x dx}{2} + \int \frac{e^x dx \cos. 2x}{2}$$

$$= \frac{e^x}{2} + \frac{1}{2} \times \int e^x dx. \cos. 2x$$

But $\int e^x dx. \cos. 2x = e^x. \cos. 2x + 2\int e^x dx. \sin. 2x$

And $2\int e^x dx. \sin. 2x = 2e^x \sin. 2x - 4\int e^x dx. \cos. 2x$

Hence $5\int e^x dx. \cos. 2x = e^x. \cos. 2x + 2e^x. \sin. 2x$

And $F = \dfrac{e^x}{2} + \dfrac{e^x.\cos.2x}{10} + \dfrac{e^x.\sin.2x}{5}$

$$= \frac{e^x}{2} + \frac{e^x}{10} \times (4 \sin. x. \cos. x + 2\cos.^2 x - 1)$$

$$= \frac{2}{5} \times e^x + \frac{e^x \cos. x}{5} \times (2 \sin. x + \cos. x)$$

To integrate $\dfrac{dx}{\sqrt{a+x-\sqrt{a^2+x^2}}} = dF$

Let $a + x - \sqrt{a^2+x^2} = u^2$

$\therefore a^2 + x^2 = (u^2 - a - x)^2$

$= u^4 + a^2 + x^2 - 2au^2 + 2ax - 2xu^2$

$\therefore\ x = \dfrac{u^4 - 2au^2}{2.(u^2 - a)} = \dfrac{u^2}{2} - \dfrac{au^2}{2\,(x^2 - a)}$

Hence $dx = udu - \dfrac{audu}{n^2 - a} + \dfrac{au^3du}{(u^2-a)^2}$

$\therefore dF = du - \dfrac{adu}{u^2 - a} + \dfrac{au^2du}{(u^2 - a)^2}$

$= du - \dfrac{adu}{u^2 - a} + \dfrac{adu}{(u^2 - a)^2} \times (u^2 - a + a)$

$= du + \dfrac{a^2du}{(u^2-a)^2}$

Again, put $\dfrac{u}{u^2 - a} = P$

Then $dP = \dfrac{du}{u^2 - a} - \dfrac{u^2du}{(u^2 - a)^2}$

$= \dfrac{du}{u^2 - a} - \dfrac{u^2 - a + a}{(u^2-a)^2}\,du$

$= - \dfrac{adu}{(u^2-a)^2}$

Hence $F = u - \dfrac{P}{a}$ which is $\therefore$ known.

To integrate $\dfrac{dx}{a+b.\cos.x} = dF$

Assume $\cos.\ x = \dfrac{1 - u^2}{1 + u^2}$; hence $dx = \dfrac{1}{\sin.x} \times \dfrac{4udu}{(1 + u^2)^2}$

Also $\sin.\ x = \sqrt{1 - \left(\dfrac{1 - u^2}{1 + u^2}\right)^2} = \dfrac{2u}{(1 + u^2)}$

And $a + b.\cos.\ x = a + b \times \dfrac{1 - u^2}{1 + u^2} = \dfrac{a+b+(a - b)u^2}{1+u^2}$

Hence $dF = \dfrac{2du}{a+b+(a-b).u^2}$ whose integral will be

a circular arc, or logarithm according as the term $(a - b) . x^2$ is positive, or negative i. e., according as a is $>$ or $<$ b.

Let a be $> b$ and $\sqrt{\frac{a-b}{a+b}} . u = v$

$$\text{Hence } dF = \frac{2}{a+b} \times \sqrt{\frac{a+b}{a-b}} \times \frac{dv}{1+v^2}$$

$$= \frac{2}{\sqrt{a^2-b^2}} \times \frac{dv}{1+v^2}$$

$$\therefore F = \frac{2}{\sqrt{a^2-b^2}} \times \tan.^{-1} v$$

But $v = \sqrt{\frac{a-b}{a+v}} . u$, and $u = \sqrt{\frac{1-\cos. x}{1+\cos. x}} = \frac{\sin. x}{1+\cos. x}$ $= \tan. \frac{x}{2}$

$$\therefore F = \frac{2}{\sqrt{a^2-b^2}} \times \tan.^{-1} \sqrt{\frac{a-b}{a+b}} . \tan. \frac{x}{2} \text{.......(1)}$$

$$= \frac{1}{\sqrt{a^2-b^2}} \times \left(2 \times \tan.^{-1} \sqrt{\frac{a-b}{a+b}} . \tan. \frac{x}{2}\right)$$

$$\text{Now } \tan. 2\theta = \frac{2 \tan.\theta}{1-\tan.^2 \theta} = \frac{2\sqrt{\frac{a-b}{a+b}} . \tan. \frac{x}{2}}{1 - \frac{a-b}{a+b} . \tan.^2 \frac{x}{2}}$$

$= \frac{\sqrt{a^2-b^2} . \sin. x}{(a+b).\cos.^2\frac{x}{2} - (a-b)\sin.^2\frac{x}{2}} = \frac{\sqrt{a^2-b^2}. \sin. x}{b+a. \cos. x}$ (θ being put $= \tan.^{-1} \sqrt{\frac{a-b}{a+b}} . \tan. \frac{x}{2}$)

$$\therefore F = \frac{1}{\sqrt{a^2-b^2}} \times \tan.^{-1} \frac{\sqrt{a^2-b^2}.\sin.x}{b + a \cos. x} \text{.......(2)}$$

$$\text{Again, } \tan.2\theta = \frac{\sqrt{a^2 - b^2} . \sin. x}{b+a. \cos. x} = \frac{\sqrt{1-\cos.^2 2\theta}}{\cos. 2\theta}$$

$$\therefore \cos.^2 2\,\theta = \frac{(b+a.\cos.x)^2}{(a^2-b^2).\sin.^2x+(b+a.\cos.x)^2}$$

$$= \frac{(b+a.\cos.x)^2}{a^2+2ab\cos.x+b^2.\cos.^2x} = \frac{(b+a.\cos.x)^2}{(a+b.\cos.x)^2}$$

$$\therefore \cos.2\,\theta = \frac{b+a.\cos.x}{a+b.\cos.x}$$

Hence, $F = \dfrac{1}{\sqrt{a^2-b^2}} \times \cos.^{-1}\dfrac{b+a.\cos.x}{a+b.\cos.x}$ which is the integral required.

For other integrals of this kind, see Translation of *Lacroix*, p. 761., &c.

595. To integrate $\dfrac{(1-x^3)^{\frac{1}{3}}}{x^5} \times dx = dF$

Assume $\dfrac{(1-x^3)^{\frac{4}{3}}}{x^4} = P$

Then $dP = \dfrac{-4dx.(1-x^3)^{\frac{4}{3}}}{x^5} - \dfrac{4dx}{x^2}(1-x^3)^{\frac{1}{3}}$

$$= \frac{-4dx}{x^5} \times (1-x^3)^{\frac{1}{3}}$$

$$\therefore F = -\frac{P}{4} = -\frac{1}{4}.\frac{(1-x^3)^{\frac{4}{3}}}{x^4}$$

To integrate $\dfrac{dx}{\sqrt{A+Bx+Cx^2}} = d.F.$

Take away the second term of the denominator, by assuming

$$x + \frac{B}{2C} = u$$

Thenec $dF = \dfrac{du}{\sqrt{C} \times \sqrt{u^2+\dfrac{A}{C}-\dfrac{B^2}{4C^2}}}$

$= \dfrac{du}{\sqrt{C} \times \sqrt{u^2 \pm Q}}$ (by putting $\dfrac{A}{C}-\dfrac{B^2}{4C^2} = \pm Q$)

$$\therefore F = \frac{1}{\sqrt{C}} \times l.(u + \sqrt{u^2 \pm Q})$$

596. $$\int \frac{x^4 dx}{x^2+a^2} = \int \left(x^2 dx - a^2 dx + \frac{a^2 dx}{a^2+x^2}\right)$$
$$= \frac{x^3}{3} - a^2 x + \tan.^{-a} x$$

To integrate $\frac{x^4 dx}{(1-x^2)^{\frac{5}{2}}} = dF_0$

Assume $\left.\begin{array}{l} \frac{x^3}{(1-x^2)^{\frac{3}{2}}} = P_1 \\ \frac{x}{(1-x^2)^{\frac{1}{2}}} = P_2 \end{array}\right\} \int \frac{x^2 dx}{(1-x^2)^{\frac{3}{2}}} = F_1$

Then $dP_1 = \frac{3x^2 dx}{(1-x^2)^{\frac{3}{2}}} + \frac{3x^4 dx}{(1-x^2)^{\frac{5}{2}}}$

$\therefore P_1 = 3 F_1 + 3 F_0$

Similarly $P_2 = \int \frac{dx}{(1-x^2)^{\frac{1}{2}}} + F_1$

$\therefore F_0 = \frac{P_1}{3} - P_2 + \sin.^{-1} x$ which is $\therefore$ known.

To integrate $\frac{dx}{(x-a)^2.(x-b)} = dF$; assume

$$\frac{1}{(x-a)^2.(x-b)} = \frac{A}{(x-a)^2} + \frac{B}{x-a} + \frac{C}{x-b}$$

Then $A.(x-b) + B.(x-a).(x-b) + C.(x-a)^2 = 1$

Let $x = a$

Then $A.(a-b) = 1$ and $\therefore A = \frac{1}{a-b}$

Let $x = b$

Then $C.(b-a)^2 = 1$ and $\therefore C = \frac{1}{(b-a)^2}$

Again $\frac{1 - \frac{1}{a-b} \times (x-b)}{x-a} = -\frac{1}{a-b} = B.(x-b) + C.(x-a)$

Let $x = a$

Then $B.(a-b) = -\frac{1}{a-b}$

$\therefore B = -\frac{1}{(a-b)^2}$

Hence $dF = \frac{dx}{(a-b).(x-a)^2} - \frac{dx}{(a-b)^2.(x-a)} + \frac{dx}{(a-b)^2(x-b)}$

$\therefore F = -\frac{1}{(a-b).(x-a)} + \frac{1}{(a-b)^2}.l.\frac{x-b}{x-a}$

Otherwise.

Put $\frac{1}{x-a} = u$

Then $\frac{dx}{(x-a)^2} = -du$

And $\frac{1}{x-b} = \frac{1}{\frac{1}{u}+a-b} = \frac{u}{1+(a-b).u}$

$\therefore dF = \frac{-udu}{1+(a-b)u} = -\frac{du}{a-b} + \frac{du}{(a-b).(1+\overline{a-b}.u)}$

$\therefore F = -\frac{u}{a-b} + \frac{1}{(a-b)^2} \times l.(1+\overline{a-b}.u)$ a result equivalent to the former.

597. To integrate $\frac{dx}{x\sqrt{bx^2-a}} = dF$,

Assume $\sqrt{bx^2-a} = u$

Then $\frac{bxdx}{\sqrt{bx^2-a}} = du$

$\therefore \frac{dx}{x\sqrt{bx^2-a}} = \frac{du}{bx^2}$

But $x^2 = \frac{u^2+a}{b}$

$\therefore dF = \frac{du}{u^2+a}$

$$\text{And } F = \frac{1}{a} \cdot \int \frac{a du}{u^2 + a} = \frac{1}{a} \tan.^{-\sqrt{a}} u$$

$$= \frac{1}{a} \tan.^{-\sqrt{a}} \sqrt{bx^2 - a}$$

To integrate $\dfrac{d\theta}{\sin.^4\theta. \cos.\theta} = d F_0$ we have

$$dF_0 = \frac{d\theta. \cos.\theta}{\sin.^4\theta.\cos.^2\theta} = \frac{d. \sin.\theta}{\sin.^4.(1-\sin.^2\theta)} = \frac{ds}{s^4.(1-s^2)}$$

$$\left.\begin{array}{l} \text{Assume} \therefore \dfrac{1}{s^3} = P_1 \\ \dfrac{1}{s} = P_2 \end{array}\right\} \int \frac{ds}{s^2(1-s^2)} = F_1$$

$$\text{Then } dP_1 = \frac{-3ds}{s^4} = \frac{-3ds \times (1-s^2)}{s^4.(1-s^2)}$$

$$= \frac{-3ds}{s.^4(1-s^2)} + \frac{3ds}{s.^2.(1-s^2)}$$

$$\therefore P_1 = -3 F_0 + 3 F_1$$

$$\text{Similarly } P_2 = -.F_1 + \int \frac{ds}{1-s^2}$$

$$\text{Hence } F_0 = -\frac{P_1}{3} - P_2 + \int \frac{ds}{1-s^2}$$

$$\text{But } \int \frac{ds}{1-s^2} = \frac{1}{2} l. \frac{1+s}{1-s} = \frac{1}{2} l. \frac{1+\sin.\theta}{1-\sin.\theta} = \frac{1}{2} l.\tan.\left(45^\circ + \frac{\theta}{2}\right)$$

$$\therefore F_0 = -\frac{1}{3\sin.^3\theta} - \frac{1}{\sin.\theta} + \frac{1}{2} l. \tan.\left(45 + \frac{\theta}{2}\right)$$

Otherwise.

$$dF = \frac{d\theta}{\sin.^4\theta. \cos.\theta} \times (\cos.^2\theta + \sin.^2\theta) = \frac{d\theta. \cos.\theta}{\sin.^4\theta}$$

$$+ \frac{d\theta}{\sin.^2\theta. \cos.\theta} = \frac{d\sin.\theta}{\sin.^4\theta} + \frac{d\theta}{\sin.^2\theta. \cos.\theta} \times (\cos.^2\theta + \sin.^2\theta)$$

$$= \frac{d \sin.\theta}{\sin.^4\theta} + \frac{d. \sin.\theta}{\sin.^2\theta} + \frac{d\sin.\theta}{1-\sin.^2\theta} \text{ which leads to the same result.}$$

To integrate $\frac{dx}{\sqrt{a^4-x^4}}$ See problem 6. page 72.

598. To integrate $\frac{dx}{\sqrt{a}-\sqrt{x}}$, assume

$$\sqrt{x}=u, \therefore x=u^2, \text{and } dx=2udu$$

$$\therefore \frac{dx}{\sqrt{a}-\sqrt{x}}=\frac{2udu}{\sqrt{a}-u}=-\frac{2udu}{u-\sqrt{a}}=-2du-\frac{2\sqrt{a}du}{u-\sqrt{a}}$$

$$\therefore \int\frac{dx}{\sqrt{a}-\sqrt{x}}=-2u-2\sqrt{a}\,.\,l.\,(u-\sqrt{a})$$

$$=-2\sqrt{x}-2\sqrt{a}.l\,.(\sqrt{x}-\sqrt{a})$$

To integrate $\frac{xdx}{\sqrt{2ax-x^2}}=d\text{F}$, we have

$$d\text{F}=\frac{adx-(adx-xdx)}{\sqrt{2ax-x^2}}$$

$$\therefore \text{F}=\int\frac{adx}{\sqrt{2ax-x^2}}-\int\frac{adx-xdx}{\sqrt{2ax-x^2}}$$

$$=\text{vers.}^{-1}x-\sqrt{2ax-x^2}$$

To integrate $dx.\log.x=d\text{F}$,
we have $\log.x=l.x\times\log.e$ (e being the base of the Naperian system)

$$\therefore d\text{F}=dx.\,lx.\times\log.e$$

$$\text{And F}=\log.e\times\int dx.\,l.x$$

$$=\log.e\times\left(x\,l.x-\int x\times\frac{dx}{x}\right)$$

$$=\log.e\times(xl.x-x)$$

599. $$\frac{dz}{1-\tan.^2z}=\frac{dz\times\cos.^2z}{\cos.^2z-\sin.^2z}$$

596. $$\int \frac{x^4 dx}{x^2+a^2} = \int \left(x^2dx - a^2dx + \frac{a^2dx}{a^2+x^2}\right)$$

$$= \frac{x^3}{3} - a^2x + \tan.^{-1} x$$

To integrate $\dfrac{x^4dx}{(1-x^2)^{\frac{5}{2}}} = dF_0$

$$\left.\begin{aligned} \text{Assume } \frac{x^3}{(1-x^2)^{\frac{3}{2}}} &= P_1, \\ \frac{x}{(1-x^2)^{\frac{1}{2}}} &= P_2 \end{aligned}\right\} \int \frac{x^2dx}{(1-x^2)^{\frac{3}{2}}} = F_1$$

Then $dP_1 = \dfrac{3x^2dx}{(1-x^2)^{\frac{3}{2}}} + \dfrac{3x^4dx}{(1-x^2)^{\frac{5}{2}}}$

$\therefore P_1 = 3F_1 + 3F_0$

Similarly $P_2 = \displaystyle\int \frac{dx}{(1-x^2)^{\frac{1}{2}}} + F_1$

$\therefore F_0 = \dfrac{P_1}{3} - P_2 + \sin.^{-1} x$ which is $\therefore$ known.

To integrate $\dfrac{dx}{(x-a)^2.(x-b)} = dF$; assume

$$\frac{1}{(x-a)^2.(x-b)} = \frac{A}{(x-a)^2} + \frac{B}{x-a} + \frac{C}{x-b}$$

Then $A.(x-b) + B.(x-a).(x-b) + C.(x-a)^2 = 1$

Let $x = a$

Then $A.(a-b) = 1$ and $\therefore A = \dfrac{1}{a-b}$

Let $x = b$

Then $C.(b-a)^2 = 1$ and $\therefore C = \dfrac{1}{(b-a)^2}$

Again $\dfrac{1 - \frac{1}{a-b}\times(x-b)}{x-a} = -\dfrac{1}{a-b} = B.(x-b) + C.(x-a)$

Let $x = a$

Then B. $(a-b) = -\dfrac{1}{a-b}$

$\therefore\ B = -\dfrac{1}{(a-b)^2}$

Hence $dF = \dfrac{dx}{(a-b).(x-a)^2} - \dfrac{dx}{(a-b)^2.(x-a)} + \dfrac{dx}{(a-b)^2\,(x-b)}$

$\therefore\ F = -\dfrac{1}{(a-b).(x-a)} + \dfrac{1}{(a-b)^2}.l.\dfrac{x-b}{x-a}$

Otherwise.

Put $\dfrac{1}{x-a} = u$

Then $\dfrac{dx}{(x-a)^2} = -du$

And $\dfrac{1}{x-b} = \dfrac{1}{\dfrac{1}{u}+a-b} = \dfrac{u}{1+(a-b).u}$

$\therefore\ dF = \dfrac{-udu}{1+(a-b)u} = -\dfrac{du}{a-b} + \dfrac{du}{(a-b).(1+\overline{a-b}\,.\,u)}$

$\therefore\ F = -\dfrac{u}{a-b} + \dfrac{1}{(a-b)^2} \times l.\ (1+\overline{a-b}.u)$ a result equivalent to the former.

597. To integrate $\dfrac{dx}{x\sqrt{bx^2-a}} = dF$,

Assume $\sqrt{bx^2-a} = u$

Then $\dfrac{bxdx}{\sqrt{bx^2-a}} = du$

$\therefore\ \dfrac{dx}{x\sqrt{bx^2-a}} = \dfrac{du}{bx^2}$

But $x^2 = \dfrac{u^2+a}{b}$

$\therefore\ dF = \dfrac{du}{u^2+a}$

$$\text{And } F = \frac{1}{a} \cdot \int \frac{ddu}{u^2 + a} = \frac{1}{a} \tan.^{-\sqrt{a}} u$$

$$= \frac{1}{a} \tan.^{-\sqrt{a}} \sqrt{bx^2 - a}$$

To integrate $\dfrac{d\theta}{\sin.^4\theta . \cos.\theta} = d F_0$ we have

$$dF_0 = \frac{d\theta . \cos.\theta}{\sin.^4\theta . \cos.^2\theta} = \frac{d.\sin.\theta}{\sin.^4 . (1 - \sin.^2\theta)} = \frac{ds}{s^4.(1 - s^2)}$$

$$\left.\begin{aligned} \text{Assume } \therefore \frac{1}{s^3} &= P_1 \\ \frac{1}{s} &= P_2 \end{aligned}\right\} \int \frac{ds}{s^2(1 - s^2)} = F_1$$

$$\text{Then } dP_1 = \frac{-3ds}{s^4} = \frac{-3ds \times (1 - s^2)}{s^4 . (1 - s^2)}$$

$$= \frac{-3ds}{s.^4(1 - s^2)} + \frac{3ds}{s.^2 . (1 - s^2)}$$

$$\therefore P_1 = -3 F_0 + 3 F_1$$

$$\text{Similarly } P_2 = -. F_1 + \int \frac{ds}{1 - s^2}$$

$$\text{Hence } F_0 = -\frac{P_1}{3} - P_2 + \int \frac{ds}{1 - s^2}$$

$$\text{But } \int \frac{ds}{1 - s^2} = \frac{1}{2} l. \frac{1 + s}{1 - s} = \frac{1}{2} l. \frac{1 + \sin.\theta}{1 - \sin.\theta} = \frac{1}{2} l.\tan.\left(45^\circ + \frac{\theta}{2}\right)$$

$$\therefore F_0 = -\frac{1}{3\sin.^3\theta} - \frac{1}{\sin.\theta} + \frac{1}{2} l.\tan.\left(45 + \frac{\theta}{2}\right)$$

Otherwise.

$$dF = \frac{d\theta}{\sin.^4\theta . \cos.\theta} \times (\cos.^2\theta + \sin.^2\theta) = \frac{d\theta . \cos.\theta}{\sin.^4\theta}$$

$$+ \frac{d\theta}{\sin.^2\theta . \cos.\theta} = \frac{d\sin.\theta}{\sin.^4\theta} + \frac{d\theta}{\sin.^2\theta . \cos.\theta} \times (\cos.^2\theta + \sin.^2\theta)$$

$$= \frac{d\sin.\theta}{\sin.^4\theta} + \frac{d.\sin.\theta}{\sin.^2\theta} + \frac{d\sin.\theta}{1 - \sin.^2\theta} \text{ which leads to the same result.}$$

To integrate $\frac{dx}{\sqrt{a^4-x^4}}$ See problem 6. page 72.

598. To integrate $\frac{dx}{\sqrt{a}-\sqrt{x}}$, assume

$\sqrt{x}=u$, $\therefore x=u^2$, and $dx=2udu$

$$\therefore \frac{dx}{\sqrt{a}-\sqrt{x}}=\frac{2udu}{\sqrt{a}-u}=-\frac{2udu}{u-\sqrt{a}}=-2du-\frac{2\sqrt{a}du}{u-\sqrt{a}}$$

$$\therefore \int\frac{dx}{\sqrt{a}-\sqrt{x}}=-2u-2\sqrt{a}\,.\,l.\,(u-\sqrt{a})$$

$$=-2\sqrt{x}-2\sqrt{a}.l\,.\,(\sqrt{x}-\sqrt{a})$$

To integrate $\frac{xdx}{\sqrt{2ax-x^2}}=d\text{F}$, we have

$$d\text{F}=\frac{adx-(adx-xdx)}{\sqrt{2ax-x^2}}$$

$$\therefore \text{F}=\int\frac{adx}{\sqrt{2ax-x^2}}-\int\frac{adx-xdx}{\sqrt{2ax-x^2}}$$

$$=\text{vers.}^{-1}x-\sqrt{2ax-x^2}$$

To integrate $dx.\log.x=d\text{F}$,
we have $\log.x=l.x\times\log.e$ (e being the base of the Naperian system)

$$\therefore d\text{F}=dx.\,lx.\times\log.e$$

$$\text{And F}=\log.e\times\int dx.\,l.x$$

$$=\log.e\times\left(x\,l.x-\int x\times\frac{dx}{x}\right)$$

$$=\log.e\times(xl.x-x)$$

599. $$\frac{dz}{1-\tan.^2z}=\frac{dz\times\cos.^2z}{\cos.^2z-\sin.^2z}$$

$$= \frac{dz \times (1 + \cos. 2z)}{2 \cos. 2z} = \frac{2dz}{4\cos. 2z} + \frac{dz}{2}$$

$$\therefore \int \frac{dz}{1 - \tan.^2 z} = \frac{1}{4} \times \int \frac{d. 2z}{\cos. 2z} + z$$

$$\text{But} \int \frac{d.2z}{\cos. 2z} = \int \frac{d2z. \cos. 2z}{1 - \sin.^2 2z}$$

$$= \int \frac{d.\sin.2z}{1 - \sin.^2 2z} = \frac{1}{2}.l. \frac{1 + \sin. 2z}{1 - \sin. 2z}$$

$$\text{Hence} \int \frac{dz}{1 - \tan.^2 z} = z + \frac{1}{8}.l. \frac{1 + \sin. 2z}{1 - \sin. 2z}$$

To integrate $\frac{dx}{1+x+x^2}$, assume

$$x + \frac{1}{2} = u$$

Then $dx = du$

And $x^2 + x + 1 = u^2 + 1 - \frac{1}{4} = u^2 + \frac{3}{4}$

$$\therefore \int \frac{dx}{1+x+x^2} = \int \frac{du}{u^2 + \frac{3}{4}}$$

$$= \frac{4}{3} \times \tan.^{-\sqrt{\frac{3}{2}}} u$$

$$= \frac{4}{3} \tan.^{-\sqrt{\frac{3}{2}}} \left(x + \frac{1}{2}\right)$$

600. $\int \frac{dx}{1 + x^2} = \tan.^{-1} x$ (*Vince, Lacroix, &c.*)

$\int \frac{dx}{\sqrt{1 + x^2}} = l. \,(x + \sqrt{1 + x^2})$ (*Vince, Lacroix*).

And $\int \frac{dx}{x\sqrt{1+x^2}} = l. \frac{x}{1 + \sqrt{1 + x^4}}$ (*Vince, Lacroix, &c.*)

601. To integrate $\dfrac{dx}{(1-x^3)^{\frac{1}{3}}} = d\text{F}$, assume $(1-x^3)^{\frac{1}{3}} = ux$

Then $1 - x^3 = u^3.x^3$

And $x^3 = \dfrac{1}{1+u^3}$

$$\therefore dx = \frac{-u^2du}{x^2(1+u^3)^2}$$

$$\therefore d\text{F} = \frac{dx}{ux} = \frac{-udu}{x^3.(1+u^3)^2}$$

$$= \frac{-udu}{1+u^3}$$

Again, assume $\dfrac{1}{1+u^3} = \dfrac{\text{A}}{u+1} + \dfrac{\text{B}u+\text{C}}{u^2-u+1}$ $\{(u+1).\times (u^2-u+1) = 1+u^3\}$

$$\left.\begin{array}{l}\text{Then A}u^2 - \text{A}u + \text{A}\\ \quad + \text{B}u^2 + \text{B}u\\ \qquad + \text{C}u + \text{C}\end{array}\right\} = 1$$

$$\left.\begin{array}{r}\therefore \text{A}+\text{B}=0\\ \text{A}+\text{C}=1\\ \text{B}-\text{A}+\text{C}=0\end{array}\right\} \quad \text{Hence} \quad \left.\begin{array}{l}\text{A} = \frac{1}{3}\\ \text{B} = -\frac{1}{3}\\ \text{C} = \frac{2}{3}\end{array}\right\}$$

$$\therefore d\text{F} = \frac{-udu}{3(u+1)} + \frac{u^2du - 2udu}{3(u^2-u+1)}$$

$$= \frac{-du}{3} + \frac{du}{3.(u+1)} + \frac{du}{3} \times \frac{(u^2-u+1-u-1)}{u^2-u+1}$$

$$= \frac{du}{3.(u+1)} - \frac{du}{3}.\frac{\overline{u-\frac{1}{2}}+\frac{3}{2}}{u^2-u+1}$$

$$\therefore \text{F} = \frac{1}{3}.\ l.(u+1) - \frac{1}{6} \times l.(u^2-u+1) - \int \frac{3du}{2.(u^2-u+1)}$$

Put $u - \dfrac{1}{2} = v$

Then $u^2 - u + 1 = v^2 + 1 - \dfrac{1}{4} = v^2 + \dfrac{3}{4}$

$$\therefore \int \frac{3du}{2.(u^2-u+1)} = \int \frac{3dv}{2\,.\,(v^2+\frac{3}{4})} = 2.\ \tan.^{-}\sqrt{\tfrac{3}{4}}\,v$$

$$\therefore\ \text{F} = \frac{1}{3}.\ l.(u+1) - \frac{1}{6}.\ l.(u^2-u+1) - 2.\ \tan.^{-}\sqrt{\tfrac{3}{4}}\left(u-\frac{1}{2}\right)$$

The above assumption applies to the general form $\dfrac{dx}{(ax^r+x^s)^{\frac{1}{t}}}$

For putting $ax^r + x^s = u^t\,x^t$ we get

$$x^{s-r} = \frac{-a}{1-u^t}$$

$$\text{And } \frac{dx}{ux} = \frac{as}{s-r} \times \frac{1}{x^{s-r}} \times \frac{u^{t-2}du}{(1-u^t)^2} = \frac{s}{s-r}\times\frac{u^{t-2}du}{1-u^t}$$

which is rational.

To integrate $e^{\sqrt{x}}xdx = d\text{F}$, let $\sqrt{x} = u$

Then $\text{F} = \int 2u^3e^u du = 2\int e^u du.u^3$

$$= 2e^u.u^3 - 6\int e^u du.u^2$$

Also $\int e^u du.u^2 = e^u.u^2 - 2\int e^u du.u$

And $\int e^u du.u = e^u u - \int e^u du = e^u.u - e^u$

Hence, $\text{F} = 2e^u u^3 - 6e^u u^2 + 12\,e^u.u - 12\,e^u$

$$= 2e^{\sqrt{x}} \times (x^{\frac{3}{2}} - 3x + 6\sqrt{x} - 6)$$

If the index of e be complicated, it will generally be more commodious to assume that index $= u$, &c. &c.

To integrate sin. mx. sin. nx. cos. px. $dx = d\text{F}$, we have

$$\sin.\ mx.\ \sin.\ nx = \frac{1}{2}\left\{\cos.\ (m-n)\,x - \cos.\ (m+n)x\right\}$$

$\therefore$ sin. mx sin. nx. cos. px $= \dfrac{1}{2}\Big\{$cos. px. cos. $(m-n)\ x$ $-$
cos. px. cos. $(m+n)\ x\Big\}$

But cos. px. cos. $(m-n)\ x = \dfrac{\cos.\ (m-n+p)\,x + \cos.(m-n-p)x}{2}$

And $\cos. px. \cos. (m+n)x = \frac{\cos. (m+n+p)x + \cos. (m+n-p)x}{2}$

Hence $dF = \frac{dx}{4} \times \{\cos. (m-n+p)x + \cos. (m-n-p)x$

$- \cos. (m+n+p)x - \cos. (m+n-p)x\}$

But $dx. \cos. (m-n+p)x = \frac{1}{m-n+p} \times d. \sin. (m-n+p)x$

&c. = &c.

$$\therefore F = \frac{1}{4}\left\{\frac{\sin. (m-n+p)x}{m-n+p} + \frac{\sin. (m-n-p)x}{m-n-p} - \frac{\sin. (m+n+p)x}{m+n+p} - \frac{\sin. (m+n-p)x}{m+n-p}\right\}.$$

By the above method we may also integrate the following general forms.

$$\left.\begin{array}{l} \sin. m_1x. \sin. m_2x. \sin. m_3x.... \sin. m_{n-1}x \sin. m_nx \times dx \\ \cos. m_1x. \cos. m_2x. \cos. m_3x.... \cos. m_{n-1}x. \cos. m_n \times dx \end{array}\right\}$$

Also that form, whose n factors are sines or cosines of the multiples of x promiscuously, may be integrated.

602. To integrate $\frac{dx.\sqrt{1-x^2}}{1+x^2} = dF$,

Assume $u = \frac{x}{\sqrt{1-x^2}}$

Then $du = \frac{dx}{\sqrt{1-x^2}} + \frac{x^2dx}{(1-x^2)^{\frac{3}{2}}}$

$$= \frac{dx}{(1-x^2)^{\frac{3}{2}}} = \frac{dx.\sqrt{1-x^2}}{(1-x^2)^2}$$

$\therefore dF = \frac{du \times (1-x^2)^2}{1+x^2}$

Also $x^2 = u^2 - u^2x^2$

$\therefore x^2 = \frac{u^2}{1+u^2}$

And $(1 - x^2)^2 = \left(1 - \frac{u^2}{1 + u^2}\right)^2 = \frac{1}{(1 + u^2)^2}$

And $1 + x^2 = 1 + \frac{u^2}{1 + u^2} = \frac{2u^2 + 1}{1 + u^2}$

Hence $dF = \frac{du}{(2u^2 + 1)(1 + u^2)}$

Again, put $\frac{1}{(1+u^2) \times (1+2u^2)} = \frac{Au + B}{1+u^2} + \frac{au + b}{1+2u^2}$

Then $B + b + (A + a) u + (2b + B) u^2 + (2A + a) u^3 = 1$

$\therefore B + b = 1, A + a = 0, 2B + b = 0,$

and $2A + a = 0$

Hence $A = 0, a = 0, b = 2, B = -1$

$\therefore dF = \frac{2du}{1+2u^2} - \frac{du}{1 + u^2}$

And $F = \sqrt{2} \tan.^{-1} \sqrt{2}.u - \tan.^{-1}u.$

To integrate $e^{ax}. \sin.^2x.dx = dF$, we have $\sin.^2x = \frac{1 - \cos.2x}{2}$

$\therefore F = \int \frac{e^{ax}dx}{2} - \int \frac{e^{ax}dx. \cos. 2x}{2}$

But $\int \frac{e^{ax}dx}{2} = \frac{1}{2a} \int e^{ax}. d.ax$

$= \frac{1}{2a}. e^{ax}$

And $\int \frac{e^{ax}dx}{2}. \cos. 2x = \frac{1}{2a}. e^{ax}. \cos. 2x + \int \frac{1}{2a}. e^{ax} \times 2dx$

$\sin. 2x = \frac{1}{2a}. e^{ax}\cos. 2x + \int \frac{1}{a} e^{ax}.dx \sin. 2x = \frac{1}{2a}. e^{ax}. \cos. 2x$

$+ \frac{1}{a^2}. e^{ax}. \sin.2x - \int \frac{2}{a^2}. e^{ax}dx \cos. 2x$

Hence $\left(\frac{1}{2} + \frac{2}{a^2}\right) . \int e^{ax} dx . \cos. 2x = \frac{1}{2a} . e^{ax} . \cos. 2x +$

$\frac{1}{a^2} e^{ax} . \sin. 2x.$

$$\therefore \frac{1}{2} \int e^{ax} dx . \cos. 2x = \frac{a^2}{a^2+4} \times e^{ax} \left\{ \frac{\cos. 2x}{2a} + \frac{\sin. 2x}{a^2} \right\}$$

$$= \frac{e^{ax}}{a^2+4} \times \frac{a . \cos. 2x + 2 \sin. 2x}{2} = \frac{e^{ax}}{a^2+4} \times \left(\frac{a}{2} - \right.$$

$\left. a \sin.^2 x + 2 \sin. x . \cos. x\right)$

Hence $F = \frac{e^{ax}}{2a} - \frac{e^{ax}}{a^2+4} \times \frac{a}{2} + \frac{e^{ax} \sin. x}{a^2+4} \times (a . \sin. x + 2 \cos. x)$

$$= \frac{2e^{ax}}{a . (a^2+4)} + \frac{e^{ax} . \sin. x}{a^2+4} . (a \sin. x + 2 \cos. x)$$

THE INTEGRATION OF DIFFERENTIAL EQUATIONS.

603. To integrate $adz = dy \,.\, \sqrt{a^2 + 4y^2}$ we have $dz = \sqrt{dx^2 + dy^2}$ (x being the abscissa of the curve)

$$\therefore\ a \,.\, \sqrt{dx^2 + dy^2} = dy \,.\, \sqrt{a^2 + 4y^2}$$

$$\therefore\ a^2 \,.\, dx^2 + a^2\, dy^2 = a^2\, dy^2 + 4y^2\, dy^2$$

$$\therefore\ a^2\, dx^2 = 4y^2\, dy^2$$

$$\therefore\ adx = 2ydy$$

and $ax = y^2$, which expresses the relation required. The curve is the *Common Parabola.*

604. To integrate $ay + \frac{bdy}{dx} + \frac{cd^2y}{dx^2} = 0$, assume (the *equation being Linear* with respect to one of the variables and its differentials)

$$y = e^{\int udx}$$

Then $\frac{dy}{dx} = u \,.\, e^{\int udx}$

and $\frac{d^2y}{dx^2} = \frac{du}{dx} \times e^{\int udx} + u^2 \,.\, e^{\int udx}$

Hence, substituting and dividing by $e^{\int udx}$

we have $a + bu + \frac{cdu}{dx} + cu^2 = 0$

$$\therefore\ dx = \frac{-cdu}{a + bu + cu^2} = \frac{-du}{\frac{a}{c} + \frac{b}{c}u + u^2} = \frac{-dv}{v^2 + \frac{a}{c} - \frac{b^2}{4c^2}}$$

$$= \frac{-dv}{v^2 \pm m^2} \left(v = u + \frac{b}{2c} \text{ and } m^2 = \frac{a}{c} - \frac{b^2}{4c^2}\right)$$

$$\therefore\ ly = \int udx = \int \frac{dv \,.\, \left(\frac{b}{2c} - v\right)}{v^2 \pm m^2} \text{ is known, \&c.}$$

605. To integrate $dx + x^3dx = dz + zdx$

Assume $z = u \,.\, \mathrm{X}$ (X being an indeterminate function of x)

Then $\mathrm{X}du + ud\mathrm{X} + u\mathrm{X}dx = (1 + x^3)\,dx$.

Now, being at liberty to make another assumption with regard to u and X, we will separate the variables by putting

$$\mathrm{X}du + u\mathrm{X}dx = 0$$

$$\left.\begin{aligned} \therefore\ du + udx &= 0 \\ \text{and } ud\mathrm{X} &= (1 + x^3)\,dx \end{aligned}\right\}$$

$$\text{Hence } \frac{du}{u} + dx = 0$$

$$\therefore\ lu = -x + c$$

$$\therefore\ ce^{-x} = u \text{ (}e\text{ being the hyperbolic base.)}$$

$$\text{Also } d\mathrm{X} = \frac{(1 + x^3)\,dx}{u} = \frac{(1 + x^3)\,dx}{ce^{-x}} = \frac{1}{c}\,e^x dx + \frac{1}{c} x^3 e^x dx.$$

$$\therefore\ \mathrm{X} = \frac{1}{c}\,e^x + \frac{1}{c}\int x^3 e^x dx$$

But $\int x^3 e^x dx = e^x x^3 - \int e^x . dx^3 = e^x x^3 - 3\int x^2 e^x dx = e^x x^3 - 3x^2 e^x + 6\int x e^x dx = e^x x^3 - 3x^2 e^x + 6xe^x - 6e^x + c'$

$$\therefore\ \mathrm{X} = \frac{1}{c}\,e^x \times (x^3 - 3x^2 + 6x - 5) + \frac{c'}{c}$$

$$\text{Hence } z = u\,.\,\mathrm{X} = e^{-x} \times e^x\,(x^3 - 3x^2 + 6x - 5) + \frac{c'}{e^x}$$

$$= x^3 - 3x^2 + 6x - 5 + \frac{c'}{e^x}$$

This is a *linear equation* of the first order. (See *Lacroix*, Art. 257.)

606. To integrate $\dfrac{pdx}{x} + \dfrac{rdy}{y} = \dfrac{x^m dx}{ay^n}$,

$$\text{Assume } x = u^{\frac{r}{p}}$$

$$\text{Then } \frac{pdx}{x} = r\,\frac{u^{\frac{r}{p}-1}}{u^{\frac{r}{p}}}\,du = \frac{rdu}{u}$$

$$\text{Also } \frac{x^m dx}{ay^n} = \frac{r}{ap} \times \frac{u^{\frac{rm+r}{p}-1}du}{y^n}$$

$$\text{Hence } \frac{rdu}{u} + \frac{rdy}{y} = \frac{r}{ap}\cdot\frac{u^{\frac{rm+r}{p}-1}du}{y^n}$$

$\therefore\ n\,u^{n-1}du \times y^n + n\,y^{n-1}dy \times u^n = \frac{n}{ap} \times u^{\frac{rm+r}{p}+n-1}\,du$ (dividing by r, and multiplying by $n\,y^n u^n$)

$$\therefore\ u^n y^n = \frac{n}{ap} \times \int u^{\frac{rm+r}{p}+n-1}du = \frac{n}{a(r\,.\,\overline{m+1} + pn)} \times$$

$$u^{\frac{r\,.\,(m+1)}{p}+n} + c$$

$$\therefore\ y^n = \frac{n}{a.(r.\overline{m)+1)}+pn)} \times u^{\frac{(r\,.\,m+1)}{p}} + \frac{c}{u^n}$$

$$= \frac{n}{a.(r.\overline{m+1} + pn)} \times x^{m+1} + \frac{c}{x^{\frac{np}{r}}}$$

607. To integrate $xdy - ydx = dx\,.\,\sqrt{x^2+y^2}$

assume $x = vy$

Then $dx = vdy + ydv$

and $\sqrt{x^2+y^2} = \sqrt{v^2y^2+y^2} = y\,.\,\sqrt{1+v^2}$

Hence $vydy - vydy - y^2dv = (vdy + ydv)\,y\,.\,\sqrt{1+v^2}$

$$\text{Or } -dv = \frac{v\,.\,\sqrt{1+v^2}\,.\,dy}{y} + dv\,.\,\sqrt{1+v^2}$$

$$\therefore\ \frac{dy}{y} = \frac{-dv}{v\sqrt{1+v^2}} - \frac{dv}{v}$$

$$\text{and } l.\,y = -\frac{1}{2}\,l.\,\frac{\sqrt{1+v^2}-1}{\sqrt{1+v^2}+1} - l.\,v + c'$$

$$= -\,l.\,\frac{\sqrt{1+v^2}-1}{v} - l.\,v + l.c$$

$$= l.\,\frac{c}{\sqrt{1+v^2}-1} = l.\,\frac{cy}{\sqrt{x^2+y^2}-y}$$

$$\therefore\ \sqrt{x^2+y^2} = c + y$$

$\therefore\ x^2 + y^2 = c^2 + 2cy + y^2$

$\therefore\ x = \sqrt{c^2+2cy}$ the relation required.

This equation is homogeneous.

608. To integrate $adx^2 = ydy^2 - bdxdy$, we have $dx^2 + \frac{b}{a}dy \times dx = \frac{y}{a}dy^2$

$$\therefore\ dx^2 + \frac{b}{a}dydx + \frac{b^2dy^2}{4a^2} = \frac{y}{a}dy^2 + \frac{b^2dy^2}{4a^2} = \frac{dy^2}{4a^2} \times (4ay + b^2)$$

$$\therefore\ dx + \frac{bdy}{2a} = \pm\frac{dy}{2a} \times \sqrt{4ay+b^2}$$

$\therefore\ x = \frac{-b}{2a}y \pm \frac{1}{6a^2}\cdot(4ay + b^2)^{\frac{3}{2}} + c$ the relation required.

609. To integrate $\sqrt{1+y^2} \times (xdy + ydx) = ydy$

we have $xdy + ydx = \frac{ydy}{\sqrt{1+y^2}}$

$$\therefore\ xy = \int\frac{ydy}{\sqrt{1+y^2}} = \sqrt{1+y^2} + c$$

$\therefore\ x = \frac{\sqrt{1+y^2}}{y} + \frac{c}{y}$ the relation required.

610. Let $y = a + bx + cx^2 + ex^3 + \&c.$

Then $\frac{dy}{ax} = b + 2cx + 3cx^2 + \&c.$

Let dx be constant

Then $\frac{d^2y}{dx^2} = 2c + 6ex + \&c.$

Let dy be constant

Then $\frac{-dy \times d^2x}{dx^3} = 2c + 6ex + \&c.$

Hence, it appears that if for d^2y in the equation where dx is constant we substitute $\frac{-dy\ d^2x}{dx}$ the equation will be transformed into one wherein dy is constant.

The same process continued, will conduct us to a rule for transforming an equation in which d^2x is constant to one in which d^2y is constant; and so on for differentials of higher orders.

To apply the above rule in the integration of $dxdy - xd^2y = xdy^2$, we have $dxdy + \frac{xdyd^2x}{dx} = xdy^2$

$$\text{Hence } \frac{dx}{x} + \frac{d^2x}{dx} = dy$$

$$\therefore y = .lx + ldx + c$$

Let $c = -.l.dy \pm a$ (since dy is constant)

$$\text{Then } y \mp a = \frac{lxdx}{dy}$$

$$\therefore e^{y \mp a} = \frac{xdx}{dy}$$

$$\therefore e^{y \mp a} dy = xdx$$

$$\text{and } e^{y \mp a} = \frac{x^2}{2} + \text{constant.}$$

611. To integrate $xdx + aydx - ydy = 1$, make dy constant and $= c$.

Then, since either dx or dy must enter each term of the equation, we will form the equation thus,

$$xdx + aydx - ydy = \frac{dy}{c}$$

$$\therefore (x + ay)\ dx - \left(y + \frac{1}{c}\right) dy = 0$$

Put $\left.\begin{aligned} x &= x' + m \\ y &= y' + n \end{aligned}\right\}$ m and n being indeterminate constants.

Then $(x' + ay' + m + an)\,dx' - \left(y' + n + \dfrac{1}{c}\right) dy' = 0$

$$\text{Again, put } \left.\begin{aligned} m + an &= 0 \\ n + \frac{1}{c} &= 0 \end{aligned}\right\} \quad \begin{aligned} \therefore n &= -\frac{1}{c} \\ m &= \frac{a}{c} \end{aligned}$$

And $x'dx' + ay'\,dx' - y'dy' = 0$ which is homogeneous. Put $\therefore y' = ux'$

Then $dy' = x'du + udx'$

$\therefore x'dx' + aux'dx' - ux'^2du - u^2x'dx' = 0$

$\therefore x'dx'\,(1 + au - u^2) = x'^2\,udu$

$$\therefore \frac{dx'}{x'} = \frac{udu}{1 + au - u^2}$$

$$\therefore lx' = \int \frac{udu}{1 + au - u^2} = \int \frac{\frac{a}{2}\,du - \left(\frac{a}{2}\,du - udu\right)}{1 + au - u^2}$$

$$= \int \frac{a}{2}\,\frac{du}{1 + au - u^2} - \frac{1}{2}\,l.\,(1 + au - u^2)$$

Again, put $u - \dfrac{a}{2} = v$

Then $1 + au - u^2 = 1 + \dfrac{a^2}{4} - v^2$

$$\text{Hence } l.\,x' = \frac{a}{2}\int \frac{dv}{1 + \frac{a^2}{4} - v^2} - \frac{1}{2}.\,l.(1 + au - u^2)$$

$$= \frac{a}{4.\sqrt{1 + \frac{a^2}{4}}}.\,l.\,\frac{\sqrt{1 + \frac{a^2}{4}} + v}{\sqrt{1 + \frac{a^2}{4}} - v} - \frac{1}{2} \times$$

$l.\,(1 + au - u^2)$

$$= l.\left\{\frac{\left(\dfrac{\sqrt{4 + a^2} + 2v}{\sqrt{4 + a^2} - 2v}\right)^{\frac{a}{2\sqrt{4 + a^2}}}}{\sqrt{1 + au - u^2}}\right\} + l.c$$

$$\therefore x^{a} \times \sqrt{1 + au - u^2} = c\left(\frac{\sqrt{4 + a^2} + 2v}{\sqrt{4 + a^2} - 2v}\right)^{\frac{a}{2\sqrt{4+a^2}}}$$

$$\text{Or } x^{a'} \times ax'y' - y'^2 = c^2\left\{\frac{(\sqrt{4 + a^2} - a)x' + 2y'}{(\sqrt{4 + a} + a)x' - 2y'}\right\}^{\frac{a}{\sqrt{4+a^2}}}$$

whence, in certain cases, the relation between x' and y' may be found, and from the equations $\left.\begin{matrix} x = x' + m \\ y = y' + n \end{matrix}\right\}$ that between x and y will be known. (See *Simpson*, vol. ii. p. 26. art. 272.)

612. To integrate $adz = (by + a^2)^{\frac{1}{2}}dy$ we have $a^2 \times (dx^2 + dy^2) = (by + a^2)\, dy^2$, x being the abscissa.

$$\therefore a^2dx^2 = bydy^2$$

$$\therefore adx = \sqrt{by}.dy$$

$$\therefore ax = \frac{2\sqrt{b}}{3}.y^{\frac{3}{2}} + c$$ which gives the relation required.

613. To integrate $\frac{d^2y}{dt^2} + a^2y = 0$, put $\frac{dy}{dt} = p$

$$\text{Then } \frac{d^2y}{dt^2} = \frac{dp}{dt} = dp \times \frac{p}{dy} = \frac{pdp}{dy}$$

$$\text{Hence } \frac{pdp}{dy} + a^2y = 0$$

$$\text{Or } pdp + a^2ydy = 0$$

$$\therefore \frac{p^2}{2} + \frac{a^2y^2}{2} + c = 0$$

$$\therefore \frac{dy^2}{dt^2} + a^2y^2 - 2c = 0$$

$$\therefore dt^2 = \frac{dy^2}{2c - a^2y^2}$$

$$\text{And } dt = \frac{dy}{\sqrt{2c - a^2y^2}} = \frac{dy}{\sqrt{2c}.\sqrt{1 - \frac{a^2y^2}{2c}}} = \frac{1}{a}.\frac{\frac{a}{\sqrt{2c}}dy}{\sqrt{1 - \frac{a^2y^2}{2c}}}$$

$$\therefore t = \frac{1}{a}. \sin.^{-1} \frac{ay}{\sqrt{2c}} + c'$$

Otherwise.

Let $y = \cos. at$, then $dy = - adt. \sin. at$, and $d^2y = - a^2dt^2 \times \cos. at$

$$\therefore \frac{d^2y}{dt^2} + a^2y = - a^2 \cos. at + a^2y = 0$$

$\therefore$ y has been rightly assumed $= \cos. at$

Again, let $y = \sin. at$, then $dy = adt \cos. at$, and $d^2y = - a^2dt^2 \times \sin. at$

$\therefore \frac{d^2y}{dt^2} + a^2y = - a^2 \sin. at + a^2y = 0$, also verifying the equation.

Hence $y = c. \cos. at + c'. \sin. at$ is the general solution.

614. To integrate $\frac{dv}{\sqrt{Av^2 + Bv + C}} + \frac{dz}{\sqrt{Az^2+Bz+C}} = 0$

we shall first simplify the expression, by multiplying by $\sqrt{A}$.

Then $$\frac{dv}{\sqrt{v^2 + \frac{B}{A}v + \frac{C}{A}}} + \frac{dz}{\sqrt{z^2 + \frac{B}{A}z + \frac{C}{A}}} = 0$$

Put $\frac{B}{A} = 2a, \frac{C}{A} = b$, and $v + a = x, z + a = y$

Then, by substitution, we get $\frac{dx}{\sqrt{x^2+b-a^2}} + \frac{dy}{\sqrt{y^2+b-a^2}} = 0$

$\therefore$ (whether $b - a^2$ be positive or negative) we have

$$l(x + \sqrt{x^2 + b - a^2}) + l(y + \sqrt{y^2 + b - a^2}) = lc'$$

$$\therefore (x + \sqrt{x^2 + b - a^2}) \times (y + \sqrt{y^2 + b - a^2}) = c'$$

Or $$v + \frac{B}{2A} + \sqrt{v^2 + \frac{B}{A}v + \frac{C}{A}} = \frac{c'}{+\frac{B}{2A} + \sqrt{z^2 + \frac{B}{A}z + \frac{C}{A}}}$$

the relation required.

Otherwise.

Let v and z be considered functions of t, and put

$$\frac{dv}{dt} = \sqrt{Av^2 + Bv + C},\ \frac{dz}{dt} = \sqrt{Az^2 + Bz + C}, \text{ and } v + z = y$$

Then $\frac{dv^2}{dt^2} + \frac{dz^2}{dt^2} = A \times (v^2 + z^2) + B.(v+z) + 2C$

Hence $\frac{d^2v + d^2z}{dt^2} = A.(v+z) + B = Ay + B$

But $\frac{d^2v + d^2z}{dt^2} = \frac{d^2y}{dt^2}$

$\therefore\ \frac{d^2y}{dt^2} = Ay + B,\ \therefore\ \frac{2dy \times d^2y}{dt^2} = 2Aydy + 2Bdy$

$\therefore\ \frac{dy^2}{dt^2} = Ay^2 + 2By + c'.$

$\therefore\ \frac{dy}{dt} = \sqrt{Ay^2 + 2By + C'}$. But $y = v + z$

And $\therefore \frac{dy}{dt} = \frac{dv}{dt} + \frac{dz}{dt} = \sqrt{Av^2 + Bv + C} + \sqrt{Az^2 + Bz + C}$

$\therefore\ \sqrt{Av^2 + Bv + C} + \sqrt{Az^2 + Bz + C} = \sqrt{A.(v+z)^2 + 2B(v+z) + C'}$
an expression which may be rationalized into
$(2C - C' - 2Avz)^2 = 4 \times (Av^2 + Bv + C) \times (Az^2 + Bz + C)$

615. To integrate $3x^2dy - 3axdy = aydx$, we must separate the variables thus, $\frac{dy}{y} = \frac{adx}{3x^2 - 3ax} = \frac{a}{3} \times \frac{dx}{x.(x-a)}$

Then, assuming $\frac{1}{x.(x-a)} = \frac{A}{x} + \frac{B}{x-a}$, we get

$A \times (x-a) + Bx = 1$; let $x = a$ and o successively.

Then $B = \frac{1}{a}$ and $A = -\frac{1}{a}$

Hence $\frac{dy}{y} = \frac{a}{3} \times \left(\frac{dx}{a(x-a)} - \frac{dx}{ax}\right) = \frac{1}{3} \times \left(\frac{dx}{x-a} - \frac{dx}{x}\right)$

$\therefore ly = \frac{1}{3} l.(x - a) - \frac{1}{3} l.x + l.c$ (making the constant a logarithm).

$$\therefore y = \left(\frac{x-a}{x}\right)^{\frac{1}{3}} \times c$$

616. To integrate $adx = \frac{xyd^2y + xdy^2}{dx}$, we have

$$\frac{adx}{x} = \frac{yd^2y + dy^2}{dx}$$

$$\therefore alx = \frac{ydy}{dx} + c \text{ (}dx\text{ being constant)}$$

$$\therefore adxlx = ydy + cdx$$

Hence $\frac{y^2}{2} + cx + c' = a \int dxlx = axlx - a\int \frac{xdx}{x}$

$$= axlx - ax$$

$$\therefore y^2 + 2.(c+a)x - 2axlx + 2c' = 0$$

617. To integrate $dx.(a+bx+cy) = dy.(d+ex+fy)$,

Assume $\left.\begin{array}{l} x = \alpha + u \\ y = \beta + v \end{array}\right\}$ α and β being constant indeterminates.

Then $du \times (a + b.\overline{\alpha+u} + c\,\overline{\beta+v}) = dv \times (d + e.\overline{\alpha+u} + f.\overline{\beta+v})$

$\therefore du.(a + b\alpha + c\beta + bu + cv) = dv.(d + e\alpha + f\beta + eu + fv).$

Assume $\left.\begin{array}{l} a + b\alpha + c\beta = 0 \\ d + e\alpha + f\beta = 0 \end{array}\right\}$ Then $\left.\begin{array}{l} \alpha = \dfrac{cd - fa}{fb - ce} \\ \text{And } \beta = \dfrac{ae - bd}{fb - ce} \end{array}\right\}$

And $budu + cvdu = eudv + fvdv$ which being homogeneous, put $u = vw$, and, by substitution, we get

$(bvw + cv)(vdw + wdv) = (evw + fv)\,dv$, and dividing by v, &c. there results finally $\frac{dv}{v} = -\frac{(bw+c)\,dw}{bw^2 + (c-e)w - f} = -\frac{1}{2} \times$

$$\frac{2bwdw + (c-e)\,dw}{bw^2 + (c-e)w - f} - \frac{1}{2} \times \frac{\overline{c+e}.dw}{bw^2 + (c-e)w - f}.$$

$$\therefore lv = -\frac{1}{2}\times l.\,(bw^2+(c-e)w-f)-\frac{c+e}{2b}\int\frac{dw}{w^2+\frac{c-e}{b}w-\frac{f}{b}}$$

Again, put $w+\dfrac{c-e}{2b}=z$

Then $w^2+\dfrac{c-e}{b}w-\dfrac{f}{b}=z^2-\dfrac{f}{b}+\dfrac{\overline{c-e}|^2}{4b^2}=z^2-\dfrac{4bf+\overline{c-e}|^2}{4b^2}$

And $\displaystyle\int\frac{dw}{w^2+\frac{c-e}{b}w+\frac{f}{b}}=\int\frac{dz}{z^2-\frac{4bf+\overline{c-e}|^2}{4b^2}}=\int\frac{dz}{z^2-r^2}$

$=\dfrac{1}{2r}\;l.\,\dfrac{z-r}{z+r}\;\left(r^2 \text{ being supposed } =\dfrac{4bf+\overline{c-e}|^2}{4b^2}\right)$

$\therefore lv.=-\dfrac{1}{2}\,l.\,(bw^2+(c-e)w-f)-\dfrac{c+e}{r}\times l\dfrac{z-r}{z+r}+\text{cor.}$

Let correction $= l.Q$

Then $v\times(bw^2+(c-e)w-f)^{\frac{1}{2}}\times\left(\dfrac{z-r}{z+r}\right)^{\frac{c+e}{r}}=Q$, which, substituting for v, w, z, their values in terms of x and y, will give the relation required.

618. For the integration of $\dfrac{pdx}{x}+\dfrac{rdy}{y}=\dfrac{x^m dx}{ay^n}$, see Problem 606.

To integrate $ay+\dfrac{bdy}{dx}+\dfrac{cd^2y}{dx^2}=0$, put $\dfrac{dy}{dx}=p$

Then $\dfrac{d^2y}{dx^2}=\dfrac{dp}{dx}$ and, by substitution, we have

$$ay+bp+c\frac{dp}{dx}=0,\text{ or } ay+bp+\frac{cpdp}{dy}=0$$

This last form being homogeneous, put $y=vp$

Then $avp+bp+\dfrac{c\,.\,pdp}{vdp+pdv}=0$

Or $(av+b)\,vdp+(av+b)\,pdv+cdp=0$

$\therefore dp\times(c+av^2+bv)=-pdv\times(av+b)$

$$\therefore \frac{dp}{p} = \frac{-(av+b)\,dv}{c+bv+av^2} = -\frac{1}{2} \times \frac{2avdv+bdv+bdv}{c+bv+av^2}$$

$$\therefore l.p = -\frac{1}{2} l.\,(c+bv+av^2) - \frac{1}{2}\int \frac{bdv}{c+bv+av^2}$$

$$\text{Now} \int \frac{bdv}{c+bv+av^2} = \frac{b}{a}\int \frac{dv}{\frac{c}{a}+\frac{b}{a}v+v^2}$$

Assume $v + \frac{b}{2a} = u$, then $v^2 + \frac{b}{a}v + \frac{c}{a} = u^2 + \frac{c}{a} - \frac{b^2}{4a^2}$

And (putting $\frac{c}{a} - \frac{b^2}{4a^2} = \pm r^2$ according as $\frac{c}{a}$ is $>$ or $< \frac{b^2}{4a^2}$)
we get

$$\int \frac{bdv}{c+bv+av^2} = \frac{b}{a}\int \frac{du}{u^2 \pm r^2} = \frac{b}{ar^2}\tan.^{-r} u \text{ or } \frac{b}{2ra}\, l\, \frac{u-r}{u+r}$$

according as + or − is taken.

$$\left.\begin{array}{l} \text{Hence } lp = -\frac{1}{2} l.\,(c+bv+av^2) - \frac{b}{2ar^2}\tan.^{-r} u + \text{cor.} \\ \text{Or } l.p = -\frac{1}{2} l.\,(c+bv+av^2) - \frac{b}{4ra} \times l\,\frac{u-r}{u+r} + \text{cor.} \end{array}\right\} \text{which}$$

values, by substituting for v, u their values in terms of p and y, we shall have the relation of p to y, or of $\frac{dy}{dx}$ to y. Let this relation be expressed by the equation $\frac{dy}{dx} = f(y)$, $f(y)$ signifying function of y.

Then $\int \frac{dy}{f(y)} = \int dx$, in which the variables are separated, will give the relation of y to x, as required.

Otherwise.

Make $y = e^{\int udx}$, then $\frac{dy}{dx} = ue^{\int udx}$, and $\frac{d^2y}{dx^2} = \frac{du}{dx}e^{\int udx} + u^2e^{\int udx}$

$$\therefore ae^{\int udx} + bue^{\int udx} + \frac{cdu}{dx}e^{\int udx} + cu^2e^{\int udx} = 0$$

$$\therefore a + bu + \frac{cdu}{dx} + cu^2 = 0$$

$$\therefore\ dx = \frac{-cdu}{a+bu+cu^2} = \frac{-du}{\frac{a}{c}+\frac{b}{c}u+u^2}$$

(1) Let $u^2+\frac{b}{c}u+\frac{a}{c} = (u-m)\times(u-n)$ then

$$\left.\begin{array}{l}\text{if } u=m, \text{ we have } y=e\int^{udx}=e\int^{mdx}=e^{mx+A}=e^A\times e^{mx}\\ \text{if } u=n, \text{ we have } y=e\int^{udx}=e\int^{ndx}=e^{nx+B}=e^B\times e^{nx}\end{array}\right\}$$

A and B the constants, which in the differentiation, disappeared.

$\therefore$ the complete integral is $y=e^A\times e^{mx}+e^B\times e^{nx}$

(2) Let $u^2+\frac{b}{c}u+\frac{a}{c}=(u-m)^2$, then $m=n$

And $y=e^{mx}\times(e^A+e^B)$

(3) Let $u^2+\frac{b}{c}u+\frac{a}{c}=(u-m'-n'\sqrt{-1})\,(u-m'+n'\sqrt{-1})$

Then $y=e^A\times e^{(m'+n'\sqrt{-1})x}+e^B\times e^{(m'-n'\sqrt{-1})x}=e^{m'x}\times(e^A\times e^{n'\sqrt{-1}.x}+e^B\times e^{-n'\sqrt{-1}\,x})$

$$\left.\begin{array}{l}\text{Now } e^{n'\sqrt{-1}.x}+e^{-n'\sqrt{-1}.x}=2\cos.\,n'x\\ \text{And } e^{n'\sqrt{-1}.x}-e^{\quad\sqrt{-1}.x}=2\sqrt{-1}.\ \sin.\,n'x\end{array}\right\}$$

$$\left.\begin{array}{l}e^{n'\sqrt{-1}.x}=\cos.\,n'x+\sqrt{-1}\,\sin.\,n'x\\ \text{And } e^{-n'\sqrt{-1}.x}=\cos.\,n'x-\sqrt{-1}\,\sin.\,n'x\end{array}\right\}$$

Hence, $y=e^{m'x}\times\{(e^A+e^B)\cos.\,n'x+(e^A-e^B)\sqrt{-1}.\sin.n'x\}$

619. To integrate $\frac{xdy-ydx}{x^2+y^2}=Kd\theta$, we have

$$\frac{\left(\frac{xdy-ydx}{x^2}\right)}{1+\frac{y^2}{x^2}}=\frac{d.\frac{y}{x}}{1+\left(\frac{y}{x}\right)^2}$$

$$\therefore\int\frac{xdy-ydx}{x^2+y^2}=\tan.^{-1}\frac{y}{x}+c'$$

And $\int Kd\theta=K\theta+c''$

$$\therefore \tan.^{-1}\frac{y}{x} = K\theta + c'' - c' = K\theta + c \; (c = c'' - c')$$

$$\therefore \frac{y}{x} = \tan.(K\theta + c)$$

$$\text{And } y = x \times \tan.(K\theta + c)$$

620. To integrate $adx + (bx + cy)\,dx = d \times dy + (bx + cy)\,ndx$, we have $adx - d \times dy + (1 - n).(bx + cy)\,dx = 0$

Let $bx + cy = u$

Then $dy = \dfrac{du - bdx}{c}$, and we get

$$(a + \overline{1-n}.u)\,dx - \frac{d}{c} \times du + \frac{db}{c} \times dx = 0$$

$$\therefore dx = \frac{\frac{d}{c} \times du}{a + \frac{db}{c} + (1-n).u}$$

$$\therefore x = \frac{d}{c.(1-n)} \times l(a + \frac{db}{c} + \overline{1-n}.u) + \text{const.}$$

$$= \frac{d}{c.(1-n)} \times l.(a + \frac{db}{c} + \overline{1-n}.bx + \overline{1-n}.cy) + l.c'$$

$$\text{Or } e^x = \frac{dc'}{c.(1-n)} \times (a + \frac{db}{c} + \overline{1-n}.bx + \overline{1-n}.cy)$$

This result exhibits the relation x and y implicitly. The method pursued in (617) may conduct the Reader to a less complicated form.

621. Let
$$\left.\begin{aligned} \int ydx &= F_1 \\ \int yxdx &= F_2 \\ \int yx^2dx &= F_3 \\ \&c. &= \&c. \\ \int yx^{n-1}dx &= F_n \end{aligned}\right\} \; F_1, F_2, \&c. \text{ being the given integrals.}$$

Then $A = \int y dx = F_1$

$$B = \int A dx = \int dx.F_1 = xF_1 - \int x dF_1 = xF_1 - F_2$$

(from the form $\int u dv = uv - \int v du$.

Now, if we observe that
$$\begin{cases} F_2 = \int x dF_1 \\ F_3 = \int x^2 dF_1 \text{ or } = \int x dF_2 \\ F_4 = \int x^3 dF_1 \text{ or} = \int x^2 dF_2 \text{ or } = \int x dF_3 \\ \&c. = \&c. \\ \text{And } F_n = \int x^{n-1} dF_1 \text{ or} = \int x^{n-2} dF_2, \text{ or} = \int x^{n-3} dF_3, \end{cases}$$
&c., we shall have no difficulty in accomplishing the successive integrations.

$$\text{For } C = \int B dx = \int x dx F_1 - \int dx F_2 = \begin{Bmatrix} \frac{x^2}{2}.F_1 - \frac{1}{2}\int x^2 dF_1 \\ - xF_2 + \int x dF_2 \end{Bmatrix}$$

But the terms on the right (not considering coefficients) are each equal to F_3; $\therefore$, collecting the coefficients, we have

$$C = \frac{x^2}{2} \times F_1 - x \times F_2 + \frac{1}{2} \times F_3$$

$$\text{Again, } D = \int C dx = \begin{Bmatrix} \frac{x^3}{2.3}.F_1 - \frac{1}{2.3}\int x^3 d F_1 \\ - \frac{x}{2}.F_2 + \frac{1}{2}.\int x^2 dF_2 \\ + \frac{x}{2}.F_3 - \frac{1}{2}\int x dF_3 \end{Bmatrix} = \frac{x^3}{2.3}.F_1 -$$

$$\frac{x^2}{2}.F_2 + \frac{x}{2}.F_3 - \frac{1}{2.3}.F_4$$

It appears, then, that the integrations are effected by integrating each term separately as if F_1, F_2, &c. were constant; thus obtaining every term but the last, and collecting the coefficients, with their signs changed, of these terms for that of the last term. The signs are alternately positive and negative, and the coefficients, equally distant from either end, are the same. It is also evident, that the index of x and that of F, in each term, are together equal to the number of terms. The law will be better shewn by actually exhibiting the forms.

$$A = F_1$$
$$B = x.F_1 - F_2$$
$$C = \frac{x^2}{1.2}.F_1 - x.F_2 + \frac{1}{1.2}.F_3$$
$$D = \frac{x^3}{1.2.3}.F_1 - \frac{x^2}{1.2}.F_2 + \frac{x}{1.2}.F_3 - \frac{x}{1.2.3}.F_4$$
$$E = \frac{x^4}{1.2.3.4}.F_1 - \frac{x^3}{1\times 1.2.3}.F_2 + \frac{x^2}{1.2\times 1.2}.F_3 - \frac{x}{1.2.3}.F_4 + \frac{1}{1.2.3.4}.F_5$$
$$F = \frac{x^5}{1....5}.F_1 - \frac{x^4}{1\times 1....4}.F_2 + \frac{x^3}{1.2\times 1.2.3}.F_3 - \frac{x^2}{1.2\times 1.2.3}.F_4 + \frac{x}{1\times 1....4}.F_5 - \frac{1}{1....5}.F_6$$
&c. = &c.
$$Q = \frac{x^{n-2}}{1...(n-2)}.F_1 - \frac{x^{n-3}}{1\times 1....(n-3)}.F_2 + \frac{x^{n-4}}{1.2\times 1...n-1}\times F_3 - \&c. .... \pm \frac{x^2}{1.2\times 1....(n-4)}.F_{n-3} \mp \frac{x}{1\times 1....(n-3)}.F_{n-2} \pm \frac{1}{1....(n-2)}.F_{n-1}$$
And $R = \frac{x^{n-1}}{1...(n-1)}.F_1 - \frac{x^{n-2}}{1\times 1....(n-2)}.F_2 + \frac{x^{n-3}}{1.2\times 1....(n-3)}.F_3 - \frac{x^{n-4}}{1.2.3\times 1....(n-4)}.F_4 + \frac{x^{n-5}}{1...4\times 1...(n-5)}.F_5$ — &c. to $\frac{n+1}{2}$th or $\frac{n}{2}$th term, according as n is odd or even, after which the coefficients recur with the same or different signs, according as (n) is odd or even.

When (n) is odd, the middle term $= \pm \frac{x^{\frac{n-1}{2}}}{\left(1.2.3....\frac{n-1}{2}\right)^2} \times F_{\frac{n+1}{2}}$

+ or — being taken according $\frac{n-1}{2}$ is even or odd, i. e., according as (n) is of the forms $4m+1$, or $4m-1$.

From the above investigation, we learn that the series

$$\frac{1}{1\ldots 2n+1}-\frac{1}{1.2\times 1\ldots .2n}+\frac{1}{1.2.3\times 1\ldots 2n-1}-\&c.\pm\frac{1}{1.2\ldots n\times 1\ldots(n+2)}$$

$$=\pm\frac{1}{2}\times\frac{1}{(1.2.3\ldots .n+1)^2}$$

622. To integrate $dy + Pydx = Qdx$ (which is the *Linear Equation.*)

Let $y = uv$

Then $dy = vdu + udv$, and the equation becomes

$vdu + udv + Puvdx = Qdx$, now, since we have assumed $y =$ the product of two new variables, we may make another assumption

$udv + Puvdx = 0$, $\therefore dv + Pvdx = 0$, and $vdu = Qdx$

and we have $\frac{dv}{v} = -Pdx$, $\therefore l.v = -\int Pdx$

$$\therefore v = e^{-\int Pdx - c}, \quad \therefore du = \frac{Qdx}{v} = Q\,dx \times e^{\int Pdx + c}$$

$$\therefore u = \int Qdx \times e^{\int Pdx + c} + c'$$

And, by substitution, we finally obtain

$$y = vu = -e^c.e^{-\int Pdx} \times \{\int Qdx \times e^c.e^{\int Pdx} + c'\}$$

623. To integrate $y - \frac{d^4y}{dz^4} = 0$, assume $y = e^{mz}$

$$\left.\begin{aligned}\text{Then } \frac{dy}{dz} &= me^{mz}\\ \frac{d^2y}{dz^2} &= m^2e^{mz}\\ \frac{d^3y}{dz^3} &= m^3e^{mz}\\ \frac{d^4y}{dz^4} &= m^4e^{mz}\end{aligned}\right\}\ \therefore \text{ substituting and dividing by } e^{mz}, \text{ we get}$$

$$m^4 = 1, \ \therefore m^2 = \pm 1, \text{ and } m = \pm 1, \text{ or } \pm\sqrt{-1},$$

$\therefore\ y = e^z$, or $= e^{-z}$, or $= e^{z\sqrt{-1}}$, or $= e^{-z\sqrt{-1}}$, and are true values of y, because they satisfy the equation. For the same reason, and because four differentiations cause to disappear, four constants,

$$y = ce^z + c'e^{-z} + Ce^{z\sqrt{-1}}\ C'e^{-z\sqrt{-1}}, \text{ is the general value.}$$

$$\left.\begin{array}{l}\text{But } e^{z\sqrt{-1}} + e^{-z\sqrt{-1}} = 2\cos. z \\ \text{And } e^{z\sqrt{-1}} - e^{-z\sqrt{-1}} = 2\sqrt{-1}.\ \sin.\ z\end{array}\right\}$$

$$\left.\begin{array}{l}\therefore\ Ce^{z\sqrt{-1}} = C\cos.\ z + C\sqrt{-1}.\ \sin.\ z \\ \text{And } C'e^{z\sqrt{-1}} = C'.\cos. z - C'.\sqrt{-1}.\ \sin. z\end{array}\right\}$$

$$\therefore\ y = ce^z + c'e^{-z} + (C+C').\ \cos.\ z + (C-C').\sqrt{-1}.\ \sin.\ z$$

The above assumption is not to be depended upon in all cases. It will, however, conduct us to true results in the general form

$\dfrac{d^ny}{dx^n} + A.\ \dfrac{d^{n-1}y}{dx^{n-1}} + \ldots.\ Ly = 0$, when A, B, C, &c. are constant.

When, however, those coefficients are functions of x and y, the assumptions are erroneous. See *Lacroix*, *Garnier*, *Euler*, &c.

624. To integrate $\dfrac{dy}{y} - \dfrac{dx}{x} = \dfrac{x^m dx}{ay\sqrt{n}}$, we have

$dy - \dfrac{y}{x}\,dx = \dfrac{x^m dx}{a\sqrt{n}}$, which is a linear equation.

Assume $\therefore\ y = uv$

Then $udv + vdu - \dfrac{uv}{x}\,dx = \dfrac{x^m dx}{a\sqrt{n}}$

Again, put $du = u \times \dfrac{dx}{x}$, then $udv = \dfrac{x^m dx}{a\sqrt{n}}$

And $\dfrac{du}{u} = \dfrac{dx}{x}$

$\therefore\ lu = lx + lc = lcx$ (the constant may be assumed any function of a constant.)

$\therefore\ u = cx$

Also $dv = \dfrac{x^m dx}{a\sqrt{n} \times u} = \dfrac{x^m dx}{ca\sqrt{n}\,x} = \dfrac{x^{m-1}dx}{ac\sqrt{n}}$

$$\therefore v = \frac{x^m}{mac\sqrt{n}} + c'$$

$$\text{Hence } y = uv = cx \times \left\{\frac{x^m}{mac\sqrt{n}} + c'\right\}$$

To integrate $xdx + ady = b.\sqrt{dx^2+dy^2}$, divide by dx

$$\text{Then } x + a.\frac{dy}{dx} = b.\sqrt{1 + \frac{dy^2}{dx^2}}$$

$$\therefore x^2 + 2ax.\frac{dy}{dx} + a^2.\frac{dy^2}{dx^2} = b^2 + b^2\frac{dy^2}{dx^2}$$

$$\therefore \frac{dy^2}{dx^2} - \frac{2ax}{b^2-a^2}.\frac{dy}{dx} = \frac{x^2-b^2}{b^2-a^2}$$

$$\therefore \left(\frac{dy}{dx}\right)^2 - \frac{2ax}{b^2-a^2}.\left(\frac{dy}{dx}\right) + \frac{a^2x^2}{(b^2-a^2)^2} = \frac{a^2x^2}{(b^2-a^2)^2} + \frac{x^2-b^2}{b^2-a^2}$$

$$= \frac{b^2}{(b^2-a^2)^2} \times (x^2+a^2-b^2)$$

$$\therefore \frac{dy}{dx} = \frac{ax}{b^2-a^2} \pm \frac{b}{b^2-a^2} \times \sqrt{x^2+a^2-b^2}$$

$$\therefore dy = \frac{axdx}{b^2-a^2} \pm \frac{b}{b^2-a^2} \times dx.\sqrt{x^2+a^2-b^2}$$

$$\therefore y = \frac{ax^2}{2.(b^2-a^2)} \pm \frac{b}{b^2-a^2} \times \int dx\sqrt{x^2+r^2}\ (r^2=a^2-b^2)$$

Again, assume $x.\sqrt{x^2+r^2} = u$

$$\text{Then } du = dx.\sqrt{x^2+r^2} + \frac{x^2dx}{\sqrt{x^2+r^2}} = dx.\sqrt{x^2+r^2} +$$

$$\frac{(x^2+r^2-r^2)\,dx}{\sqrt{x^2+r^2}} = 2dx\sqrt{x^2+r^2} - \frac{r^2dx}{\sqrt{x^2+r^2}}$$

$$\therefore \int dx\sqrt{x^2+r^2} = \frac{u}{2} + \frac{r^2}{2}\int\frac{dx}{\sqrt{x^2+r^2}}$$

$$= x\sqrt{x^2+r^2} + \frac{r^2}{2}.\ l.\ (x + \sqrt{x^2+r^2})$$

$$\text{Hence } y = \frac{1}{2.(b^2-a^2)} \times (x^2 \pm bx\sqrt{x^2+a^2-b^2}) \mp$$

$$\frac{br^2}{2\,(b^2-a^2)}\, l.\,(x+\sqrt{x^2+a^2-b^2}) + \text{cor.}$$

The equation $ydx - xdy = dx.\sqrt{x^2+y^2}$, being homogeneous, assume $y = xv$; then $vxdx - xvdx - x^2dv = dx.\sqrt{x^2 + x^2v^2}$

$$\therefore \frac{dx}{x} = -\frac{dv}{\sqrt{1+v^2}}.$$

$$\therefore lx = -l.(v + \sqrt{1+v^2}) + lc$$

$$\therefore x = \frac{c}{v+\sqrt{1+v^2}} = \frac{c}{\frac{y}{x}+\sqrt{1+\frac{y^2}{x^2}}}$$

$$= \frac{cx}{y+\sqrt{x^2+y^2}}$$

$$\therefore c = y + \sqrt{x^2+y^2}$$

And $x^2 + y^2 = c^2 + y^2 - 2cy$

$x^2 = c^2 - 2cy$. See *Lacroix*, p. 385.

625. To integrate $y^2dy = 3yxdx - x^2dy$, assume $x = vy$ (the equation being homogeneous.)

Then $y^2dy = 3\,vy^2.\,d.(vy) - v^2y^2dy$

$$\therefore dy = 3v^2.dy + 3yvdv - v^2dy = 2v^2dy + 3yvdv$$

$$\therefore dy \times (1 - 2v^2) = 3yvdv$$

Or $\dfrac{dy}{y} = \dfrac{3vdv}{1-2v^2}$

$$\therefore ly = -\frac{3}{4}\int\frac{-4vdv}{1-2v^2} = l.\,c - \frac{3}{4}.\,l.\,(1-2v^2)$$

$\therefore y = \dfrac{c}{(1-2v^2)^{\frac{3}{4}}} = \dfrac{cy^{\frac{3}{2}}}{(y^2-2x^2)^{\frac{3}{4}}}$. Hence $y^2 - c^{\frac{4}{3}}\,y^{\frac{2}{3}}$ $= 2\,x^2$.

626. $x^2dy + 2xydy = b^2dx - y^2dy$, we have

$$dy \times (x^2 + 2xy + y^2) = b^2dx$$

Or $dy = \dfrac{b^2dx}{(x+y)^2}$

Put $x+y=u$; then $dx=du-dy$

$$\therefore\ dy=\frac{b^2.(du-dy)}{u^2}=\frac{b^2du}{u^2}-\frac{b^2dy}{u^2}$$

$$\therefore\ dy\times\left(1+\frac{b^2}{u^2}\right)\left(=dy\times\frac{u^2+b^2}{u^2}\right)=\frac{b^2du}{u^2}$$

$$\therefore\ dy=\frac{b^2du}{b^2+u^2}$$

$$\therefore\ y=\tan.^{-1}u+c=\tan.^{-1}(x+y+c')$$

$$\therefore\ x+y+c'=\tan.\ y \text{ to radius } b$$

$$\therefore\ x=\tan.(y)-y-c'$$

To integrate $cx^2dx+ydx=ady$, which is a linear equation,

Assume $y=uv$, then $dy=udv+vdu$

And $cx^2dx+uvdx=audv+avdu$

Again, assume $uvdx=avdu$, then $cx^2dx=audv$

$$\therefore\ dx=\frac{adu}{u}$$

$$\text{And } \frac{x}{a}=lu+lc'=l.uc'$$

$$\therefore\ u=\frac{e^{\frac{x}{a}}}{c'}$$

$$\text{Also } dv=\frac{cx^2dx}{au}=\frac{cc'}{a}x^2dx.e^{-\frac{x}{a}}$$

$$\therefore\frac{v}{cc'}=\int -x^2\times\left(-\frac{dx}{a}e^{-\frac{x}{a}}\right)=-x^2e^{-\frac{x}{a}}-\int\left(-2xdx.e^{-\frac{x}{a}}\right)$$

$$=-x^2e^{-\frac{x}{a}}-2axe^{-\frac{x}{a}}+\int -2a^2\times\left(-\frac{dx}{a}e^{-\frac{x}{a}}\right)=-x^2e^{-\frac{x}{a}}$$

$$-2axe^{-\frac{x}{a}}-2a^2e^{-\frac{x}{a}}+c''$$

$$\therefore\ y=uv=ce^{\frac{x}{a}}\times\left\{c''-2a^2e^{-\frac{x}{a}}-2axe^{-\frac{x}{a}}-x^2e^{-\frac{x}{a}}\right\}$$

627. To integrate $y^ndy-a^{n-1}xdy+c^{n-1}ydx=0$, which is *Linear*, with respect to x, assume $x=uv$.

Then $y^{n-1}dy - a^{n-1}\frac{dy}{y} \times uv + c^{n-1}vdu + c^{n-1}udv = 0$

Again, assume $\frac{a^{n-1}dy}{y}. uv = c^{n-1}vdu$, then $c^{n-1}udv = -y^{n-1}dy$

$$\therefore a^{n-1} \times \frac{dy}{y} = c^{n-1} \times \frac{du}{u}$$

$$\therefore c^{n-1}\, l.u = a^{n-1}\, ly + lc'$$

$$\therefore u^{(c^{n-1})} = c'y^{(a^{n-1})}$$

$$\therefore u = (c')^{c^{-n+1}} \times y^{\left(\frac{a}{c}\right)^{n-1}} = c''.y^m \text{ (putting } (c')^{c^{-n+1}} = c''$$

and $\left(\frac{a}{c}\right)^{n-1} = m)$

$$\therefore dv = \frac{-y^{n-1}dy}{c^{n-1}u} = -\frac{1}{c''.c^{n-1}} \times y^{n-m-1}dy$$

$$\therefore v = -\frac{1}{c''.c^{n-1}(n-m)} \times y^{n-m} + C$$

$$\therefore x = uv = c''y^m \times \left\{C - \frac{y^{n-m}}{c''.c^{n-1}(n-m)}\right\}$$

To integrate $a^2dy^2 + bxdy^2 = c^2d^2y$, multiply by $\frac{dx}{dy^2}$,

Then $a^2dx + bxdx = c^2dx \times \frac{d^2y}{dy^2} = c^2dx \times \frac{d(dy)}{dy^2}$

$$\therefore a^2x + \frac{bx^2}{2} = -c^2dx \times \frac{1}{dy} + C \text{ (} dx \text{ being constant)}$$

Hence $-dy = \frac{c^2dx}{\frac{bx^2}{2} + a^2x - C} = \frac{c^2}{2b} \times \frac{dx}{x^2 + \frac{2a^2}{b}x - \frac{2C}{b}}$

Again, assume, $x + \frac{a^2}{b} = u$

Then $x^2 + \frac{2a^2}{b} - \frac{2C}{b} = u^2 - \left(\frac{a^4}{b^2} + \frac{2C}{b}\right) = u^2 - r^2$

$$\therefore -dy = \frac{c^2}{4rb} \times \frac{2rdu}{u^2 - r^2}$$

And $-y = \frac{c^2}{4rb} \times l.\frac{u-r}{u+r} + l.c'$ which expresses the relation required.

Since $\frac{dx}{x} - \frac{dy}{y} : \frac{dx}{y} - \frac{dy}{x} :: n : 1$, we have

$ydx - xdy = nxdx - nydy$, which is homogeneous.

Put $\therefore y = xv$, and our equation becomes

$vxdx - vxdx - x^2dv = nxdx - nxv \times d.(xv)$

$\therefore xdv = -ndx + nv^2dx + nvxdv$

$$\therefore dv \times \frac{1-nv}{v^2-1} = \frac{ndx}{x}$$

$$\therefore \int\frac{dv}{v^2-1} - \int\frac{nvdv}{v^2-1} = n\,lx + lc$$

$$\text{or } \frac{1}{2}l.\frac{v-1}{v+1} - \frac{n}{2}l.(v^2-1) = nl.x + lc$$

$$\therefore \frac{v-1}{(v+1).(v^2-1)^n} = c^2.x^{2n}$$

$$\text{or } \frac{1}{(v+1)^{n+1}.(v-1)^{n-1}} = c^2x^{2n}$$

$$\text{or } \frac{1}{\left(\frac{y}{x}+1\right)^{n+1}\left(\frac{y}{x}-1\right)^{n-1}} = \frac{x^{2n}}{(y+x)^{n+1}.(y-x)^{n-1}} = c^2\,x^{2n}$$

$$\therefore (y+x)^{n+1}.(y-x)^{n-1} = \frac{1}{c^2}$$

$\therefore (y^2 - x^2)^{n+1} = \frac{(y-x)^2}{c^2}$ an equation expressing the relation between y and x, and $\therefore$ the nature of the curve.

Otherwise.

Since $\frac{dx}{x} - \frac{dy}{y} : \frac{dx}{y} - \frac{dy}{x} :: n : 1$, componendo and dividendo we get $\frac{dx-dy}{x} + \frac{dx-dy}{y} : \frac{dx+dy}{x} - \frac{dx+dy}{y} :: n+1 : n-1$

$\therefore (dx-dy).(x+y) : (dx+dy)(y-x) :: n+1 : n-1$

$$\therefore \frac{dx-dy}{x-y} = -\frac{n+1}{n-1} \times \frac{dx+dy}{x+y}$$

$$\therefore l.(x-y) = -\frac{n+1}{n-1} \times l.(x+y) + lc$$

$\therefore (x-y)^{n-1} = \frac{c'}{(x+y)^{n+1}}$ a conclusion, equivalent to the former.

628. To integrate $d^2y\sqrt{ay} = dx^2$, we have

$$\frac{d^2y}{dx^2} = \frac{1}{\sqrt{a}} \cdot \frac{1}{y^{\frac{1}{2}}}$$

$$\therefore \frac{dy.d.(dy)}{dx^2} = \frac{1}{\sqrt{a}} \cdot \frac{dy}{y^{\frac{1}{2}}}$$

$$\therefore \frac{dy^2}{2dx^2} = \frac{2}{\sqrt{a}} . \sqrt{y} + c$$

$$\therefore \frac{dy^2}{dx^2} = \frac{2(c\sqrt{a}+2\sqrt{y})}{\sqrt{a}}$$

$$\therefore \frac{dy}{dx} = \pm \sqrt{2}\frac{\sqrt{c\sqrt{a}+2\sqrt{y}}}{a^{\frac{1}{4}}}$$

$$\therefore \frac{a^{\frac{1}{4}}dy}{\sqrt{c\sqrt{a}+2\sqrt{y}}} = \pm \sqrt{2}.\,dx$$

Now, put $2\sqrt{y} = u$ and $c\sqrt{a} = b$

Then $y = \frac{u^2}{4}$ $\therefore dy = \frac{udu}{2}$

And $\frac{a^{\frac{1}{4}}}{2^{\frac{3}{2}}} \times \frac{udu}{\sqrt{b+u}} = \pm dx$

$$\therefore \pm x = \frac{a^{\frac{1}{4}}}{2^{\frac{3}{2}}} \times \int \frac{udu}{\sqrt{b+u}} = \frac{a^{\frac{1}{4}}}{2^{\frac{3}{2}}} \int \frac{(b+u-b)du}{\sqrt{b+u}}$$

$$= \frac{a^{\frac{1}{4}}}{2^{\frac{3}{2}}} \times \left(-\int \frac{bdu}{\sqrt{b+u}} + \int du\sqrt{b+u} \right)$$

$$= \frac{ba^{\frac{1}{4}}}{2^{\frac{1}{2}}}\sqrt{b+u} + \frac{a^{\frac{1}{4}}}{3.2^{\frac{1}{2}}}(b+u)^{\frac{3}{2}} + c'$$

$$= \frac{c.a^{\frac{3}{4}}}{2^{\frac{1}{2}}}.\sqrt{c\sqrt{a}+2\sqrt{y}} + \frac{a^{\frac{1}{4}}}{3.2^{\frac{1}{2}}}(c.\sqrt{a}+2\sqrt{y})^{\frac{3}{2}}$$

$+ c'$ the relation required.

629. To integrate $ay^m dy = cx^n dy - ay\, x^{n-1} dx$, assume $x^n = u$, then we have

$$ay^m dy = cudy - \frac{ay}{n} du$$

$\therefore\ ny^{m-1}dy = \frac{cn}{a}.u\frac{dy}{y} - du$, which being linear with respect to u, assume $u = vz$

Then $ny^{m-1}dy = \frac{cn}{a}\ \frac{vz\,dy}{y} - zdv - vdz$.

Again, put $\frac{cn}{a}\ \frac{vz\,dy}{y} = vdz.$

Then $\frac{dz}{z} = \frac{cn}{a}.\ \frac{dy}{y}$, and $\therefore\ dv = \frac{-ny^{m-1}dy}{z}$

Also $l.\,z = \frac{cn}{a}\ ly + l.\mathrm{C}$ (the correction being $= l.\mathrm{C}$)

$$\therefore\ z = \mathrm{C} \times y^{\frac{cn}{a}}$$

$$\text{And } v = \int\frac{-ny^{m-1}dy}{z} = \int\frac{-ny^{m-\frac{cn}{a}-1}}{\mathrm{C}} = \frac{-n}{\mathrm{C}\times\left(m-\frac{cn}{a}\right)}$$

$\times\ y^{m-\frac{cn}{a}} + \mathrm{C}'$

$$= \frac{an}{\mathrm{C}\times(cn-am)}\, y^{\frac{am-cn}{a}} + \mathrm{C}'$$

$$\text{Hence } x^n = u = vz = \mathrm{C}\times y^{\frac{cn}{a}} \times \left\{\frac{an}{\mathrm{C}\times cn-am}\, y^{\frac{am-cn}{a}} + \mathrm{C}'\right\}$$

$\therefore\ x = \left(\frac{an}{cn-am} \times y^m + \mathrm{C.C}'\,y^{\frac{cn}{a}}\right)^{\frac{1}{n}}$ which expresses the algebraic relation required.

630. To integrate $az^2y^{n-2}dy = dF$, having given $dz = (b + cy^n)^m dy$.

By the form $\int udv = uv - \int vdu$, we get

$$F = \frac{a}{n-1} z^2 y^{n-1} - \frac{2a}{n-1} \int y^{n-1} z dz = \frac{a}{n-1} z^2 y^{n-1} - \frac{2a}{n-1} \times$$
$$\int z (b + cy^n)^m y^{n-1} dy$$

But $\int z (b+cy^n)^m . y^{n-1} dy = \dfrac{z.(b + cy^n)^{m+1}}{cn.(m+1)} - \displaystyle\int \frac{dz.(b+cy^n)^{m+1}}{cn.(m+1)}$

$$= \frac{z.(b + cy^n)^{m+1}}{cn.(m+1)} - \int \frac{(b+cy^n)^{2m+1} dy}{cn.(m+1)}$$

$$\therefore F = \frac{a}{n-1} z^2 y^{n-1} - \frac{2a}{n-1} . \left\{ \frac{z.(b + cy^n)^{m+1}}{cn.(m+1)} - \int \frac{(b+cy^n)^{2m+1} dy}{cn.(m+1)} \right\}$$

$$= \frac{a}{n-1} z^2 y^{n-1} - \frac{2a}{cn.(n-1).(m+1)} \times \left\{ z . (b + cy^n)^{m+1} \right.$$
$$\left. - \int (b+cy^n)^{2m+1} dy \right\}$$

We have, therefore, reduced the integration of the function to that of $(b + cy^n)^{2m+1} dy$; which may be effected in finite terms, when (m) is an integer, by expanding $(b + cy^n)^{2m+1}$ by the Binomial Theorem, and integrating each term multiplied by dy separately. If (m) be not an integer, assume

$$\left.\begin{array}{l} P_1 = (b + cy^n)^{2m+1} y \\ P_2 = (b + cy^n)^{2m} y \\ P_3 = (b + cy^n)^{2m-1} y \\ \&c. = \&c. \end{array}\right\} \quad \left.\begin{array}{l} F_1 = \int (b + cy^n)^{2m+1} dy \\ F_2 = \int (b + cy)^{2m} dy \\ \&c. = \&c. \end{array}\right\}$$

$$\begin{aligned} \text{Then } dP_1 &= (n.\overline{2m+1}+1).\, dF_1 - bn.(2m+1).dF_2 \\ dP_2 &= (n.\ 2m + 1)\ dF_2 - bn.\ 2m .\ dF_3 \\ dP_3 &= (n.\overline{2m-1}+1).\, dF_3 - bn.(2m-1).\, dF_4 \\ \&c. &= \&c. \end{aligned}$$

whence by integration and successive substitutions, F_1 will be reduced to its simplest form.

631. To integrate $(x - y).\,dxdy = y.\,dx^2 + (a - x).dy^2$, divide by dx^2

Then $(x - y).\,\frac{dy}{dx} = y + (a - x).\,\left(\frac{dy}{dx}\right)^2$. Put $\frac{dy}{dx} = p$.

Hence $y.\,(1 + p) = px \times (1 + p) - ap^2$

(A) $\therefore y = px - \frac{ap^2}{1+p}$, which, being of the form $y = px - f(p)$ (that of *Clairaut*) is integrable by differentiation.

For $dy = pdx + xdp - d.\,\left(\frac{ap^2}{1+p}\right)$

But $dy = p.dx$.

$$\therefore\ xdp = d.\,\left(\frac{ap^2}{1+p}\right) = \frac{2apdp}{1+p} - \frac{ap^2dp}{(1+p)^2}$$

$$\text{(B)}\ \therefore dp.\,\left(x + \frac{ap^2}{(1+p)^2} - \frac{2ap}{1+p}\right) = 0$$

$$\therefore dp = 0, \text{ and } x + \frac{ap^2}{(1+p)^2} - \frac{2ap}{1+p} = 0 = x.(1+p)^2 - 2ap - ap^2$$

$\therefore p = c$, and the general solution is (by equation A)

$$y = cx - \frac{ac^2}{1+c}$$

A particular solution, involving no arbitrary constant, may be obtained by eliminating p by means of equation (A), and the other factor of equation (B).

The result will be $(x + y)^2 = 4ay$.

To integrate $\frac{d^2x}{dz^2} + x + \cos.\,mz = 0$, assume

$x = u.\,\sin.\,z + v.\,\cos.\,z$

Then $\frac{dx}{dz} = \frac{du}{dz}.\,\sin.\,z + \frac{dv}{dz}.\,\cos.\,z + u.\cos.\,z - v.\,\sin.\,z$

Assume $\frac{du}{dz}.\,\sin.\,z + \frac{dv}{dz}.\,\cos.\,z = 0$

$\therefore\ \frac{dx}{dz} = u.\,\cos.\,z - v.\,\sin.\,z$

$$\therefore \frac{d^2x}{dz^2} = \frac{du}{dz}.\cos. z - \frac{dv}{dz}\sin. z - (u\sin. z + v.\cos. z)$$

$$\therefore \frac{d^2x}{dz^2} + x + \cos. mz = \frac{du}{dz}.\cos. z - \frac{dv}{dz}.\sin. z + \cos. mz$$

$$\left.\begin{array}{r}\text{Hence, we have } du.\cos. z - dv.\sin. z = -dz.\cos. mz \\ \text{And } du.\sin. z + dv.\cos. z = 0\end{array}\right\}$$

$$\therefore du = -\frac{dv.\cos. z}{\sin. z} = -\frac{dz.\cos. mz + dv.\sin. z}{\cos. z}$$

$$\therefore dv.(\sin.^2 z + \cos.^2 z) = dz.\cos. mz \times \sin. z$$

$$\therefore dv = dz.\sin. z.\cos. mz$$

$$= \frac{1}{2}dz.(\sin. \overline{m+1}\, z - \sin. \overline{m-1}.z) \text{ by Trigonometry.}$$

$$= \frac{1}{2}dz.\sin. \overline{m+1}.\, z - \frac{1}{2}dz\sin. \overline{m-1}\, z$$

$$\text{Hence } du = \frac{-dv.\cos.^2}{\sin. z} = -dz.\cos. z.\cos. mz =$$

$$-\frac{1}{2}dz\ (\cos. \overline{m+1}.z + \cos. \overline{m-1}\,.z) = -\frac{1}{2}dz.\cos. (m+1)z$$

$$-\frac{1}{2}dz.\cos. (m-1)\, z$$

Integrating, we have $v = -\dfrac{1}{2.(m+1)}\cos.(m+1)z + \dfrac{1}{2.(m-1)}$
$\times \cos. (m-1)\, z + c$

And $u = -\dfrac{1}{2.(m+1)}.\sin. (m+1)z - \dfrac{1}{2.(m-1)}.\sin.(m-1)z + c'$

$$\therefore x = u.\sin. z + v.\cos. z = -\frac{1}{2.(m+1)}(\sin.z.\sin.\overline{m+1}.z +$$

$$\cos.z.\cos.\overline{m+1}z) + \frac{1}{2(m-1)}.(\cos.z.\cos.\overline{m-1}.z - \sin. z.\sin.\overline{m-1}\, z)$$

$$+ c.\cos.z + c'\sin.z = -\frac{1}{2.(m+1)}\cos. mz + \frac{1}{2.(m-1)}.\cos. mz$$

$+ c.\cos. z. + c'.\sin.z = \dfrac{\cos. mz}{m^2 - 1} + c.\cos. z + c'.\sin. z$ the value required.

This Differential Equation belongs to the Problem of the Three Bodies.

632. To integrate $\frac{d^2y}{dx^2} = \frac{m}{(a-y)^2}$ we have dx constant,

$$\therefore \frac{dy \cdot d^2y}{dx^2} = \frac{mdy}{(a-y)^2}$$

$$\text{And } \frac{dy^2}{2.dx^2} = \frac{+m}{a-y} + c = \frac{m+ac-cy}{a-y}$$

$$\therefore \frac{dx}{dy} = \pm\sqrt{\frac{2.(a-y)}{m+ac-cy}} = \pm\sqrt{\frac{2}{c}} \cdot \sqrt{\frac{a-y}{b-y}}$$

(putting $b = \frac{m+ac}{-c}$)

$$\therefore x = \pm\sqrt{\frac{2}{c}} \cdot \int\sqrt{\frac{a-y}{b-y}}.dy$$

Assume $y - \frac{a+b}{2} = u$ and substitute,

$$\text{Then } x = \pm\sqrt{\frac{2}{c}} \cdot \int\sqrt{\frac{u+\frac{b-a}{2}}{u-\frac{b-a}{2}}}.\ du = \pm\sqrt{\frac{2}{c}} \times$$

$$\int\frac{udu + \frac{b-a}{2}du}{\sqrt{u^2 - \frac{\overline{b-a}|^2}{4}}} = \pm\sqrt{\frac{2}{c}} . \int\frac{udu}{\sqrt{u^2-r^2}} \pm \sqrt{\frac{2}{c}} \times$$

$$\int\frac{rdu}{\sqrt{u^2-r^2}} \text{(putting } r = \frac{b-a}{2}\text{)} = \pm\sqrt{\frac{2}{c}}.\sqrt{u^2-r^2} \pm \sqrt{\frac{2}{c}} \times$$

$l. (u + \sqrt{u^2-r^2}) + C'$, which values of x being designated by ρ and ρ' we shall have

$x - \rho = 0$, $x - \rho' = 0$, and $(x - \rho).(x - \rho') = 0$ each fulfilling the conditions of the Differential Equation.

To integrate $d^2y + Aydx^2 = X\,dx^2$, or $\frac{d^2y}{dx^2} + Ay = X$

which is $\therefore$ a *Linear Equation of the Second Order*, we will multiply by $e^{-mx}dx$.

Then $e^{-mx}dx \times \left(\dfrac{d^2y}{dx^2} + Ay\right) = e^{-mx}dx \times X = dP$ (by hyp.)

Assume $P + C = e^{-mx}.\left(a.\dfrac{dy}{dx} + b.y\right)$ and differentiating we have $dP = -me^{-mx}dx.\left(a\dfrac{dy}{dx} + b.y\right) + e^{-mx}.\left(a\dfrac{d^2y}{dx} + bdy\right)$

$$= e^{-mx}dx \times \left(a.\frac{d^2y}{dx^2} + \overline{b-am}.\frac{dy}{dx} - bmy\right)$$

$\therefore$ equating coefficients of like terms in the two values of dP we get $a = 1, b - am = 0$, and $-bm = A$

Whence $-\dfrac{A}{m} - m = 0$, or $A + m^2 = 0$ and $\therefore m = \pm\sqrt{-A}$

$\therefore a = 1, b = \dfrac{-A}{m} = \pm\sqrt{-A}$

Hence then we have

$$\frac{dy}{dx} + \sqrt{-A}.y = e^{x\sqrt{-A}}.\left(\int e^{-x\sqrt{-A}}dx\,X + C\right)$$

And $\dfrac{dy}{dx} - \sqrt{-A}.y = e^{-x\sqrt{-A}}.\left(\int e^{+x\sqrt{-A}}dx\,X + C\right)$

$$\therefore y = \frac{1}{2\sqrt{-A}} \times \left(e^{x\sqrt{-A}}.\int e^{-x\sqrt{-A}}dxX - e^{-x\sqrt{-A}}\int e^{x\sqrt{-A}}dxX\right)$$

$+ \dfrac{C}{2\sqrt{-A}} \times \left(e^{x\sqrt{-A}} - e^{-x\sqrt{-A}}\right)$ which, although in an imaginary form, admits of being *realized*, by means of the form $e^{\pm\theta.\sqrt{-1}} = \cos.\theta \pm \sqrt{-1}.\sin.\theta$

By the same process may be integrated, a Linear Equation of the $(n)^{th}$ order, or one of the form,

$$Ay + A_1.\frac{dy}{dx} + A_2.\frac{d^2y}{dx^2} + \ldots\ldots A_n.\frac{d^ny}{dx^n} = X$$

The values of m will, in this case, be found by the resolution of the equation $A + A_1\,m. + A_2m^2 + \ldots. A_n.m^n = 0$.

To integrate $\frac{dz}{dx} - \frac{y}{x} \cdot \frac{dz}{dy} = -\frac{y^2}{x^2}$, which is a particular case of the general Partial Differential Equation, $\frac{dz}{dx} + \text{M} \cdot \frac{dz}{dy}$ $= \text{N}$, (M and N being any functions whatever of x, y, z,) we will assume the integral of this latter form to be

$\text{F}(x, y, z) = 0$ (F being the characteristic of a determinate function)

$$\left.\begin{array}{l} \text{Now we have } \frac{dz}{dx} = -\frac{d\text{F}(x)}{dx} \times \frac{dz}{d.\text{F}(z)} \\ \text{And } \frac{dz}{dy} = -\frac{d\text{F}(y)}{dy} \cdot \frac{dz}{d.\text{F}(z)} \end{array}\right\} \text{(See \textit{Lacroix} or \textit{Garnier}.)}$$

And by substitution, we get

$$\frac{-d\text{F}(x)}{dx} \times \frac{dz}{d.\text{F}(z)} - \text{M} \times \frac{d\text{F}(y)}{dy} \times \frac{dz}{d.\text{F}(z)} = \text{N}$$

$$\therefore \frac{d\text{F}(x)}{dx} + \text{M} \frac{d.\text{F}(y)}{dy} + \text{N} \frac{d.\text{F}(z)}{dz} = 0 \text{.... (A)}$$

But $d.\text{F}(x, y, z) = d.\text{F}(x) + d.\text{F}(y) + d.\text{F}(z)$

$$= d.\text{F}(y) + d\text{F}(z) - \text{M}.dx.\frac{d\text{F}(y)}{dy} - \text{N}.dx.\frac{d\text{F}(z)}{dz} = 0$$

$$\therefore d\text{F}(y) - \text{M}dx.\frac{d\text{F}(y)}{dy} + d\text{F}(z) - \text{N}dx\frac{d.\text{F}(z)}{dz} = 0$$ which

equation will be verified if we put

$$\left.\begin{array}{r} dy - \text{M}dx = 0 \\ \text{And } dz - \text{N}dx = 0 \end{array}\right\} \text{...... B}$$

Now by integrating the equations (B), which are of the first degree and $\therefore$ contain two arbitrary constants c and c', we may express any two of the variables x, y, z, each in terms of the third, the constants c, c' and those contained in M, N. Hence and by substitution, we shall express $\text{F}(x, y, z) = 0$ in terms of these same quantities. But $\because$ $d.\text{F}(x, y, z) = 0$, this third variable will disappear, and $\text{F}(x, y, z)$ will be a function of c, c'.

$\therefore \varphi(c, c') = 0$ (φ representing an arbitrary function.)

(C) $\therefore c' = \varphi(c)$, an equation which will give us the integral required.

In the Problem we have $M = -\frac{y}{x}$, and $N = -\frac{y^2}{x^2}$

$\therefore$ the equations (B) become

$$\left.\begin{aligned} dy + \frac{y}{x}\,dx = 0 \\ dz + \frac{y^2}{x^2}\,dx = 0 \end{aligned}\right\}$$

From the former we deduce $xdy + ydx = 0$

$\therefore xy = c$

Hence the latter becomes

$$dz + \frac{c^2}{x^4}\,dx = 0$$

$$\therefore z - \frac{c^2}{3x^3} = c'$$

$$\therefore z = \frac{c^2}{3x^3} + c' = \frac{x^2y^2}{3x^3} + \phi'(c) = \frac{y^2}{3x} + \phi'.(xy)$$

For a more complete discussion of problems of this nature, the Reader may consult with great advantage *Lacroix's* Differential and Integral Calculus, and the collection of Examples on the same subject by the translators of that work.

633. To integrate $xdy - ydx - (x^3 + 1).\,dx = 0$.

First $dy - y.\,\frac{dx}{x} = \frac{x^3+1}{x}\,dx$, which being Linear with respect to y assume

$$y = u\,v$$

Then $vdu + udv - uv\,\frac{dx}{x} = \frac{x^3+1}{x}\,dx$; again put

$$udv - uv\,\frac{dx}{x} = 0, \text{ or}$$

$$\frac{dv}{v} = \frac{dx}{x}$$

$\therefore l.v = l.x + l.c = l.cx$ (the constant being put $= lc$)

$\therefore\ v = cx.$

Hence $du = \dfrac{x^2+1}{x.v}\,dx = \dfrac{x^2+1}{cx^2}\,dx = \dfrac{dx}{c} + \dfrac{dx}{cx^2}$

$\therefore\ u = \dfrac{x}{c} - \dfrac{1}{cx} + c'$

Hence $y = uv = cx \times \left(\dfrac{x}{c} - \dfrac{1}{cx} + c'\right) = x^2 + cc'x - 1$

$= x^2 + c''x - 1$ ($c'' = cc'$ being the constant proper for the integration of an equation of the first order.)

To integrate $(dx^2 + dy^2)^{\frac{3}{2}} + 2a^{\frac{1}{2}}\sqrt{a-x}.\,dx.\,d^2y = 0$, divide by dx^3, &c.

Then $\dfrac{d^2y}{dx^2} \times 2a^{\frac{1}{2}}\sqrt{a-x} = -\left(1 + \dfrac{dy^2}{dx^2}\right)^{\frac{3}{2}}$

$\therefore\ \dfrac{d^2y}{dx^2} = -\dfrac{1}{2a^{\frac{1}{2}}.\sqrt{a-x}}\,.\left(1 + \dfrac{dy^2}{dx^2}\right)^{\frac{3}{2}}$

Put $\dfrac{dy}{dx} = p$, then $\dfrac{d^2y}{dx^2} = \dfrac{dp}{dx}$ (dx being constant)

And we have $\dfrac{dp}{dx} = -\dfrac{1}{2a^{\frac{1}{2}}.\sqrt{a-x}} \times .(1+p^2)^{\frac{3}{2}}$

$\therefore\ \dfrac{dp}{(1+p^2)^{\frac{3}{2}}} = \dfrac{-dx}{2a^{\frac{1}{2}}.\sqrt{a-x}}$

Again, assume $P = \dfrac{p}{(1+p^2)^{\frac{1}{2}}}$, then $dP = \dfrac{dp}{(1+p^2)^{\frac{1}{2}}} - \dfrac{p^2dp}{(1+p^2)^{\frac{3}{2}}}$

$= \dfrac{dp}{(1+p^2)^{\frac{3}{2}}}$. Hence then we have

$$\frac{p}{(1+p^2)^{\frac{1}{2}}} = -\int\frac{dx}{2a^{\frac{1}{2}}.\sqrt{a-x}} = \frac{1}{a^{\frac{1}{2}}}.\sqrt{a-x} + c$$

$$= \frac{1}{\sqrt{a}}.\ (\sqrt{a-x} + c')$$

$$\therefore\ ap^2 = (\sqrt{a-x}+c')^2 + p^2.\,(\sqrt{a-x}+c')^2$$

$$\therefore p = \pm \frac{\sqrt{a-x}+c'}{\sqrt{a-(\sqrt{a-x}+c')^2}}$$

$$\therefore dy = \pm \frac{\sqrt{a-x}+c'}{\sqrt{a-(\sqrt{a-x}+c')^2}} \times dx$$

Now, put $\sqrt{a-x}+c' = u$

Then $a - x = (u-c')^2$

$$\therefore dx = 2du.(u-c')$$

And $dy = \pm \dfrac{2udu.(u-c')}{\sqrt{a-u^2}} = \pm \left(\dfrac{2u^2du}{\sqrt{a-u^2}} - \dfrac{2c'udu}{\sqrt{a-u^2}} \right)$

$$\therefore y = \pm 2\int \frac{u^2du}{\sqrt{a-u^2}} \pm 2c'.\sqrt{a-u^2}$$

Again, put $v = u.\sqrt{a-u^2}$

Then $dv = du.\sqrt{a-u^2} - \dfrac{u^2du}{\sqrt{a-u^2}}$

$$= \frac{adu}{\sqrt{a-u^2}} - \frac{2u^2du}{\sqrt{a-u^2}}$$

$$\therefore \int \frac{2u^2du}{\sqrt{a-u^2}} = a\int \frac{\frac{du}{\sqrt{a}}}{\sqrt{1-\frac{u^2}{a}}} - v = a.\sin.^{-1}\frac{u}{\sqrt{a}} - v$$

$$\therefore y = \pm 2c'.\sqrt{a-u^2} \mp u.\sqrt{a-u^2} \pm a.\sin.^{-1}\frac{u}{\sqrt{a}}\ c''$$

in which, substituting for u in terms of x, we shall have the relation required.

634. To integrate $e^x dx - \dfrac{ydy}{e^x} = dy - ydx$.

Put $e^x = u$

Then $e^x dx = du$, $\therefore x = \dfrac{du}{e^x} = \dfrac{du}{u}$

$$\therefore du - \frac{ydy}{u} = dy - \frac{ydu}{u}$$

$\therefore udu - ydy = udy - ydu$, which being homogeneous, the variables may be separated by assuming $y = uv$,&c. In this case, however, it appears better to transform thus, $du. (u + y) = dy\ (u + y)$

$$\therefore du = dy$$

And $y \therefore = u + c = e^x + c$

635. To integrate $\sqrt{x}.\ dy = \sqrt{y}.\ dx + \sqrt{y}.\ dy$, which is homogeneous, assume

$$x = uy$$

Then $\sqrt{uy}.\ dy = \sqrt{y}.\ d.\ uy + \sqrt{y}\ dy$

$$\therefore \sqrt{u}.\ dy = udy + ydu + dy$$

$$\therefore \frac{dy}{y} = \frac{-du}{u - \sqrt{u} + 1}$$

Again, assume $\sqrt{u} - \frac{1}{2} = v$

Then $u - \sqrt{u} + 1 = v^2 + 1 - \frac{1}{4} = v^2 + \frac{3}{4}$

Also $u = \left(v + \frac{1}{2}\right)^2$

$$\therefore du = 2dv.\left(v + \frac{1}{2}\right) = 2vdv + dv$$

$$\therefore \frac{dy}{y} = \frac{-2vdv}{v^2 + \frac{3}{4}} - \frac{dv}{v^2 + \frac{3}{4}}$$

$$\therefore ly = -l.\left(v^2 + \frac{3}{4}\right) - \frac{4}{3}.\tan.^{-} \sqrt{\tfrac{4}{3}}\, v + c$$

$$= -l.\left(\frac{x}{y} - \sqrt{\frac{x}{y}} + 1\right) - \frac{4}{3}.\tan.^{-} \sqrt{\tfrac{4}{3}}\left(\sqrt{\frac{x}{y}} - \frac{1}{2}\right)$$

$+ c$, which expresses, in a transcendental form, the relation of x and y.

To integrate $x.\frac{dz}{dx} + y.\frac{dz}{dy} = n.\sqrt{x^2+y^2}$, which is a *Partial*

Differential Equation of the First Order, and of the form exhibited in the last equation of No. 632, we will take from thence the equations (B)

$$\left.\begin{array}{l} dy - \mathrm{M}dx = 0 \\ dz - \mathrm{N}dx = 0 \end{array}\right\}$$

Here $\mathrm{M} = \frac{y}{x}$ and $\mathrm{N} = n\frac{\sqrt{x^2+y^2}}{x}$

$$\left.\begin{array}{l} \therefore\ dy - \frac{y}{x}\,dx = 0 \\ \text{And } dz - \frac{n.\sqrt{x^2+y^2}}{x}.\,dx = 0 \end{array}\right\}$$

From the first we have

$$\frac{xdy-ydx}{x^2} = 0$$

$\therefore\ \frac{y}{x} = c.\quad \therefore$ substituting in the second

$$dz = \frac{n.\sqrt{x^2+c^2x^2}}{x}dx = n.\sqrt{1+c^2}\times dx$$

$$\therefore\ z = n.\sqrt{1+c^2}\,x + c'$$

But, by No. 632, $c' = \varphi(c) = \varphi\left(\frac{y}{x}\right)$, ($\varphi$ representing an arbitrary function.)

$$\therefore\ z = n.\sqrt{1+\frac{y^2}{x^2}}\times x + \varphi.\left(\frac{y}{x}\right)$$

$= n.\sqrt{x^2+y^2} + \varphi.\left(\frac{y}{x}\right)$ a result which we may easily verify.

636. To integrate $x^2d^2y = aydx^2$, we have $\frac{d^2y}{dx^2} = \frac{ay}{x^2}$, which being linear with respect to y and its differentials, we shall assume $y = e^{\int udx}$

$$\therefore\ \frac{dy}{dx} = u.e^{\int udx}$$

And $\frac{d^2y}{dx^2} = \frac{du}{dx}.e^{\int udx} + u^2.e^{\int udx}$

Hence $\frac{du}{dx}.e^{\int udx} + u^2.e^{\int udx} = \frac{ae^{\int udx}}{x^2}$

$\therefore \frac{du}{dx} + u^2 = \frac{a}{x^2}$, which being of the form $\frac{du}{dx} + Au^2 = B.x^m dx$

(Riccati's Equation) we will put $u = \frac{1}{z}$

Then $\frac{du}{dx} = \frac{-dz}{z^2dx}$ and $u^2 = \frac{1}{z^2}$

$$\therefore -\frac{dz}{dx} \times \frac{1}{z^2} + \frac{1}{z^2} = \frac{a}{x^2}$$

$\therefore \frac{dz}{dx} = 1 - \frac{az^2}{x^2}$, which is homogeneous.

Put $\therefore z = xv$, and we have $\frac{dz}{dx} = v + \frac{xdv}{dx}$

And $\frac{az^2}{x^2} = \frac{ax^2v^2}{x^2} = av^2$

$$\therefore v + \frac{xdv}{dx} = 1 - av^2$$

$$\therefore \frac{dx}{x} = \frac{dv}{1 - v - av^2} = \frac{-\frac{1}{a}dv}{v^2 + \frac{v}{a} - \frac{1}{a}}$$

Again, put $v + \frac{1}{2a} = w$

Then $v^2 + \frac{v}{a} - \frac{1}{a} = w^2 - \frac{1}{a} - \frac{1}{4a^2} = w^2 - r^2$

$$\therefore \frac{dx}{x} = \frac{1}{a}.\ \frac{dw}{r^2 - w^2}\ \left(r^2 = \frac{1}{a} + \frac{1}{4a^2}\right)$$

And $\therefore lx = \frac{1}{2ar}.\ l.\ \frac{r+w}{r-w} + l.c$

$$= l.c\left(\frac{r+w}{r-w}\right)^{\frac{1}{2ar}}$$

$$\therefore x = c \times \left(\frac{r+w}{r-w}\right)^{\frac{1}{2ar}}$$

$$\therefore \frac{r+w}{r-w} = \left(\frac{x}{c}\right)^{2ar}$$

Hence $w = r.\dfrac{x^{2ar} - c^{2ar}}{x^{2ar} + c^{2ar}} \quad \therefore v = r.\dfrac{x^{2ar} - c^{2ar}}{x^{2ar} + c^{2ar}} - \dfrac{1}{2a}$

$$\therefore z = rx.\frac{x^{2ar} - c^{2ar}}{x^{2ar} + c^{2ar}} - \frac{x}{2a}$$

And $u = \dfrac{1}{z} = \dfrac{2a}{x} \times \dfrac{x^{2ar} + c^{2ar}}{\overline{2ar-1}.x^{2ar} - \overline{2ar+1}.c^{2ar}}$

Again, if $\dfrac{2ar+1}{2ar-1} = \rho$, we have

$$ly = \int udx = \frac{2a}{2ar-1}.\int\frac{x^{2ar-1}dx}{x^{2ar} - \rho} + \frac{2ac^{2ar}}{2ar-1}.\int\frac{dx}{x.(x^{2ar} - \rho)}$$

$$= \frac{1}{r.(2ar-1)} \times l.(x^{2ar} - \rho) + \frac{2ac^{2ar}}{2ar-1}.\int\frac{dx}{x.(x^{2ar} - \rho)}$$

Put $x^{2ar} = s$

Then $\dfrac{dx}{x.(x^{2ar} - \rho)} = \dfrac{1}{2ar}.\dfrac{ds}{s.(s-\rho)}$

Now, to integrate $\dfrac{ds}{s.(s-\rho)}$, we assume $\dfrac{1}{s.(s-\rho)} = \dfrac{A}{s} + \dfrac{B}{s-\rho}$

$$\therefore As - A\rho + Bs = 1$$

$$\left.\begin{array}{l}\therefore A + B = 0\\ \text{And } A\rho = -1\end{array}\right\} \quad \therefore A = -\frac{1}{\rho} \text{ and } B = \frac{1}{\rho}$$

Hence then we finally get

$$ly = \frac{1}{r.(2ar-1)}.l.(x^{2ar} - \rho) + \frac{c^{2ar}}{r.(2ar-1)} \times \frac{1}{\rho} \times l\frac{s-\rho}{s} + lc'$$

$$= l.\left\{(x^{2ar} - \rho)^{\frac{1}{r.(2ar-1)}} \times \left(\frac{s-\rho}{s}\right)^{\frac{c^{2ar}}{r.(2ar+1)}} + c'\right\}$$

$\therefore y = (x^{2ar} - \rho)^{\left(\frac{1}{r.(2ar-1)} + \frac{c^{2ar}}{r.(2ar+1)}\right)} \times x^{\frac{2ac^{2ar}}{2ar+1}} \times c'$, the reduction of which complicated form we shall leave to be accomplished by the Reader.

637. To integrate $\dfrac{\left(1+\dfrac{dy^2}{dx^2}\right)^{\frac{3}{2}}}{\dfrac{d^2y}{dx^2}} = \dfrac{a^2}{2x}$, put $\dfrac{dy}{dx} = p$

Then $\because$ dx is constant, we have

$\dfrac{d^2y}{dx^2} = \dfrac{dp}{dx}$ and substituting, &c. we get

$$\frac{dp}{dx} \times \frac{a^2}{2x} = (1 + p^2)^{\frac{3}{2}}$$

$$\therefore \frac{dp}{(1 + p^2)^{\frac{3}{2}}} = \frac{2xdx}{a^2}$$

Assume $P = \dfrac{p}{\sqrt{1 + p^2}}$

Then $dP = \dfrac{dp}{\sqrt{1 + p^2}} - \dfrac{p^2dp}{(1 + p^2)^{\frac{3}{2}}}$

$$= \frac{dp}{(1 + p^2)^{\frac{3}{2}}}$$

$$\therefore \frac{x^2}{a^2} = \int dP = P + c = \frac{p}{\sqrt{1 + p^2}} + c$$

$$\therefore \left(\frac{x^2 - a^2c}{ca^2}\right)^2 = \frac{p^2}{1 + p^2}$$

Hence $p = \pm \dfrac{x^2 - a^2c}{x.\sqrt{2ca^2 - x^2}}$

$$\therefore dy = \pm \frac{xdx}{\sqrt{2ca^2 - x^2}} \mp \frac{a^2cdx}{x\sqrt{2ca^2 - x^2}}$$

$$\therefore y = \mp \sqrt{2ca^2 - x^2} \mp \frac{a^2c}{2\sqrt{2ca^2}} \times \int \frac{2\sqrt{2a^2c}.dx}{x.\sqrt{2ca^2 - x^2}}$$

$$= \mp \sqrt{2ca^2 - x^2} \mp \frac{a\sqrt{c}}{2\sqrt{2}}.l.\frac{\sqrt{2ca^2} - \sqrt{2ca^2 - x^2}}{\sqrt{2ca^2} + \sqrt{2ca^2 - x^2}} + c'$$

$$= \mp \sqrt{2ca^2 - x^2} \mp \frac{a\sqrt{c}}{\sqrt{2}}.l.\frac{a\sqrt{2c} - \sqrt{2ca^2 - x^2}}{x} + c'$$

To integrate $\dfrac{dx}{\sqrt{1 - x^2}} + \dfrac{dy}{\sqrt{1 - y^2}} = 0$, we have

$\sin.^{-1}x + \sin.^{-1}y = C = \sin.^{-1}C'$ (by a supposition which, from C being an arbitrary constant, we are at liberty to make)

Let $\sin.^{-1}x = X$ and $\sin^{-1}y = Y$

Then $C' = \sin.(X + Y) = \sin. X. \cos. Y + \cos. X. \sin. Y$

But $x = \sin. X$, and $y = \sin Y$

$$\therefore C' = x.\sqrt{1-y^2} + y\sqrt{1-x^2}$$

In the same manner may be integrated the forms

$$\left.\begin{array}{llll} (1).\ d.\cos.^{-1}x & + d.\cos.^{-1}y & + d.\cos.^{-1}z & + \&c. = 0 \\ (2).\ d.\text{vers.}^{-1}x & + d.\text{vers.}^{-1}y & + d.\text{vers.}^{-1}z & + \&c. = 0 \\ (3).\ d.\tan.^{-1}x & + d.\tan.^{-1}y & + d.\tan.^{-1}z & + \&c. = 0 \\ (4).\ d.\cot.^{-1}x & + d.\cot.^{-1}y & + d.\cot.^{-1}z & + \&c. = 0 \\ (5).\ d.\sec.^{-1}x & + d.\sec.^{-1}y & + d.\sec.^{-1}z & + \&c. = 0 \\ (6).\ d.\text{cosec.}^{-1}x & + d.\text{cosec.}^{-1}y & + d.\text{cosec.}^{-1}z & + \&c. = 0 \\ & & & \&c. = \&c. \end{array}\right\}$$

which although Transcendental in form, will thus be rendered Algebraical. The general form would be

$d.f^{-1}(x) + d.f^{-1}(y) + \ldots\ldots = 0$ ($f^{-1}(x)$ being the inverse function of (x), or that number whose function expressed by (f) $= x$). Hence we should have

$$f^{-1}x + f^{-1}y + \&c. = f^{-1}c$$

$\therefore c = f.\{f^{-1}x + f^{-1}y + \ldots\}$ which for the above six, and some other forms may be algebraically expressed. Take, for example, the third form.

$$\text{Then } C = \tan.(\tan.^{-1}x + \tan.^{-1}y + \ldots\&c.) = \frac{S_1 - S_3 + S_5 - S_7 + \ldots}{1 - S_2 + S_4 - \ldots}$$

(by a well known theorem) S_1 being = the sum, $\tan.(\tan.^{-1}x) + \tan.(\tan.^{-1}y)$ &c. S_2 the sum of the products of every two, S_3 that of every three, &c. &c.

But $\tan.(\tan.^{-1}x) = x$, $\tan.(\tan.^{-1}y) = y$ &c.

$$\therefore C = \frac{x + y + \ldots - (xyz + xyw + \&c.) + (xyzwv + \&c.) - \&c.}{1 - (xy + xz + yz + yw + \ldots) + (xyzw + \ldots) - \&c.}$$

Again, take the 5th form.

$$\text{Then } C = \sec.(\sec.^{-1}x + \sec.^{-1}y + \ldots)$$

$$= \sqrt{1 + \tan.^2(\sec^{-1}x + \sec.^{-1}y + \&c.)}$$

Now $\tan.(\sec.^{-1} x + \sec.^{-1} y + \ldots..) = \dfrac{S_1 - S_3 + \&c.}{1 - S_2 + S_4 - \&c.}$; S_1 being the sum of $\tan.(\sec.^{-1}x)$, $\tan.(\sec.^{-1}y)$, &c. &c., and if we put $\sec.^{-1}x = \theta$ we have $x = \sec.\theta = \sqrt{1+\tan.^2\theta}$

$\therefore \tan.\theta = \tan.(\sec.^{-1}x) = \sqrt{x^2 - 1}$

$\therefore S_1 =$ the sum of $\sqrt{x^2 - 1}$, $\sqrt{y^2 - 1}$, &c.

$S_2 = \ldots\ldots$ the products of every two of them

&c. = &c.,

$$\therefore C^2 = 1 + \left(\frac{S_1 - S_3 + \&c.}{1 - S_2 + S_4 - \&c.}\right)^2 = \frac{(1 - S_2 + S_4 - \ldots)^2 + (S_1 - S_3 + \ldots)^2}{(1 - S_2 + S_4 - \&c.)^2}$$

which is purely algebraical, and which will terminate when the number of variables is limited. For another method of treating functions whose variables are separated, and which are of a transcendental form, the Reader may consult the Examples on the Differential and Integral Calculus, by *G. Peacock*, A. M., &c. page 345.

To integrate $xy - \dfrac{d^2z}{dxdy} = 0$ which is a Partial Differential Equation of the Second Order, we have first

$\dfrac{d^2z}{dxdg} = xy$, and considering x as constant, we have

$\dfrac{dz}{dx} = \int xydy = \dfrac{xy^2}{2} + C = \dfrac{xy^2}{2} + \varphi x$ (φx being an arbitrary function.) Now let y be constant,

Then $z = \int dx \int xydy = \displaystyle\int \frac{y^2xdx}{2} + \int dx\varphi x$

$= \dfrac{y^2x^2}{4} + \varphi'(x) + \varphi''.(y)$ (φ' and φ'' being arbitrary functions).

638. To integrate $\dfrac{dy}{dx} = a\sin.x + by$ we have $dy - bydx$

$= adx \sin. x$, which being a Linear Equation, we will assume $y = uv$

Then $udv + vdu - buvdx = adx. \sin. x$.

Now, put $vdu - buvdx = 0$

$$\therefore \frac{du}{u} = bdx$$

$$\therefore l.u = bx + l.c$$

$\therefore \frac{u}{c} = e^{bx}$ (e being the hyperbolic base.)

$$\therefore u = ce^{bx}$$

But udv also $= a \sin. x.dx$

$$\therefore dv = \frac{a \sin. x \times dx}{ce^{bx}}$$

And $v = \frac{a}{c}. \int dx \sin.xe^{-bx} = -\frac{a}{c}.\cos.x \times e^{-bx} - \frac{ab}{c}.\int dx \cos. x\ e^{-bx}$

$$= -\frac{a}{c}.\cos. x \times e^{-bx} - \frac{ab}{c}.\sin. x \times e^{-bx} - \frac{ab^2}{c}.\int dx \sin.x \times e^{-bx}$$

But $\int dx \sin. x \times e^{-bx} = \frac{c}{a} v$

$$\therefore v = -\frac{a}{c}. e^{-bx} \cos. x - \frac{ab}{c} e^{-bx} \sin. x + b^2 v - \text{const.}$$

$$\therefore v = -\frac{a}{c.(b^2+1)} \times e^{-bx} \times \left\{\cos. x + b \sin. x\right\} + \text{const.}$$

Hence $y = uv = \frac{-a}{b^2+1} \times (\cos. x + b \sin. x) + cc' e^{bx}$

Otherwise.

The equation when multiplied by e^{-bx} evidently becomes integrable; for we have

$$e^{-bx}dy - be^{-bx}dx \times y = a\, e^{-bx} \times \sin. x \times dx$$

Or $ye^{-bx} = a \int e^{-bx}dx. \sin. x$

$\therefore y = \frac{a}{e^{-bx}} \int e^{-bx}dx. \sin. x$, which may be integrated as before.

To integrate $\frac{d^2y}{dx^2} - \frac{2dy}{dx} + 2y = 0$, make $y = e^{\int udx}$

Then $\frac{dy}{dx} = ue^{\int udx}$

And $\frac{d^2y}{dx^2} = u^2e^{\int udx} + \frac{du}{dx}.e^{\int udx}$

$\therefore u^2e^{\int udx} + \frac{du}{dx}.e^{\int udx} - 2ue^{\int udx} + 2e^{\int udx} = 0$

$\therefore u^2 - 2u + 2 = -\frac{du}{dx}$

$\therefore -x = \int\frac{du}{u^2 - 2u + 2}$

Put $u - 1 = v$. Then $u^2 - 2u + 2 = v^2 + 2 - 1 = v^2 + 1$

$\therefore c - x = \int\frac{dv}{v^2 + 1} = \tan.^{-1} v$

$\therefore v = \tan.(c - x) = u - 1$

$\therefore \int udx = \int dx + \int dx.\tan.(c-x) = x - \int d.(c-x).\tan.(c-x)$

$= x - \int\frac{\tan.(c-x).d.\tan.(c-x)}{1+\tan.^2.(c-x)}$

$= x - \frac{1}{2}.\ l.\ \overline{1+\tan.^2(e-x)} + C'c$

$= x - l.\sqrt{1+\tan.^2(c-x)} - l.c'$ (c' being arbitrary.)

$= x - l.\ \overline{c'.\sec.(c-x)}$

$\therefore y = e^{\int ud} = e^{x - l.(c'\sec.\overline{c-x})} = \frac{e^x}{e^{l.(c'\sec.\overline{c-x})}}$

But, putting $e^{l.(c'\sec.\overline{c-x})} = m$ we have

$\therefore l.(c.'\sec.\overline{c-x}) = l.m$

$\therefore c'.\sec.\overline{c-x} = m = e^{l.(c'.\sec.\overline{c-x})}$

$\therefore y = \frac{e^x}{c'.\sec.(c-x)} = \frac{e^x.\cos.(c-x)}{c'}$

From the above operation it appears that the hyperbolic

logarithm of a function is the inverse of an Exponential of the *same* function, and *vice versâ*. To prove this generally, let $\varphi\ (x)$ represent a^x. Then $\varphi^{-1}.(\varphi x) = x = \log.\ a^x = \log.(\varphi x)$ (a being the base of the system) $\therefore\ \varphi^{-1}u$ has the same meaning log. as u, or $\varphi^{-1}u = \log.\ u$.

Also $\log.^{-1}u = \varphi^{-(-1)}u = \varphi\ u$. Hence then we may conveniently employ the symbol ($\log.^{-1}$) to express Exponential Functions. The very circumscribed limits of this work prevent us from pursuing this subject to the extent we could wish. It belongs indeed to the General Theory of Functions, which the Reader will find very ably laid down in the writings of Mr. Babbage.

639. To integrate $(a+y).\frac{dx}{dy} = x + y - x.\frac{dy}{dx}$, suppose dy constant and differentiate, then

$$dy.\frac{dx}{dy} + \frac{d^2x}{dy}.(a+y) = dx + dy - \frac{dx.dy}{dx} + \frac{xdy\,d^2x}{dx^2}$$

$$\therefore\ \frac{d^2x}{dy}.(a+y) = \frac{xdyd^2x}{dx^2}$$

$$\therefore\ a + y = \frac{dy^2}{dx^2}.\,x$$

$$\therefore\ \frac{dx^2}{x} = \frac{dy^2}{a+y}$$

$$\therefore\ \frac{dx}{\sqrt{x}} = \frac{dy}{\sqrt{a+y}}$$

$$\therefore\ \sqrt{x} = \sqrt{a+y} + c$$

$$\therefore\ a + y = c^2 - 2c\sqrt{x} + x$$

$$\therefore\ y = c^2 - a - 2c\sqrt{x} + x$$

640. To integrate $\frac{ydx - xdy}{(x+y)^2} = d\text{F}$, put $y = ux$

Then $dy = udx + xdu$ and we have

$$dF = \frac{uxdx - xudx - x^2du}{(x+ux)^2} = \frac{-x^2du}{(x+ux)^2} = \frac{-du}{(1+u)^2}$$

$$\therefore F = \frac{1}{1+u} + C = \frac{1}{1+\frac{y}{x}} + C = \frac{x}{x+y} + C$$

To integrate $xdy - ydx = \dfrac{2xdy - ydx}{\sqrt{x^2+y^2}}$,

Put $x = vy$

Then $vydy - y.d(vy) = \dfrac{2vydy - y.d(vy)}{\sqrt{v^2y^2 + y^2}} = \dfrac{vdy - ydv}{\sqrt{1+v^2}}$

Or $-y^2dv = \dfrac{vdy - ydv}{\sqrt{1+v^2}}$

$$\therefore \quad dv.\sqrt{1+v^2} = \frac{ydv - vdy}{y^2} = d.\frac{v}{y}$$

$$\therefore \quad \frac{v}{y} = \int dv.\sqrt{1+v^2}$$

Now assume $v.\sqrt{1+v^2} = u$

Then $du = dv.\sqrt{1+v^2} + \dfrac{v^2dv}{\sqrt{1+v^2}}$

$$= 2dv.\sqrt{1+v^2} - \frac{dv}{\sqrt{1+v^2}}$$

$$\therefore \quad \frac{v}{y} = \frac{u}{2} + \frac{1}{2}.\int\frac{dv}{\sqrt{1+v^2}}$$

$$= \frac{1}{2}.\ v.\ \sqrt{1+v^2} + \frac{1}{2}.l.\left(v+\sqrt{1+v^2}\right) + C$$

Or $\dfrac{x}{y^2} = \dfrac{1}{2}.\dfrac{x}{y}.\sqrt{1+\dfrac{x^2}{y^2}} + \dfrac{1}{2}l.\left(\dfrac{x}{y} + \sqrt{1+\dfrac{x^2}{y^2}}\right) + C$

$$\therefore x = \frac{1}{2}.x\sqrt{y^2+x^2} + \frac{1}{2}.y^2.l.\frac{x+\sqrt{y^2+x^2}}{y} + Cy^2$$

641. To integrate $\dfrac{dy}{dx} - \dfrac{a+2x-y}{a-x+2y}$, let

$$\left.\begin{array}{l} a + 2x - y = u \\ a - x + 2y = v \end{array}\right\} \text{ Then } \begin{array}{l} y = \dfrac{u + 2v - 3a}{3} \\ x = \dfrac{v + 2u - 3a}{3} \end{array}$$

$$\therefore \frac{du + 2dv}{dv + 2du} = \frac{u}{v} \text{ which is homogeneous.}$$

Put $\therefore u = vz$

$$\text{Then } \frac{zdv + vdz + 2dv}{dv + 2vdz + 2zdv} = \frac{vz}{v} = z$$

$$\therefore zdv + vdz + 2dv = zdv + 2zvdz + 2z^2dv$$

$$\therefore vdz \,.\, (1 - 2z) = dv \,.\, (2z^2 - 2)$$

$$\therefore \frac{dv}{v} = \frac{(1 - 2z) \,.\, dz}{2z^2 - 2}$$

$$\therefore l.\, v = \frac{1}{2} \int \frac{dz}{z^2 - 1} - \int \frac{zdz}{z^2 - 1}$$

$$= \frac{1}{4} \,.\, l.\, \frac{z-1}{z+1} - \frac{1}{2} \,.\, l.\, (z^2 - 1) + l.\, c$$

$$= l.c. \left(\frac{z - 1}{z + 1}\right)^{\frac{1}{4}} \times \frac{1}{(z^2-1)^{\frac{1}{2}}}$$

$$\therefore v^4 = c^4 \,.\, \frac{z-1}{z+1} \times \frac{1}{(z^2 - 1)^2} = \frac{c^4}{(z + 1)^3 \,.\, (z - 1)} =$$

$$\frac{c^4}{\left(\dfrac{u}{v} + 1\right)^3 . \left(\dfrac{u}{v} - 1\right)} = \frac{c^4 v^4}{(u + v)^3 \ (u - v)}$$

$$\therefore c^4 = (u + v)^3 \,.\, (u - v) = (2a + x + y)^3 \,.\, (3x + 3y) =$$
$$3 \,.\, (2a + x + y)^3 \,.\, (x + y)$$

642. To integrate $x^m \,.\, (ydx + xdy) = y^n \,.\, (ydx - xdy)$, we will divide both sides by xy.

$$\text{Then } x^m \left(\frac{dx}{x} + \frac{dy}{y}\right) = y^n . \left(\frac{dx}{y} - \frac{dy}{y}\right).$$

$$\left.\begin{array}{l} \text{Put } x^m = u \\ \text{and } y^n = v \end{array}\right\} \text{ then } \left.\begin{array}{l} \dfrac{dx}{x} = \dfrac{1}{m} \cdot \dfrac{du}{u} \\ \dfrac{dy}{y} = \dfrac{1}{n} \cdot \dfrac{dv}{v} \end{array}\right\}$$

$$\therefore u \cdot \left(\frac{1}{m} \cdot \frac{du}{u} + \frac{1}{n} \cdot \frac{dv}{v}\right) = v \cdot \left(\frac{1}{m} \cdot \frac{du}{u} - \frac{1}{n} \cdot \frac{dv}{v}\right)$$

which equation being homogeneous, assume $u = vw$.

Then, $w\left(n \cdot \dfrac{vdw + wdv}{vw} + m\dfrac{dv}{v}\right) = n \cdot \dfrac{vdw+wdv}{vw}$ $- \dfrac{mdv}{v}$

$$\therefore nvwdw + nw^2dv + mw^2dv = nvdw + nwdv - mwdv$$

$$\therefore vdw \times (nw - n) = wdv \cdot (n - m - mw - nw) \cdot$$

$$\therefore \frac{dv}{v} = \frac{n \cdot (w-1) \cdot dw}{w \cdot (\overline{n - m} - \overline{n + m} \cdot w)} = \frac{n}{n+m} \cdot \frac{w-1}{w} \times \frac{dw}{a - w} \text{ (putting } \frac{n-m}{n+m} = a)$$

$$= \frac{n}{n+m} \cdot \left\{\frac{dw}{a-w} - \frac{dw}{w \cdot (a-w)}\right\}$$

Now, put $\dfrac{1}{w \cdot (a - w)} = \dfrac{A}{w} + \dfrac{B}{a-w}$

Then $A \cdot (a - w) + B\,w = 1$; let $w = a$ and 0 successively, and from the corresponding results we get

$$B = \frac{1}{a}, A = \frac{1}{a}$$

Hence $\dfrac{dv}{v} = \dfrac{n}{n+m} \cdot \left\{\dfrac{a-1}{a} \cdot \dfrac{dw}{a-w} - \dfrac{1}{a} \cdot \dfrac{dw}{w}\right\}$

$$\therefore lv = \frac{n}{a \cdot (n+m)} \cdot \{\overline{1-a} \cdot l.\, w\,(w-a) - l. + l.\, c\}$$

$$= \frac{n}{a \cdot (n+m)} \cdot \left\{l.\, c\,\frac{(w-a)^{1-a}}{w}\right\}$$

$\therefore v^{\frac{a.(n+m)}{n}} = c \cdot \dfrac{(w-a)^{1-a}}{w}$ $\therefore$ substituting we have

$$v^{\frac{n-m}{n}} = \frac{c}{w} \cdot (w-a)^{\frac{2m}{n+m}} = \frac{cv}{u} \cdot \frac{(u - av)^{\frac{2m}{n+m}}}{v^{\frac{2m}{n+m}}}$$

$$= \frac{cv^{\frac{n-m}{n+m}}}{u} \cdot (u-av)^{\frac{2m}{n+m}}$$

$$\text{Or } y^{n-m} = \frac{c \cdot y^{\frac{n^2-mn}{n+m}}}{x^m} \cdot (x^m - ay^n)^{\frac{2m}{n+m}}$$

$$\therefore y^{\frac{n-m}{n+m}} = \frac{c}{x^m} \cdot (x^m - ay^n)^{\frac{2m}{n+m}}$$

$$\text{Hence } y^{\frac{n-m}{2}} = \frac{c^{\frac{(n+m)}{2n}}}{n+m} \cdot \frac{(n+m)\,x^m - (n-m)\,y^n}{x^{\frac{n+m}{2}}}$$

$$= c' \cdot \frac{(n+m)\,x^m - (n-m)^r\,y^n}{x^{\frac{n+m}{2}}}$$

By the above process many other general forms may be integrated. All those, for instance, capable of being reduced to the *very general* one,

$$x_1^{m_1} \cdot \left\{a_1 \frac{dx_1}{x_1} + b_1 \frac{dx_2}{x_2} + \ldots\right\} + x_2^{m_2} \cdot \left\{a_2 \frac{dx_1}{x_1} + b_2 \frac{dx_2}{x_2} + \ldots\right\}$$

$$+ x_3^{m_3} \cdot \left\{a_3 \frac{dx_3}{x_3} + b_3 \cdot \frac{dx_2}{x_2} + \ldots\right\} + \text{\&c. \&c. \&c.} = 0, \text{ the}$$

logarithmic forms $\frac{dx_1}{x_1}$ &c. being taken in any order whatever.

We should here assume $x_1^{m_1} = u_1$, $x_2^{m_2} = u_2$, &c. $=$ &c.; then from the very elegant and useful property of *the Differential of the power of a Variable divided by that power, being always of that same form with respect to the Variable itself*, we obtain infallibly an Homogeneous Equation, the integration of whicn is attended with no difficulty. The student is recommended to commit this theorem to memory.

To integrate $d^3y - 3d^2ydx + 3dydx^2 - ydx^3 = 0$, or

$\frac{d^3y}{dx^3} - \frac{3d^2y}{dx^2} + \frac{3dy}{dx} - y = 0$, which is a particular case of the general *Linear Equation of the Third Order*,

$\frac{d^3y}{dx^3} + A\frac{d^2y}{dx^2} + B\frac{dy}{dx} + Cy = X$, A, B, C being constants, and X a function of x.

$$\left.\begin{aligned} \text{Assume } y &= e^{mx}\int y_1 dx \\ y_1 &= e^{m_1x}\int y_2 dx \\ y_2 &= e^{m_2x}\int y_3 dx \end{aligned}\right\} \ldots\ldots\ldots (A)$$

Then since $\frac{dy}{dx} = e^{mx}\{m\int y_1 dx + y_1\}$

$$\frac{d^2y}{dx^2} = e^{mx}\left\{m^2\int y_1 dx + 2my_1 + \frac{dy_1}{dx}\right\}$$

$$\frac{d^3y}{dx^3} = e^{mx}\left\{m^3\int y_1 dx + 3m^2y_1 + 3m\frac{dy_1}{dx} + \frac{d^2y_1}{dx^2}\right\}$$

By substituting, &c., in the above equation, we get

$$\frac{d^2y_1}{dx^2} + (3m^2+2Am+B)\frac{dy_1}{dx} + (3m+A)y_1 + (m^3+Am^2+Bm+C) \times \int y_1 dx = \frac{X}{e^{mx}}$$

Now by assigning to (m) its values in the equation

$$m^3 + Am^2 + Bm + C = 0 \ldots\ldots\ldots (m)$$

we reduce the integration to that of the *Linear Equation* of the *Second Order*

$$\frac{d^2y_1}{dx^2} + (3m^2 + 2Am + B)\frac{dy_1}{dx} + (3m + A)y_1 = \frac{X}{e^{mx}}$$

By exactly the same process, with regard to the second and third assumption of (A), we obtain

$$\left\{\begin{aligned} &m_1^2 + A_1m_1 + B_1 = 0 \ldots\ldots\ldots (m_1) \\ &\frac{dy_2}{dx} + (2m_1 + A_1)y_1 = \frac{X}{e^{mx} \times e^{m_1x}} \end{aligned}\right\}$$

And $\left\{\begin{aligned} &m_2 + A_2 = 0 \ldots\ldots (m_2) \\ &y_3 = \frac{X}{e^{mx} \times e^{m_1x} \times e^{m_2x}} \end{aligned}\right\}$ A_1, B_1, A_2, being put $= 3m^2 + 2Am + B$, $3m + A$, and $2m_1 + A_1$ respectively.

Hence then we get

$$y_2 = e^{m_2 x} \int y_3 dx = e^{m_2 x} \int \frac{X dx}{e^{mx + m_1 x + m_2 x}}$$

$$y_1 = e^{m_1 x} \int y_2 dx = e^{m_1 x} \int e^{m_2 x} dx \int \frac{X dx}{e^{mx + m_1 x + m_2 x}}$$

$$y = e^{mx} \int y_1 dx = e^{mx} \int e^{m_1 x} dx \int e^{m_2 x} dx \int \frac{X dx}{e^{mx + m_1 x + m_2 x}}$$

Let now a_1, a_2, a_3 be the roots of equation (m)

b_1, b_2 . . . . (m_1)

And c_1 . . . . (m_2)

and diminish the roots of (m) by any one of its roots, for instance, a_1, by substituting for (m) the quantity

$$u + a_1 = m \quad .$$

Then the resulting equation will be

$$u^3 + (3a_1 + A)u^2 + (3a_1^2 + 2Aa_1 + B).u + (a_1^3 + Aa_1^2 + Ba_1 + C) = 0$$

But since a_1 is a root of equation (m),

$$a_1^3 + Aa_1^2 + Ba_1 + C = 0, \text{ we have } \therefore$$

$u^2 + (3a_1 + A)u + (3a_1^2 + 2Aa_1 + B) = 0$ whose coefficients being precisely of the same form as those of equation (m_1), the equations are identical, or

$$m_1 = u = m - a_1$$

$$\therefore \left. \begin{aligned} b_1 &= a_2 - a_1 \\ b_2 &= a_3 - a_1 \end{aligned} \right\}$$

In the same manner it may be shewn that

$$m_2 = m_1 - b_1, \text{ or that}$$

$$c_1 = (a_3 - a_1) - (a_2 - a_1) = a_3 - a_2$$

Hence then, and substituting $a_1, a_2 - a_1, a_3 - a_2$ for m, m_1, m_2 respectively, we obtain

$$y = e^{a_1 x} \int e^{\overline{a_2 - a_1}.x} dx \int e^{\overline{a_3 - a_2}.x} dx \int \frac{X dx}{e^{a_3 x}},$$ which will be the complete integral, because of the introduction of three constants by the three successive integrations.

By an extension of this process, the Student will be enabled to integrate the very general *Linear Equation*

$$\frac{d^n y}{dx^n} + A \frac{d^{n-1} y}{dx^{n-1}} + \ldots\ldots \; P \frac{dy}{dx} + Qy = X$$

It will be found that

$$y = e^{a_1 x}\int e^{(a_2-a_1)x}dx \int e^{(a_3-a_2)x}dx \int e^{(a_4-a_3)x}dx \ldots\ldots \int \frac{\mathrm{X}dx}{e^{a_n x}}$$

a_1, a_2 ..... a_n being the roots of the equation

$$m^n + \mathrm{A}m^{n-1} + \mathrm{B}m^{n-1} + \ldots\ldots \mathrm{P}m + \mathrm{Q} = 0$$

In the problem before us we have

$$m^3 - 3m^2 + 3m - 1 = 0$$

$$\text{Or } (m-1)^3 = 0$$

$$\therefore \text{ in this case, } a_1 = a_2 = a_3 = 1$$

and consequently $y = e^x \int e^0 dx \int e^0 dx \int \frac{0 \times dx}{e^x}$

But since $d.$ (const.) $= 0$, we have

$$\int \frac{0 \times dx}{e^x} = \int (0) = \text{const.} = \mathrm{C}_1, \text{ and } e^0 = 1$$

$$\therefore y = e^x \int dx \int dx \times \mathrm{C}_1$$

$$= e^x \int (\mathrm{C}_1 x + \mathrm{C}_2)$$

$$= e^x \left\{ \frac{\mathrm{C}_1 x^2}{2} + \mathrm{C}_2 x + \mathrm{C}_3 \right\}$$

A more commodious although less scientific method, may be devised for the integration of all those *Linear Equations* whose constant coefficients are the same as those of $(1 + x)^n$ expanded, (n) indicating the order of the equation. The general equation of this class is

$$\frac{d^n y}{dx^n} + n. \frac{d^{n-1}y}{dx^{n-1}} + \frac{n.(n-1)}{2}. \frac{d^{n-2}y}{dx^{n-1}} + \ldots\ldots = \mathrm{X}$$

Putting $y = e^{mx} \int y_1 dx$, differentiating, &c. we obtain

$$m^n + nm^{n-1} + \ldots\ldots 1 = 0 = (m+1)^n$$

$$\text{and } \frac{d^{n-1}y_1}{dx^{n-1}} + \mathrm{A}_1 \frac{d^{n-2}y_1}{dx^{n-2}} + \mathrm{A}_2 \frac{d^{n-3}y_1}{dx^{n-3}} + \ldots\ldots = \frac{\mathrm{X}}{e^{mx}}$$

And it will be seen that A_1, A_2, &c. will vanish from being $= n.\,(m+1)^{n-1}, n.\,(n-1).(m+1)^{n-2}$ &c. The integration will $\therefore$ be reduced to that of

$\frac{d^{n-1}y_1}{dx^{n-1}} = \frac{\mathrm{X}}{e^{mx}}$, and of $y = e^{mx} \int y_1 dx$ which present no difficulties.

If any of the roots of the general equation

$$m^n + Am^{n-1} + Bm^{n-2} + \ldots = 0$$

be imaginary, the integral may be rendered real from the circumstance of imaginary roots entering by pairs, which enables us to use the forms

$$\cos.\,\theta \pm \sqrt{-1}\ \sin.\,\theta = e^{\pm\sqrt{-1}}$$

successfully for that purpose.

To integrate $x\dfrac{dz}{dx} + y\dfrac{dz}{dy} = z$, which is a *Partial* Differential Equation, *Linear* with respect to z and its *differential*, assume

$$z = e^u$$

Then $dz = e^u du$, and substituting, &c. we get

$$\left.\begin{aligned} \frac{xdu}{dx} + \frac{ydu}{dy} &= 1 \\ \text{Also } dx.\frac{du}{dx} + dy.\frac{du}{dy} &= du \end{aligned}\right\}$$

$$\text{Hence } du = \frac{du}{dx} \times dx + \frac{dy}{y}.\left(1 - \frac{xdu}{dx}\right)$$

$$= \frac{dy}{y} + \frac{du}{dx} \times \frac{ydx - xdy}{y}$$

$$\therefore\ d.\,(u - ly) = \frac{ydu}{dx} \times \frac{ydx - xdy}{y^2}$$

$$= \frac{ydu}{dx} \times d.\left(\frac{x}{y}\right)$$

Now the first member of this equation being a complete differential, it is necessary that the other should be. (*Lacroix.*)

Put $\therefore\ \dfrac{ydu}{dx} = \dfrac{d.\,\varphi\left(\frac{x}{y}\right)}{d.\left(\frac{x}{y}\right)}$, $\varphi\left(\dfrac{x}{y}\right)$ being an arbitrary function of $\dfrac{x}{y}$

Thence we have $u - ly = \varphi\left(\dfrac{x}{y}\right)$

But $z = e^u$

$\therefore u = lz$

$\therefore l.\frac{z}{y} = \varphi\left(\frac{x}{y}\right)$

$\therefore z = y \times e^{\varphi.\left(\frac{x}{y}\right)} = y \times \varphi'\left(\frac{x}{y}\right)$, because $\varphi'\left(\frac{x}{y}\right)$ represents an *arbitrary* function of $\left(\frac{x}{y}\right)$

By a similar process we may integrate the general form $x\frac{dz}{dx} + y\frac{dz}{dy} + t\frac{dz}{dt} + u\frac{dz}{du} + \ldots. = nz$. The integral will be

$$z = y^n \times \varphi\left(\frac{x}{y}, \frac{t}{y}, \frac{u}{y}, \ldots..\right)$$

FINITE DIFFERENCES, OR INCREMENTS.

643. We shall here use the notation Δz, $\Delta^2 z$ &c. instead of $\underset{\cdot}{z}$, $\underset{\cdot\cdot}{z}$ &c., in conformity with that adopted in the *Differential Calculus*.

Let $z, z_1, z_2 \ldots z_n, z_{n+1}$ &c. denote the successive values of z, as constantly augmented by the Increment Δz.

Then $z_n = z + n \Delta z$

$z_{n+1} = z + (n+1) \Delta z$

&c. = &c.

Hence $\dfrac{1}{z} = \dfrac{z_n}{zz_n} = \dfrac{z + n\Delta z}{zz_n} = \dfrac{1}{z_n} + \dfrac{n\Delta z}{zz_n}$

$$\frac{1}{zz_n} = \frac{z_{n+1}}{zz_n z_{n+1}} = \frac{z+(n+1)\Delta z}{zz_n z_{n+1}} = \frac{1}{z_n z_{n+1}} + \frac{(n+1)\Delta z}{zz_n z_{n+1}}$$

$$\frac{1}{zz_n z_{n+1}} = \frac{z_{n+2}}{zz_n z_{n+1} z_{n+2}} = \frac{z+(n+2)\Delta z}{zz_n z_{n+1} z_{n+2}} = \frac{1}{z_n z_{n+1} z_{n+2}} + \frac{(n+2)\Delta z}{zz_n z_{n+1} z_{n+2}}$$

&c. = &c.

By substitution we $\therefore$ get

$$\frac{1}{z} = \frac{1}{z_n} + \frac{n\Delta z}{z_n z_{n+1}} + \frac{n.(n+1)\Delta z^2}{z_n z_{n+1} z_{n+2}} + \&c.$$

644. Putting the constant increment of $v = \Delta v$ we $v_{-3} v_{-1} v_2 = v_2 (v - \Delta v)(v_1 - 4\Delta v) = v v_1 v_2 - (v_1 v_2 + 4v v_2) \times \Delta v + 4v_2 \Delta v^2$

But $v_1 v_2 = v_1 (v + 2 \Delta v) = v v_1 + 2v_1 . \Delta v$

And $4vv_2 = 4v(v_1 + \Delta v) = 4vv_1 + 4v . \Delta v$

$\therefore v_{-3} v_{-1} v_2 = v v_1 v_2 - 5v v_1 \Delta v - 2(v_1 + 2v) \Delta v^2 + 4v \; \Delta v^2$

Again $v_1 + 2v = v + \Delta v + 2v = 3v + \Delta v$

And $4 v_2 = 4(v + 2 \Delta v) = 4v + 8 \Delta v$

Hence then we finally get

$$v_{-3} v_{-1} v_2 = vv_1v_2 - 5vv_1 . \Delta v - 2v \Delta v^2 + 6 \Delta v^3.$$

Otherwise.

Assume $v_{-3} v_{-1} v_2 = A . v v_1 v_2 + B v v_1 + C v + D$

Then we have

$$(v - 3 \Delta v).(v - \Delta v)(v + 2 \Delta v) = A . v.(v + \Delta v)(v + 2 \Delta v) + Bv (v + \Delta v) + C v + D$$

And by reduction

$$v^3 - 2 \Delta v . v^2 - 5 \Delta v^2 . v + 6 \Delta v^3 = A . v^3 + \left.\begin{matrix} 3 A \Delta v \\ + B \end{matrix}\right\} v^2 \left.\begin{matrix} + 2 A \Delta v^2 \\ + B \Delta v \\ + C \end{matrix}\right\} v + D$$

$\therefore$ equating coefficients of like powers of (v) we have

$$\left.\begin{aligned} A &= 1 \\ 3 A \Delta v + B &= -2 \Delta v \\ 2 A \Delta v^2 + B \Delta v + C &= -5 \Delta v^2 \\ D &= 6 \Delta v^3 \end{aligned}\right\} \therefore \left.\begin{aligned} A &= 1 \\ B &= -5 \Delta v \\ C &= -2 \Delta v^2 \\ D &= 6 \Delta v^3 \end{aligned}\right\} \text{as before,}$$

645. To difference z log. z we have

$$\Delta (z \log. z) = (z + \Delta z) \log. (z + \Delta z) - z \log. z$$
$$= (z + \Delta z) \{\log. (z + \Delta z) - \log. z\} + \Delta z . \log . z$$
$$= (z + \Delta z) . \log . \left(1 + \frac{\Delta z}{z}\right) + \Delta z . \log . z$$
$$= (z + \Delta z) \left\{ \frac{\Delta z}{z} - \frac{(\Delta z)^2}{2z^2} + \frac{(\Delta z)^3}{3z^3} - \dots \right\} \frac{1}{M} + \Delta z . \log . z$$

$\therefore$ multiplying and collecting the coefficients of like powers of Δz we have $\Delta (z \log. z) =$

$$= \Delta z . \log . z + \frac{1}{M} \times \left\{ \Delta z + \frac{(\Delta z)^2}{1.2.z} - \frac{(\Delta z^3)}{2.3z^2} + \frac{(\Delta z)^4}{3.4z^3} - \&c. \right\}$$

the difference required.

646. $$\frac{1}{z_m} = \frac{z_{m-1}}{z_m z_{m-1}} = \frac{z_m - \Delta . z_{m-1}}{z_m z_{m-1}} = \frac{1}{z_{m-1}} - \frac{\Delta . z_{m-1}}{z_m z_{m-1}}$$

Similarly $\dfrac{1}{z_{m-1}} = \dfrac{1}{z_{m-2}} - \dfrac{\Delta . z_{m-2}}{z_{m-1} z_{m-2}}$

$$\frac{1}{z_{m-2}} = \frac{1}{z_{m-3}} - \frac{\Delta . z_{m-3}}{z_{m-2}\, z_{m-3}}$$

&c. = &c.

$$\frac{1}{z_3} = \frac{1}{z_2} - \frac{\Delta . z_2}{z_3 z_2}$$

$$\frac{1}{z_2} = \frac{1}{z_1} - \frac{\Delta z_1}{z_2\, z_1}$$

$$\frac{1}{z_1} = \frac{1}{z} - \frac{\Delta z}{z_1 z}$$

Hence, and by substitution we get

$$\frac{1}{z_m} = \frac{1}{z} - \frac{\Delta z}{z\, z_1} - \frac{\Delta z_1}{z_1 z_2} - \frac{\Delta z_2}{z_2 z_3} - \ldots\ldots - \frac{\Delta z_{m-1}}{z_{m-1} z_m}$$

But $\Delta z_1 = \Delta .(z + \Delta z) = \Delta z + \Delta^2 z$

$\Delta z_2 = \Delta (z_1 + \Delta z_1) = \Delta z_1 + \Delta .(\Delta z + \Delta^2 z) = \Delta z + 2\Delta^2 z + \Delta^3 z$

$\Delta z_3 = \Delta (z_2 + \Delta z_2) = \Delta z_2 + \Delta .(\Delta z_2) = \Delta z + 3\Delta^2 z + 3\Delta^3 z$
$+ \Delta^4 z$

&c. = &c.

$\Delta z_{m-1} = \Delta z + (m-1)\Delta^2 z + (m-1)\dfrac{m-2}{2}\Delta^3 z + \ldots (m-1) \times$
$\Delta_{m-1} z + \Delta^m z.$

Hence $\dfrac{1}{z_m} = \dfrac{1}{z} - \dfrac{\Delta z}{z z_1} - \dfrac{\Delta z + \Delta^2 z}{z_1 z_2} - \dfrac{\Delta z + 2\Delta^2 z + \Delta^3 z}{z_2 z_3}$
— &c. ....to $(m + 1)$ terms.

The Student may apply to this Problem the two Theorems

$$\left\{ \begin{array}{l} \Delta^n u_x = u_{x+n} - n . u_{x+n-1} + \dfrac{n.(n-1)}{2} u_{x+n-2} - \&c. \\ u_{x+n} = u_x + n \Delta u_x + n . \dfrac{n-1}{2} . \Delta^2 u_x + \&c. \end{array} \right\}$$ See

Appendix to the Translation of *Lacroix*, where this subject is treated in an able and perspicuous manner.

647. Here we have $\Delta\, l.x = l.(x + \Delta x) - lx = l. \dfrac{x + \Delta x}{x}$

$$= l.\left(1 + \frac{\Delta x}{x}\right)$$

Put $1 + \frac{\Delta x}{x} = \frac{1+y}{1-y}$

Then $y = \frac{\Delta x}{2x + \Delta x}$

And $\Delta . lx = l. \frac{1+y}{1-y} = l. (1 + y) - l. (1 - y)$

$$= \left\{ \begin{array}{l} y - \frac{y^2}{2} + \frac{y^3}{3} - \frac{y^4}{4} - \ldots\ldots \\ y + \frac{y^2}{2} + \frac{y^3}{3} + \frac{y^4}{4} - \ldots\ldots \end{array} \right\}$$

$$= 2 \left\{ y + \frac{y^3}{3} + \frac{y^5}{5} + \ldots. \right\}$$

$$= 2 \left\{ \frac{\Delta x}{2x + \Delta x} + \frac{1}{3} \cdot \left(\frac{\Delta x}{2x + \Delta x} \right)^3 + \frac{1}{5} \cdot \left(\frac{\Delta x}{2x + \Delta x} \right)^5 + \ldots. \right\}$$

INTEGRAL CALCULUS OF FINITE DIFFERENCES.

648. To find the integral of x_4 we have

$x_4 = x + 4 = x + 4 \Delta x$ (Δx being $= 1$)

$\therefore \Sigma x_4 = \Sigma(x) + 4x + C$

Assume $\Sigma x = Ax^2 + Bx + F$

Then $x = \Delta(Ax^2 + Bx + F) = A.\overline{x+1}|^2 + B.(x+1) - Ax^2 - Bx$

$= 2Ax + A + B$

$$\left.\begin{array}{r}\therefore\ 2A = 1\\ \text{And } A + B = 0\end{array}\right\} \quad \begin{array}{l}\therefore A = \frac{1}{2}\\ B = -\frac{1}{2}\end{array}$$

Hence $\Sigma x_4 = \frac{1}{2}x^2 + \frac{7}{2}x + \text{const.}$

Otherwise.

$\Sigma x_4 = \Sigma(x + 4) = \Sigma x + 4\Sigma\Delta x$

$= \frac{(x-1).x}{2} + 4x + C = \frac{x^2}{2} + \frac{7x}{2} + C$, by the rule of successive factors, which is known to every Student.

649. First let us prepare the difference in proper factors. We have $(x-m).(x-5m) = (x-m).(x-2m-3m) = (x-m).\times(x-2m) - 3m.(x-m)$

And $(x-2m).(x-5m) = (x-2m)(x-3m-2m) = (x-2m).\times(x-3m) - 2m.(x-2m)$.

Then $\Sigma\left\{(x-m).(x-5)+\frac{(x-2m)(x-5m)}{2}\right\}=\Sigma(x-m)\times$

$(x-2m)+\frac{1}{2}\Sigma(x-2m)(x-3m)-3m.\Sigma.(x-m)-m\Sigma(x-2m)$

$=\frac{(x-m)(x-2m)(x-3m)}{3m}+(x-2m)(x-3m)\times(x-4m)\times\frac{1}{6m}-\frac{3}{2}\times$

$(x-m)(x-2m)-\frac{1}{2}.(x-2m)(x-3m)+\text{const.}$

$$=\frac{(x-2m)^2(x-3m)}{2m}-(x-2m).(2x-3m)+C$$

$$=\frac{x^3}{2m}-\frac{11}{2}x^2+15mx-12m^2+C.$$

Otherwise.

The difference may be reduced to the form

$\frac{3x^2}{2}-\frac{19m}{2}.x+10m^2=\Delta(Ax^3+Bx^2+Cx+D)$ by hypothesis

$$=3Am.x^2+(3Am+2Bm)x+(Am^3+Bm^2+Cm)$$

Hence by comparing coefficien's of homologous terms we get

$A=\frac{1}{2m}$, $B=-\frac{11}{2}$, $C=15m$, and $\therefore$

$\Sigma.\left(\frac{3x^2}{2}-\frac{19m}{2}.x+10m^2\right)=\frac{x^3}{2m}-\frac{11}{2}x^2+15mx+$ arb. const. the same as before.

650. Here we have

$$\frac{1}{VV_2V_5}=\frac{V_1V_3V_4}{VV_1V_2V_3V_4V_5}$$

Now assume $V_1V_3V_4=A\times VV_1V_2+B.VV_1+C.V+D$

Then $(V+1)(V+3).(V+4)=A.V(V+1)(V+2)+B.V.(V+1)$ $+CV+D$ supposing $\Delta V=1$ as in the enunciation.

$$\therefore V^3+8V^2+19V+12=A.V^3+\left.\begin{matrix}3A\\+B\end{matrix}\right\}.V^2+\left.\begin{matrix}2A\\+B\\+C\end{matrix}\right\}.V+D$$

$$\therefore A = 1$$
$$B = 8 - 3A = 5$$
$$C = 19 - B - 2A = 12$$
$$D = 12$$

$$\text{Hence } \Sigma.\frac{1}{V\,V_2 V_5} = \Sigma\,\frac{VV_1V_2 + 5VV_1 + 12V + 12}{VV_1V_2V_3V_4V_5}$$

$$= \Sigma\frac{1}{V_3V_4V_5} + 5.\Sigma\,\frac{1}{V_2V_3V_4V_5} + 12.\Sigma.\frac{1}{V_1\,V_2V_3\,V_4\,V_5}$$

$$+\ 12.\ \Sigma.\ \frac{1}{VV_1V_2V_3V_4V_5} = -\frac{1}{2V_3V_4} - \frac{5}{3V_2\,V_3\,V_4}$$

$$-\frac{12}{4V_1\,V_2\,V_3\,V_4} - \frac{12}{5VV_1\,V_2\,V_3\,V_4} + \text{cor.} = -\frac{1}{30}$$

$$\frac{15V^3 + 95V^2 + 170V + 72}{VV_1V_2V_3V_4} + \text{cor.}$$

651. To integrate $\dfrac{4z+3k}{z.(z+k).\,(z+2k)} = \Delta\,u_z$, we have

$$u_z = \Sigma\frac{4}{(z+k).(z+2k)} + \Sigma.\ \frac{3k}{z.(z+k)(z+2k)}$$

$$= -\frac{4}{k}\cdot\frac{1}{z+k} - \frac{3}{2}\cdot\frac{1}{z.(z+k)} + \text{const.}$$

$$= C - \frac{1}{2k}\cdot\frac{8z+3k}{z.(z+k)}$$

652. To integrate $\dfrac{1}{z}$, put $u = lz$

Then $\Delta u = \Delta.\, lz = l.\,(z + \Delta z) - lz = l.\,\dfrac{z+\Delta z}{z}$

$$= l.\left(1 + \frac{\Delta z}{z}\right)$$

$$= \frac{\Delta z}{z} - \frac{(\Delta z)^2}{2z^2} + \frac{(\Delta z)^3}{3z^3} - \text{\&c. by}$$

logarithms.

$$\therefore \frac{1}{z} = \frac{\Delta lz}{\Delta z} + \frac{\Delta z}{2z^2} - \frac{(\Delta z)^2}{3z^3} + \frac{(\Delta z)^3}{4z^4} - \frac{(\Delta z)^4}{5z^5} + \&c.$$

$$\therefore \Sigma \frac{1}{z} = \frac{lz}{\Delta z} + \Sigma . \left\{ \frac{\Delta z}{2z^2} - \frac{(\Delta z)^2}{3z^3} + \frac{(\Delta z)^3}{4z^4} - \dots \right\}$$

$$= \frac{lz}{\Delta z} + \Sigma . \left\{ \frac{a}{z^2} + \frac{b}{z^3} + \frac{c}{z^4} + \dots \right\} \text{ by supposition.}$$

Let now $\Sigma . \left(\frac{a}{z^2} + \frac{b}{z^3} + \dots \right) = \frac{A}{z} + \frac{B}{z^2} + \frac{C}{z^3} + \frac{D}{z^4} + \dots$

then we have by differencing both sides of the equation

$$\frac{a}{z^2} + \frac{b}{z^3} + \frac{c}{z^4} + \frac{d}{z^5} + \dots = -\frac{A\Delta z}{z^2} + \frac{A\Delta z^2 - 2B\Delta z}{z^3}$$

$$\frac{-A\Delta z^3 + 3B\Delta z^2 - 3C\Delta z}{z^4} + \frac{A\Delta z^4 - 4B\Delta z^3 + 6C\Delta z^2 - 4\Delta z}{z^5}$$

$-$ &c. Hence equating coefficients of like powers of z &c. we get

$$A = \frac{-a}{\Delta z} = -\frac{1}{2}$$

$$B = \frac{A\Delta z^2 - b}{2\Delta z} = -\frac{\Delta z}{12}$$

$$C = \frac{-A\Delta z^3 + 3B\Delta z^2 - c}{3\Delta z} = 0$$

$$D = \frac{A\Delta z^4 - 4B\Delta z^3 + 6C\Delta z^2 - d}{4\Delta z} = \frac{\Delta z^3}{120}$$

&c. = &c. the law of continuation being evident.

$$\text{Hence } \Sigma \left(\frac{1}{z} \right) = \frac{lz}{\Delta z} - \frac{1}{2z} - \frac{\Delta z}{12z^2} + \frac{\Delta z^3}{120z^4} - \frac{\Delta z^5}{252.z^6}$$

$+$ &c. $+$ const.

653. The integration of $\frac{1}{r^2}$ may, in a series, thus be effected.

$$\frac{1}{r^2} = \frac{1}{r} \times \frac{r_1}{rr_1} = \frac{1}{r} \times \frac{r + \Delta r}{rr_1} = \frac{1}{rr_1} + \frac{\Delta r}{r^2 . r_1}$$

Also $\frac{1}{r^2} = \frac{1}{r} \times \frac{r_2}{rr_2} = \frac{1}{r} \times \frac{r+2\Delta r}{rr_2} = \frac{1}{rr_2} + \frac{2\Delta r}{r^2 . r_2}$

And $\frac{1}{r^2} = \frac{1}{r} \times \frac{r_3}{rr_3} = \text{........} = \frac{1}{rr_3} + \frac{3\Delta r}{r^2 . r_3}$

&c. = &c. to infinity.

Hence by substituting the values of $\frac{1}{r^2}$ in these several equations we get

$$\frac{1}{r^2} = \frac{1}{rr_1} + \frac{\Delta r}{rr_1 r_2} + \frac{2(\Delta r)^2}{rr_1 r_2 r_3} + \frac{2 . 3 . (\Delta r)^3}{rr_1 r_2 r_3 r_4} +$$

$\frac{2 . 3 . 4(\Delta r)^4}{r \ldots . r_5}$ &c., each term of which being rendered separately integrable, by the common rule, we have

$$\Sigma . \frac{1}{r^2} = C - \frac{1}{\Delta r \times r} - \frac{1}{2rr_1} - \frac{2\Delta r}{3} \times \frac{1}{rr_1 r_2} -$$

$$\frac{2 . 3 \Delta r^2}{4} . \frac{1}{rr_1 r_2 r_3} - \text{\&c.}$$

This method will hold good in other functions, such as $\frac{1}{r^3}$, $\frac{1}{r^4}$ &c. The more general way, however, is to assume a series of terms consisting of the successive factors with indeterminate coefficients. See *Emerson's Increments*, and *Mr. Herschel's Examples of the Applications of the Calculus of Finite Differences.*

654. We will find the first integral, second, third, &c., in succession, taking care at each step to introduce an arbitrary constant, or such a function of x that its Difference may be zero.

In one copy we have x_v; in another xv. We will resolve both cases.

$$\Sigma . x_v = \frac{x_v x_{v-1}}{2h} + C_1 \text{ (h being the const. Increment of x)}$$

$$\therefore\ \Sigma^2 . x_v = \frac{x_v\, x_{v-1}\, x_{v-2}}{2 . 3\, h^2} + \frac{C_1\, x}{h} + C_2$$

$$\Sigma^3 . x_v = \frac{x_v\, x_{v-1}\, x_{v-2} x_{v-3}}{2 . 3 . 4\, h^3} + \frac{C_1}{h} . \Sigma x + \frac{C_2 x}{h} + C_3$$

$$= \frac{x_v \dots x_{v-3}}{2 . 3 . 4 h^3} + \frac{C_1}{2h^2} . x\, x_{-1} . + \frac{C_2 x}{h} + C_3$$

$$\Sigma^4 . x_v = \frac{x_v \dots x_{v-4}}{2.3.4.5 h^4} + \frac{C_1}{2.3 h_3} . x\, x_{-1}\, x_{-2} + \frac{C_2}{2h^2} . x\, x_{-1}$$

$$+ C_3 \frac{x}{h} + C_4$$

&c. = &c.

$$\Sigma^n . x_v = \frac{x_v\, x_{v-1} \dots x_{v-n}}{2.3.4.5 \dots (n+1) . h^n} + c_1 . x\, x_{-1}\, x_{-2} \dots x_{-(n-2)} +$$

$$C_2 . x\, x_{-1} \dots x_{-(n-3)} + c_3 . x\, x_{-1} \dots x_{-(n-4)} + c_4 . x\, x_{-1} \dots _{-(n-5)}$$
$$+ \dots + c_{n-2} . x\, x_{-1} + c_{n-1}\, x + c_n,$$

c_1, c_2, c_3, being known functions of the arbitrary constants C_1, C_2 ...., and the determinate constants h, 1, 2, 3, &c., are themselves arbitrary constants, or such functions of x as we described above.

We have exhibited $\Sigma^n x_v$ *merely* in terms of the successive values of x.

Secondly, because $\Delta . PQ = (P + \Delta P)(Q + \Delta Q) - PQ = P \Delta Q + Q_1 \Delta P$,

$\therefore\ \Sigma (P \Delta Q) = PQ - \Sigma . (\Delta P . Q_1)$ (which is similar to $\int u dv = uv - \int v du$)

$$\text{Similarly } \Sigma\ (\Delta P\ .\ Q_1) = \Delta P . \Sigma Q_1 - \Sigma . (\Delta^2 P . \Sigma Q_2)$$
$$\Sigma\, (\Delta^2 P . \Sigma Q_2) = \Delta^2 P . \Sigma^2 Q_2 - \Sigma . (\Delta^3 P . \Sigma^2 Q_3)$$
$$\&c. = \&c.$$

$\therefore \Sigma . (P \Delta Q) = PQ - \Delta P . \Sigma Q_1 + \Delta^2 P . \Sigma^2 Q_2 - \Delta^3 P . \Sigma^3 Q_3 + \dots$ (a) which is a Theorem from which others of greater generality may be deduced. Thus, applying the Theorem to the Integration of each term, and taking their sum, we have

$$\left.\begin{array}{r}\Sigma^2.(P\Delta Q)=P.\Sigma Q-\Delta P.\Sigma^2 Q_1+\Delta^2 P.\Sigma^3 Q_2-\Delta^3 P.\Sigma^4 Q_3+\ldots\\ -\Delta P.\Sigma^2 Q_1+\Delta^2 P.\Sigma^3 Q_2-\Delta^3 P.\Sigma^4 Q_3+\ldots\\ +\Delta^2 P.\Sigma^3 Q_2-\Delta^3 P.\Sigma^4 Q_3+\ldots\\ -\Delta^3 P.\Sigma^4 Q_3+\ldots\end{array}\right\}$$

$$= P.\Sigma Q-2\Delta P.\Sigma^2 Q_1+3.\Delta^2 P.\Sigma^3 Q_2-\ldots$$

$$\ldots m\Delta^{m-1} P.\Sigma^m Q_{m-1}\pm\ldots(b)$$

By proceeding in this manner with this latter form, and so on, we get

$$\Sigma^3.(P\Delta Q)=P.\Sigma^2 Q-(1+2)\,\Delta P.\Sigma^3 Q_1+(1+2+3)\,\Delta^2 P\times \Sigma^4 Q_2-\ldots\frac{(1+m)m}{2}.\Delta^{m-1}P\,\Sigma^{m+1}\,Q_{m-1}\ldots\ldots$$

$$\Sigma^4.(P.\Delta Q)=P.\Sigma^3 Q-(1+\overline{1+2})\,\Delta P.\Sigma^4 Q_1+\ldots\frac{m(m+1)(m+2)}{2.3}\times \Delta^{m-1}P.\Sigma^{m+2}.Q_{m-1}\ldots$$

$$\Sigma^5.(P.\Delta Q)=P.\Sigma^4 Q-(1+\overline{1+\overline{1+2}})\,\Delta P.\Sigma^5 Q_1+\ldots \frac{m.(m+1)(m+2)(m+3)}{2.3.4}\Delta^{m-1}P.\Sigma^{m+3}Q_{m-1}\ldots$$

&c. = &c.

The coefficients of the general terms are derived from one another as follows:

We see by the actual process that the $(m)^{th}$ terms of the respective series are of the form

± 1 ($\pm$ according as m is odd or even)

$\pm(1+1\ldots\ldots m \text{ terms})=m$

$\pm(1+2+3+\ldots m)=\dfrac{m.(m+1)}{2}$ (by taking the Integral)

$$\pm\left\{1+3+6+\ldots\frac{m.(m+1)}{2}\right\}=\frac{m.(n+1)(n+2)}{2.3}$$

(Integrating)

$$\pm\left\{1+4+10+\ldots\frac{m.(m+1)(m+2)}{2.3}\right\}=\frac{m.(m+1).(n+2).(n+3)}{2.3.4}$$

(integrating)

&c. = &c.

Hence then it appears that the coefficient of the $(m)^{th}$ term of the $(n)^{th}$ integral $\Sigma^n (P \Delta Q)$, is

$$T_m = \pm \frac{m.(m+1).(m+2).....(m+n-2)}{1.2.3.....n-1}$$

And we $\therefore$ get

$\Sigma^n (P \Delta Q) = T_1 \times P \Sigma^{n-1} Q + T_2 \times \Delta P . \Sigma^n Q_1 + T_3 \times \Delta^2 P . \Sigma^{n+1} Q_2 + T_4 \times \Delta^3 P . \Sigma^{n+2} Q_3 + ... + T_m \times \Delta^{m-1} P . \Sigma^{n+m-2} Q_{m-1} +$ &c., T_1, T_2, T_3 .... expressing the values of T_m, when (m) is 1, 2, 3 ..... respectively.

Now in order to resolve the problem in question $\Sigma^n (xv)$,

$$\left.\begin{array}{l} \text{put } x = P \\ v = \Delta Q \end{array}\right\} \text{Then we get } \left.\begin{array}{l} Q = \Sigma v \\ Q_1 = \Sigma v_1 \\ \text{\&c.} = \text{\&c.} \end{array}\right\} \text{\&c. and} \therefore \text{have}$$

$\Sigma^n (xv) = T_1 \times x \Sigma^n v + T_2 \times \Delta x \Sigma^{n+1} v_1 + T_3 \times \Delta^2 x \times \Sigma^{n+2} v_2 +$ &c.

$+ T_m \times \Delta^{m-1} x . \Sigma^{n+m-1} v_{m-1} +$ ..... a Theorem which reduces the $(n)^{th}$ Integral of the product of any two quantities, to the resolution of the general term

$$T_m \times \Delta^{m-1} x . \Sigma^{n+m-1} v_{m-1}$$

which implies the product $T_m \times$ {the $(m-1)^{th}$ difference of one of the quantities} $\times$ {$(n+m-1)^{th}$ Integral of the $(m-1)^{th}$ succeeding value of the other quantity}. Having found this term by common rules, and the integration of the form Σx_v, already exhibited in this discussion, we may proceed to find each term of the series for $\Sigma^n (xv)$, by putting $m = 1, 2, 3, 4$ .... in succession. The sum of the results, annexed to the $(m+1)^{th}$ terms of the forms for $\Sigma (xv)$, $\Sigma^2 (xv)$, ... $\Sigma^n (xv)$, which we omitted writing down for the sake of brevity, will be the complete Integral of (xv).

Having said so much already on this subject, we must leave the rest to the student, who will be amply gratified by applying this very important Theorem to the differences

$$x^2 . \cos. x\theta,\ x^2 . (x+1) . \sin. (x\theta)^p,\ e^x \cos. x\theta$$
$$x^p . a^x, \text{ \&c. \&c. \&c.}$$

The general coefficients expressed by T_m are those of the several orders of figurates.

655. To integrate x^3 we have

$$x^3 = x^2 . (x - 1 + 1) = x^2 . (x - 1) + x^2$$
$$= x . (x - 1) . (x + 1 - 1) + x . (x - 1 + 1)$$
$$= (x - 1) . x . (x + 1) + x$$
$$\therefore \Sigma . x^3 = \frac{(x - 2) . (x - 1) . x . (x + 1)}{4} + \frac{(x - 1) . x}{2} + C$$

$(\Delta\ x = 1)$

$$= \frac{x^4}{4} - \frac{x^3}{2} + \frac{x^2}{4} + C$$
$$\text{or} = \frac{x^2 . (x - 1)^2}{4} + C$$

To integrate $e^x \cos. x\theta$ we have,

$$\cos. x\theta = \frac{e^{x\theta\sqrt{-1}} + e^{-x\theta\sqrt{-1}}}{2}$$
$$\therefore \Sigma\, (e^x \cos. x\theta) = \Sigma\left\{\frac{e^{x.(1+\theta\sqrt{-1})}}{2} + \frac{e^{x.(1-\theta\sqrt{-1})}}{2}\right\}$$

Now, generally, $\Delta\ e^u = e^{u+\Delta u} - e^u = e^u . (e^{\Delta u} - 1)$

$$\therefore e^u = \frac{\Delta e^u}{e^{\Delta u} - 1}$$

and $\Sigma . e^u = \Sigma\ \dfrac{\Delta e^u}{e^{\Delta u} - 1} = \dfrac{e^u}{e^{\Delta u} - 1} + \text{cons.}$ provided Δu be constant.

$\therefore$ Supposing Δx, and $\therefore\ \Delta\ (x . \overline{1 + \theta\sqrt{-1}})$ and $\Delta . (x . \overline{1 - \theta\sqrt{-1}})$ constant, we have

$$\Sigma\, (e^x . \cos. x\ \theta) = \frac{1}{2} \times \frac{e^{x.\overline{1+\theta.\sqrt{-1}}}}{e^{(1+\theta\sqrt{-1})\Delta x} - 1} + \frac{1}{2} \times \frac{e^{x.\overline{1-\theta.\sqrt{-1}}}}{e^{(1-\theta\sqrt{-1})\Delta x} - 1} + C$$

$$= \frac{1}{2} e^x \times \left\{\frac{e^{x\theta\sqrt{-1}}}{e^{(1+\theta\sqrt{-1})\Delta x} - 1} + \frac{e^{-x\theta\sqrt{-1}}}{e^{(1-\theta\sqrt{-1})\Delta x} - 1}\right\} + C$$

$$= \frac{e^x}{\left\{e^{\overline{1+\theta\sqrt{-1}}.\Delta x}-1\right\}\left\{e^{\overline{1-\theta\sqrt{-1}}.\Delta x}-1\right\}} \times$$

$$\left\{\frac{e^{(x\theta-\theta\Delta x)\sqrt{-1}}+e^{-(x\theta-\theta\Delta x)\sqrt{-1}}}{2} \times e^{\Delta x} -\right.$$

$$\left.\frac{e^{x\theta\sqrt{-1}}+e^{-x\theta\sqrt{-1}}}{2}\right\} + \mathrm{C} = \frac{e^x}{e^{2\Delta x}-2e^{\Delta x}.\cos.(\theta\Delta x)+1} \times$$

$\left\{e^{\Delta x} \times \cos.(\overline{x-\Delta x}.\theta) - \cos. x\theta\right\} + \mathrm{C}$, which is by no means an inelegant result.

This method applies to the general form $e^x . \cos.^m x\theta \times \sin.^n (x\theta)$.

656. To integrate the Equation $u_{x+2} - \mathrm{A}u_{x+1} + \mathrm{B}u_x = 0$

Assume $u_x = \mu^x + \mathrm{K}$ (μ and K being independent of x)

$$\text{Then } \left.\begin{aligned} u_{x+1} &= \mu^{x+1} + \mathrm{K} \\ u_{x+2} &= \mu^{x+2} + \mathrm{K} \end{aligned}\right\} \therefore \text{ substituting, we get}$$

$$\mu^{x+2} - \mathrm{A}\mu^{x+1} + \mathrm{B}\mu^x + \mathrm{K}.(1-\mathrm{A}+\mathrm{B}) = 0$$

$$\text{Put K} = 0$$

$$\text{Then } \mu^{x+2} - \mathrm{A}\mu^{x+1} + \mathrm{B}\mu^x = 0$$

$$\therefore \mu^2 - \mathrm{A}\mu + \mathrm{B} = 0$$

$$\text{Hence } \mu = \frac{\mathrm{A} \pm \sqrt{\mathrm{A}^2 - 4\mathrm{B}}}{2}$$

$$\therefore u_x = \mu^x + \mathrm{K} = \left\{\frac{\mathrm{A}+\sqrt{\mathrm{A}^2-4\mathrm{B}}}{2}\right\}^x + \mathrm{K}$$

$$\text{Or} = \left\{\frac{\mathrm{A}-\sqrt{\mathrm{A}^2-4\mathrm{B}}}{2}\right\}^x + \mathrm{K} \text{ which}$$

being particular integrals, the general integral will be

$$u_x = \mathrm{C}.\left\{\frac{\mathrm{A}+\sqrt{\mathrm{A}^2-4\mathrm{B}}}{2}\right\}^x + \mathrm{C}_1.\left\{\frac{\mathrm{A}-\sqrt{\mathrm{A}^2-4\mathrm{B}}}{2}\right\}^x$$

See the *Appendix* to the *Translation of Lacroix*, or *Garnier*, or *Du Bourguet*.

657. To integrate $\frac{x}{a^x}$ we have the form

$$\Sigma . uv = u\Sigma v - \Sigma(\Delta u\, \Sigma v_1) \text{ which gives}$$

$$\Sigma \frac{x}{a^x} = x\,\Sigma . \frac{1}{a^x} - \Sigma(\Delta\, x\, \Sigma\, v_1)$$

$$= \frac{x}{a^x} \times \frac{a^{\Delta x}}{1-a^{\Delta x}} - \frac{\Delta x}{a^{\Delta x}}\, \Sigma^2 . \frac{1}{a^x}$$

$$= \frac{a^{\Delta x}}{1-a^{\Delta x}} \times \frac{1}{a^x} \times \left\{ x - \frac{\Delta x}{a^{\Delta x}} . \frac{a^{\Delta x}}{1-a^{\Delta x}} \right\}$$

$$= \frac{a^{\Delta x}}{1-a^{\Delta x}} \times \frac{1}{a^x} \times \left\{ x - \frac{\Delta x}{1-a^{\Delta x}} \right\} + C.$$

SERIES.

658. Since the Scale of Relation is $f + g$, we have

$$\begin{aligned} a &= a \\ bx &= bx \\ cx^2 &= fbx^2 + gax^2 \\ dx^3 &= fcx^3 + gbx^3 \\ \&c. &= \&c. \end{aligned}$$

Hence $S = a + bx + fx . (S - a) + gx^2 . S$ (S being the sum of the infinite series)

$$\therefore S = \frac{a + bx - afx}{1 - fx - gx^2} = \frac{a + (b - af)x}{(1 - mx) . (1 - nx)}$$ (by supposition)

Hence assuming, $\frac{1}{(1 - mx) . (1 - nx)} = \frac{A}{1 - mx} + \frac{B}{1 - nx}$

$$= \frac{(A + B) - (mB + nA) . x}{(1 - mx)(1 - nx)}$$

we have $\left.\begin{aligned} A + B &= 1 \\ nA + mB &= 0 \end{aligned}\right\}$ and $\therefore \left.\begin{aligned} A &= \frac{m}{m - n} \\ \text{and } B &= -\frac{n}{m - n} \end{aligned}\right\}$ m and n being found from the equation $1 - fx - gx^2 = 0$.

Hence $S = \frac{m}{m - n} \times \frac{a + (b - af)x}{1 - mx} - \frac{n}{m - n} \times \frac{a + (b - af)x}{1 - nx}$

$$= \frac{m . (a + \overline{b - af} . x)}{m - n} \times (1 + mx + m^2x^2 + m^3x^3 + \ldots . \infty)$$

$$- \frac{n . (a + \overline{b - af} . x)}{m - n} \times (1 + nx + n^2x^2 + n^3x^3 + \ldots . \infty)$$

Which are two Geometric Series, whose first terms are $\frac{m(a+\overline{b-af}\cdot x)}{m-n}$, $-\frac{n.(a+\overline{b-af}\cdot x)}{m-n}$, and common ratios mx, nx, respectively.

When $m = n$, or $f^2 = -4g$, the series becomes

$$S = \frac{a+(b-af)\cdot x}{\left(1-\frac{fx}{2}\right)^2} = \frac{a+(b-af)x}{1-\frac{fx}{2}} \times \frac{1}{1-\frac{fx}{2}}$$

$$= \frac{a+(b-af)\cdot x}{1-\frac{fx}{2}} \times \left(1 + \frac{fx}{2} + \frac{f^2x^2}{2^2} + \frac{f^3x^3}{2^3} + \&c.\right)$$

so that in the case of the denominator of the generating fraction having equal roots, the Series may be expressed by *one Geometrical Series*. See No. 671.

In the same manner a Series whose scale of relation is $f_1 + f_2 + f_3 \ldots\ldots f_n$ may be resolved into (n) Geometrical Series, when the roots of the equation

$$1 - f_1 x - f_2 x^2 - \ldots\ldots f_n x^n = 0$$

are unequal, and into one, when those roots are equal.

This Problem may be solved by the method of Finite Differences. See *Appendix to Translation of Lacroix*, p. 534.

659. It is well known that

$$x - \frac{x^2}{2} + \frac{x^3}{3} - \&c. = l.\,(1+x) = -l.\left(\frac{1}{1+x}\right)$$

$$= -l.\left(1 - \frac{x}{1+x}\right).$$

$$\therefore\ x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots = \frac{x}{1+x} + \frac{1}{2}\cdot\left(\frac{x}{1+x}\right)^2 +$$

$\frac{1}{3}\cdot\left(\frac{x}{1+x}\right)^3 + \&c.$, the form required.

This might have been determined by the method of indeterminate coefficients.

660. Given $a + b = s$, and $ab = p$, to find $a^n + x^n$ in a series.

$$\begin{aligned} \text{We have } a^2 + 2ab + b^2 &= s^2 \\ \text{and } \quad 4ab &= 4p \\ \therefore a^2 - 2ab + b^2 &= s^2 - 4p \\ \therefore a - b &= \pm \sqrt{s^2 - 4p} \\ \text{and } a + b &= s \end{aligned}$$

$$\text{Hence } a = \frac{s \pm \sqrt{s^2 - 4p}}{2} = \frac{s \pm v}{2}$$

$$\text{and } b = \frac{s \mp \sqrt{s^2 - 4p}}{2} = \frac{s \mp v}{2} \; (v = \sqrt{s^2 - 4p})$$

$$\left.\begin{aligned} \therefore 2^n a^n &= s^n \pm n s^{n-1} v + n . \frac{n-1}{2} . s^{n-2} v^2 \pm s^{n-3} v^3 + \&c. \\ 2^n b^n &= s^n \mp n s^{n-1} v + n . \frac{n-1}{2} s^{n-2} v^2 \mp s^{n-3} v^3 + \&c. \end{aligned}\right\}$$

$$\text{Hence } a^n + b^n = \frac{1}{2^{n-1}} \times \left\{ s^n + n . \frac{n-1}{2} . s^{n-2} v^2 + n \times \frac{n-1}{2} . \frac{n-2}{3} . \frac{n-3}{4} s^{n-4} v^4 + n . \frac{n-1}{2} . \frac{n-2}{3} . \frac{n-3}{4} . \frac{n-4}{5} \times \frac{n-5}{6} s^{n-6} v^6 + \&c. \right\}$$

the expression required.

Otherwise,

This problem may also be solved by the method of summing the n^{th}. powers of the roots of the equation

$x^2 - sx + p = 0$ whose roots are a and b.

661. To find the value of $a^n + b^n$, when a and b are roots of the equation.

$$x^2 - px + 1 = 0$$

$$\text{We have } \left.\begin{aligned} a + b &= p \\ \text{and } ab &= 1 \end{aligned}\right\}$$

Hence $a^n + b^n = a^n + \dfrac{1}{a^n}$ and the above methods may

be applied to obtain the series required. A better method, however, readily suggests itself;

$$\text{Put } a + \frac{1}{a} = 2 \cos.\ \theta$$

Then $a^n + \frac{1}{a^n} = 2 \cos.\ n\theta$ which being expanded according to the powers of cos. θ will evidently give the series required. This expansion is effected in different ways by Authors. The most simple and obvious, perhaps, is the following:

From the form

$$\text{Cos.}\,(A + B) + \cos.\,(A - B) = 2 \cos. A . \cos. B \text{ we get}$$

$$\text{Cos. } n\theta = 2 \cos.\ \theta . \cos.\ (n - 1)\ \theta - \cos.\ (n - 2)\ \theta$$
$$= p . \cos.\ (n - 1)\ \theta - \cos.\ (n - 2)\ \theta$$

$$\text{Similarly Cos. } (n - 1)\,\theta = p . \cos.\ (n - 2)\ \theta - \cos.\,(n - 3)\ \theta$$
$$\text{Cos. } (n - 2)\ \theta = p . \cos.\ (n - 3)\ \theta - \cos.\ (n - 4)\ \theta$$
$$\&c. = \&c.$$
$$\text{Cos. } 2\theta = p . \cos.\ \theta$$

Hence by substitution we get

$$\text{Cos. } n\theta = p . \cos.\ (n - 1)\ \theta - \cos.\ (n - 2)\ \theta$$
$$= (p^2 - 1) . \cos.\,(n-2)\ \theta - p \cos.\,(n-3)\ \theta$$
$$= (p^3 - 2p) \cos.\ (n-3)\ \theta - (p^2 - 1) . \cos.\,(n-4)\ \theta$$
$$= (p^4 - 3p^2 + 1) \cos.\ (n-4)\ \theta - (p^3 - 2p) . \cos.\ (n-5)\ \theta$$
$$= \&c.$$
$$= \{p^{n-1} - (n - 2) . p^{n-3} + \&c.\} \cos.\ \theta - \{p^{n-2} - (n-3)\ p^{n-4} + \&c.\} \cos.\ 0 .$$
$$= \{ p^{n-1} - (n-2) . p^{n-3} + \ldots\}\frac{p}{2} - \{p^{n-2} - (n-3) \times p^{n-4} + \ldots\}$$

$$\text{Hence } 2 \cos.\ n\,\theta = p^n - n . p^{n-2} + n . \frac{n-3}{2} p^{n-4} - n . \frac{n-4}{2} . \frac{n-5}{2} p^{n-6} + \ldots$$

$$\therefore a^n + b^n = p^n - \&c.$$

On this subject the reader may consult *Woodhouse's Trig.* and a paper by Mr. Knight in the third vol. of *Leybourne's Repository*.

662. To resolve $\frac{1}{1+x^n}$ into quadratic divisors.

Put $x^n + 1 = 0$ and first suppose n even.

Then $x^n = -1 = \cos.(2p+1)\pi$

$$= \cos.(2p+1)\pi \pm \sqrt{-1}\,.\,\sin.(2p+1)\pi$$

$$\therefore x = \cos.\frac{2p+1}{n}\pi \pm \sqrt{-1}\,.\,\sin.\frac{2p+1}{n}\pi$$ a form exhibiting the n values of x in pairs, corresponding to $p = 0$, $1, 2$, &c. $\frac{n}{2} - 1$

Hence $x^n + 1 = \left(x - \cos.\frac{\pi}{n} - \sqrt{-1}\sin.\frac{\pi}{n}\right).\left(x - \cos.\frac{\pi}{n} + \sqrt{-1}.\sin.\frac{\pi}{n}\right) \times \left(x - \cos.\frac{3\pi}{n} - \sqrt{-1}.\sin.\frac{\pi}{n}\right).\left(x - \cos.\frac{3\pi}{n} + \sqrt{-1}\sin.\frac{\pi}{n}\right) \times$ &c. $= \left(\overline{x - \cos.\frac{\pi}{n}}\right|^2 + \sin.^2\frac{\pi}{n}\right) \times \left(\overline{x - \cos.\frac{3\pi}{n}}\right|^2 + \sin.^2\frac{3\pi}{n}\right) \times$ &c. $= (x^2 - 2x.\cos.\frac{\pi}{n} + 1) \times (x^2 - 2x.\cos.\frac{3\pi}{n} + 1) \times (x^2 - 2x.\cos.\frac{5\pi}{n} + 1) \times$ &c. to $\frac{n}{2}$ terms.

Again, let n be odd.

Then $x = \cos.\frac{2p+1}{n}\pi \pm \sqrt{-1}.\ \sin.\frac{2p+1}{n}\pi$, and the pairs of values corresponding to $p = 0, 1, 2 \ldots\ldots \frac{n-3}{2}$, are $\cos.\frac{\pi}{n} \pm \sqrt{-1}.\sin.\frac{\pi}{n}$, $\cos.\frac{3\pi}{n} \pm \sqrt{-1}\sin.\frac{3\pi}{n}$, &c....., the other value of x being evidently $= -1$.

Hence, as before, we get

$$x^n + 1 = (x+1).(x^2 - 2x.\cos.\frac{\pi}{n} + 1).(x^2 - 2x\cos.\frac{3\pi}{n} + 1) \times \text{\&c.}$$ there being $\frac{n-1}{2}$ of the quadratic factors.

663. Here, by the form

$$\cos.(A+B)+\cos.(A-B)=2\cos.A.\cos.B$$

we have

$$\cos.4x=2\cos.x.\ \cos.3x-\cos.2x$$

$$\cos.3x=2\cos.x.\ \cos.2x-\cos.x$$

$$\cos.2x=2\cos.x.\ \cos.x-1$$

$$\therefore \cos.4x=(2^2.\cos.^2x-1).\cos.2x-2\cos.^2x$$

$$=(2^2.\cos.^2x-1).\ 2\cos.^2x-2\cos.^2x-2^2.\cos.^2x+1$$

$$=2^3.\cos.^4x-\overset{2.2}{\cancel{4}}\cos.^2x+1.$$ See Number 661.

664. Here we have

$$lx=l.\frac{1+x}{1+\frac{1}{x}}=l.(1+x)-l.\left(1+\frac{1}{x}\right)$$

$$\text{But } l.(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\ldots\ldots$$

$$\text{And } l.(1+x^{-1})=x^{-1}-\frac{x^{-2}}{2}+\frac{x^{-3}}{3}-\frac{x^{-4}}{4}+\ldots..$$

$$\therefore lx=x-x^{-1}-\frac{x^2-x^{-2}}{2}+\frac{x^3-x^{-3}}{3}-\ldots..$$

665. Since by the above problem

$$lx=x-x^{-1}-\frac{x^2-x^{-2}}{2}+\frac{x^3-x^{-3}}{3}-\ldots..$$

Put $x=e^{z\sqrt{-1}}$

$$\text{Then } z\sqrt{-1}=e^{z\sqrt{-1}}-e^{-z\sqrt{-1}}-\frac{e^{2z\sqrt{-1}}-e^{-2z\sqrt{-1}}}{2}$$

$$+\frac{e^{3z\sqrt{-1}}-e^{-3z\sqrt{-1}}}{3}-\ldots..$$

$$=2\sqrt{-1}\sin.z-2.\sqrt{-1}.\frac{\sin.2z}{2}+2\sqrt{-1}.\frac{\sin.3z}{3}-\ldots..$$

$$\therefore \frac{z}{2}=\sin.z-\frac{\sin.2z}{2}+\frac{\sin.3z}{3}-\ldots..$$

666. Since $2\cos. x = e^{x\sqrt{-1}} + e^{-x\sqrt{-1}}$, we have

$$2^n.\cos.^n x = (e^{x\sqrt{-1}} + e^{-x\sqrt{-1}})^n = e^{nx\sqrt{-1}} + e^{-nx\sqrt{-1}} + n.(e^{(n-2)x\sqrt{-1}} + e^{-(n-2)x\sqrt{-1}}) + n.\frac{n-1}{2}.(e^{(n-4)x\sqrt{-1}} + e^{-(n-4)x\sqrt{-1}}) + \ldots\ldots \text{ to}$$

$\frac{n+1}{2}$ or $\frac{n}{2}$ terms, according as n is odd or even; in the latter case we have a middle term, viz. the $\left(\frac{n}{2} + 1\right)^{\text{th}}$ term in which no power of e appears.

First, let n be odd.

Then $2^{n-1}\cos.^n x = \cos. nx + n.\cos.(n-2)x + n.\frac{n-1}{2} \times$

$$\cos.(n-4)x + \&c. + \ldots n.\frac{n-1}{2}.\frac{n-2}{3}\ldots.\frac{\frac{n+3}{2}}{\frac{n-1}{2}} \times \cos. x$$

Let n be even.

Then $2^{n-1}.\cos.^n x = \cos. nx + n.\cos.(n-2)x + \ldots. n.\frac{n-1}{2}\ldots.$

$$\frac{\frac{n}{2}+2}{\frac{n}{2}-1}\cos. 2x + \text{middle term.}$$

$$\text{This } \left(\frac{n}{2}+1\right)^{\text{th}} \text{ term} = \frac{1}{2} \times \frac{n.(n-1)(n-2)\ldots.\left(\frac{n}{2}+1\right)}{1\ .\ 2\ .\ 3\ \ldots\ldots.\ \frac{n}{2}}$$

$$= 2^{\frac{n}{2}-1} \times \frac{1.3.5\ldots.n-1}{1.2.3\ \ldots..\ \frac{n}{2}}$$

$$\text{For } \frac{n.(n-1).(n-2)\ldots.\left(\frac{n}{2}+1\right)}{1\ .\ 2\ .\ 3\ \ldots..\ \frac{n}{2}} = \frac{2n.(2n-2).(2n-4)\ldots(n+2)}{2.4.6\ \ldots.\ n}$$

$$\times\frac{1.3.5....n-1}{1.3.5....n-1}=2^{\frac{n}{2}}.\frac{n.(n-1).(n-2)....\frac{n}{2}+1}{1.2.3\;....\;(n-1)\;.\;n}\times 1.3.5\,....\,(n-1)$$

$$=2^{\frac{n}{2}}.\frac{1.3.5....(n-1)}{1.2.3\;.....\;\frac{n}{2}}$$

Hence then we have

$$(\cos.\,x)^n=\frac{1}{2^{n-1}}\times\{\cos.\,nx+n.\cos.\,(n-2)\,x+n.\frac{n-1}{2}.\cos.\,(n-4)x$$

$$+\;....\;+\;\frac{n.\,(n-1)\;....\;\frac{n+3}{2}}{1.2\;....\;\frac{n-1}{2}}.\;\cos.\;x\}\text{ when }n\text{ is odd.}$$

$$\text{And }(\cos.\,x)^n=\frac{1}{2^{n-1}}\times\;\{\cos.nx+n.\cos.\,(n-2)\,x+...$$

$$\frac{n.\,(n-1)....\frac{n}{2}+2}{2.3\;.....\;\frac{n}{2}-1}.\;\cos.\,2x\}+\frac{1}{2^{\frac{n}{2}}}\times\frac{1\;.\;3\;.\;5\;....\;n-1}{1.2\;....\;\frac{n}{2}}$$

667. Let the arc be x, then, by Taylor's theorem,

$$u'=\sin.\,(x+h)=\sin.\,x+\frac{d.\sin.\,x}{dx}.\,h+\frac{d^2.\sin.x}{dx^2}.\,\frac{h^2}{1.2}+\text{\&c.}$$

$$\left.\begin{array}{l}\text{Now }\dfrac{d.\sin.\,x}{dx}=\cos.\;x\;=1\\ \therefore\;\dfrac{d^2.\sin.x}{dx^2}=-\sin.\,x=0\\ \quad\;\dfrac{d^3.\sin.\,x}{dx}=-\cos.x=-1\\ \quad\;\text{\&c.}=\text{\&c.}\qquad=0\end{array}\right\}\text{ when }x=0$$

$$\therefore\;\sin.\,(h)=h-\frac{h^3}{1.2.3}+\frac{h^5}{1.2....5}-\text{\&c.}$$

Let $h=x$

$$\text{Then }\sin.\,x=x-\frac{x^3}{1.2.3}+\frac{x^5}{1.2.3.4.5}-.....$$

668. Put $z^n - 1 = 0$

Then $z^n = 1 = \cos. 2p\pi \pm \sqrt{-1}. \sin. 2p\pi$ (p being any integer.)

And $z = \cos. \frac{2p}{n}\pi \pm \sqrt{-1} \sin. \frac{2p}{n}\pi$ (by Demoivre's theorem,)

a form exhibiting the n values of z in pairs, corresponding to those of p, viz. 0, 1, 2, 3 ..... $\frac{n}{2}$. These values are

$$\cos. 0 \pm \sin. 0 = 1$$

$$\cos. \frac{2\pi}{n} \pm \sqrt{-1} \sin. \frac{2\pi}{n}$$

$$\cos. \frac{4\pi}{n} \pm \sqrt{-1}. \sin. \frac{4\pi}{n}$$

$$\cos. \frac{6\pi}{n} \pm \sqrt{-1}. \sin. \frac{6\pi}{n}$$

&c. &c.

$$\cos. \pi \pm \sqrt{-1}. \sin. \pi = -1$$

Hence, by the Theory of Equations, we get

$$z^n - 1 = (z+1).(z-1) \times \left(z - \cos. \frac{2\pi}{n} + \sqrt{-1} \sin. \frac{2\pi}{n}\right) \times$$
$$\left(z - \cos. \frac{2\pi}{n} - \sqrt{-1}. \sin. \frac{2\pi}{n}\right) \times \left(z - \cos. \frac{4\pi}{n} + \sqrt{-1}. \sin. \frac{4\pi}{n}\right) \times$$
$$\left(z - \cos. \frac{4\pi}{n} - \sqrt{-1}. \sin. \frac{4\pi}{n}\right) \times \&c. = (z^2 - 1).\left(z^2 - 2z. \times \cos. \frac{2\pi}{n} + 1\right).\left(z^2 - 2z. \cos. \frac{4\pi}{n} + 1\right) \times \&c. \text{ to } \frac{n}{2} \text{ terms.}$$

Assume
$$\frac{1}{z^n - 1} = \frac{P}{z-1} + \frac{Q}{z+1} + \frac{A_2 + B_2 z}{z^2 - 2z. \cos. \frac{2\pi}{n} + 1} +$$
$$\frac{A_3 + B_3 z}{z^2 - 2z. \cos. \frac{4\pi}{n} + 1} + \dots + \frac{A_{\frac{n}{2}} + B_{\frac{n}{2}} z}{z^2 - 2z. \cos. \frac{n-2}{n}\pi + 1} = \frac{A_1 + B_1 z}{D_1}$$
$$+ \frac{A_2 + B_2 z}{D_2} + \dots \frac{A_{\frac{n}{2}} + B_{\frac{n}{2}} z}{D_{\frac{n}{2}}}$$
by supposition. Hence, reducing to a common denominator, we have

$\{P(z+1) + Q.(z-1)\} \times D_2 . D_3 D_4 \dots D_{\frac{n}{2}} + (A_2 + B_2 z) D_1 \times$
$D_3 D_4 \dots D_{\frac{n}{2}} + (A_3 + B_3 z) D_1 D_2 D_4 D_5 \dots D_{\frac{n}{2}} + \dots + (A_{\frac{n}{2}} + B_{\frac{n}{2}} z) \times$
$D_1 D_2 D_3 \dots D_{\frac{n}{2}-1} = 1$

Now, substituting for z each of its values $(a_2, a_3 \dots)$ in $D_2 D_3 \dots D_{\frac{n}{2}}$ we get

$$A_2 + B_2 a_2 = \frac{1}{d_2}$$

$$A_3 + B_3 a_3 = \frac{1}{d_3}$$

$$\&c. = \&c.$$

$$A_{\frac{n}{2}} + B_{\frac{n}{2}} a_{\frac{n}{2}} = \frac{1}{d_{\frac{n}{2}}}$$

d_2, d_3, &c. denoting the values of $D_1 . D_3 . D_4 \dots D_{\frac{n}{2}}$, $D_1 D_2 D_4 \dots D_{\frac{n}{2}}$ &c. resulting from those respective substitutions.

Now, the roots $a_2, a_3 \dots$ being imaginary, and $\therefore d_2, d_3$, &c. also of that form, we may at once determine A_2, B_2; A_3, B_3, &c., by equating the real, and the imaginary terms in these equations. It only remains then to determine P and Q, which may be readily done by substituting for x the values 1, -1 successively.

669. To resolve $\frac{1}{z^{2n} - 2lz^n + 1}$ into quadratic factors,

we will put $z^{2n} - 2lz^n + 1 = 0$

Then $z^n = l \pm \sqrt{l^2 - 1}$

$= l \pm \sqrt{-1} . \sqrt{1 - l^2}$

Let $\cos. \theta \therefore = l$. (since l is less than unit)

Then $z^n = \cos. \theta \pm \sqrt{-1} . \sin. \theta = \cos. (\theta + 2p\pi) \pm \sqrt{-1} \times \sin. (\theta + 2p\pi)$.

Hence $z = \cos.\dfrac{\theta+2p\pi}{n} \pm \sqrt{-1}.\sin.\dfrac{\theta+2p\pi}{n}$ a form which exhibits the (n) pairs of roots of the equation corresponding to the values 0, 1, 2, 3, ... $n-1$ of p.

Hence then we have

$z^{2n}-2lz^n+1 = \left(z-\cos.\dfrac{\theta}{n}-\sqrt{-1}.\sin.\dfrac{\theta}{n}\right).\left(z-\cos.\dfrac{\theta}{n}+\sqrt{-1}\times\right.$ $\left.\sin.\dfrac{\theta}{n}\right) \times \left(z-\cos.\dfrac{\theta+2\pi}{n}-\sqrt{-1}.\sin.\dfrac{\theta+2\pi}{n}\right)\left(z-\cos.\dfrac{\theta+2\pi}{n}\right.$ $\left.+\sqrt{-1}.\sin.\dfrac{\theta+2\pi}{n}\right) \times$ &c. $=\left(z^2-2z.\cos.\dfrac{\theta}{n}+1\right)\left(z^2-2z\times\right.$ $\left.\cos.\dfrac{\theta+2\pi}{n}+1\right)\left(z^2-2z\cos.\dfrac{\theta+4\pi}{n}+1\right) \times$ &c. which is Cote's Theorem.

670. Putting log. cos. A $= f(\text{A}) = u$, we have, by Maclaurin's Theorem,

$$u = u_0 + \frac{d_0u}{d\text{A}}.\text{A} + \frac{d_0^2u}{d\text{A}^2}.\frac{\text{A}^2}{1.2} + \frac{d_0^3u}{d\text{A}^3}.\frac{\text{A}^3}{1.2.3} + \ldots.$$

$\dfrac{d_0u}{d\text{A}}, \dfrac{d_0^2u}{d\text{A}^2}$ being the values of the differential coefficients on the supposition that A $= 0$.

$$\text{Now}\ \frac{\text{M}du}{d\text{A}} = \frac{-\sin.\text{A}.}{\cos.\text{A}} = -\tan.\text{A}.$$

$$\frac{\text{M}d^2u}{d\text{A}^2} = -\frac{1}{\cos.^2\text{A}}$$

$$\frac{\text{M}d^3u}{d\text{A}^3} = \frac{-2\sin.\text{A}}{\cos.^3\text{A}}$$

$$\frac{\text{M}d^4u}{d\text{A}^4} = \frac{-2}{\cos.^2\text{A}} - \frac{6\sin.^2\text{A}}{\cos.^4\text{A}} = \frac{4}{\cos.^2\text{A}} - \frac{6}{\cos.^4\text{A}}$$

&c. = &c.

Hence $u_0 = \log.\cos.0 = \log.(1) = 0$

$$\frac{d_0u}{d\text{A}} = 0$$

$$\frac{d_0^2 u}{dA^2} = -\frac{1}{M}.\ \frac{1}{\cos.^2 0} = -\frac{1}{M}$$

$$\frac{d_0^3 u}{dA^3} = 0$$

$$\frac{d_0^4 u}{dA^4} = \frac{1}{M}.(4-6) = \frac{-2}{M}$$

&c. = &c.

$$\text{Hence } u = -\frac{1}{M}.\left\{\frac{A^2}{1.2} + \frac{2A^4}{1.2.3.4} + \ldots\ldots\right\}$$

$$= -\frac{1}{M}.\left\{\frac{A^2}{2} + \frac{A^4}{3.4} + \ldots\ldots\right\}$$

671. Let $A + By + Cy^2 + Dy^3 + Ey^4 + \ldots = S$ be the recurring Series, y being a quantity perfectly arbitrary, and A, B, C, &c., the absolute values of the terms.

$$A = A$$
$$By = By$$
$$Cy^2 = Cy^2$$
$$Dy^3 = fCy^3 - gBy^3 + eAy^3$$
$$Ey^4 = fDy^4 - gCy^4 + eBy^4$$
$$Fy^5 = fEy^5 - gDy^5 + eCy^5$$
&c. = &c.

Hence $S = A + By + Cx^2 + fy.(S - By - A) - gy^2.(S - A) + ey^3.S$

$$\therefore\ S = \frac{A + By + Cy^2 - fy.(By + A) + gy^2 A}{1 - fy + gy^2 - ey^3}$$

$$\therefore\ S = \frac{(C - fB + qA)y^2 - (B - fA)\,y + A}{1 - fy + gy^2 - ey^3}$$

Put $x = \dfrac{1}{y}$. Then $y = \dfrac{1}{x}$ and substituting, we get

$$S = x \times \frac{Ax^2 - (B - fA)\,x + C - fB + qA}{x^3 - fx^2 + gx - e}$$

Now, supposing n, m, r the three *unequal roots* of

$$x^3 - fx^2 + gx - e = 0,$$

Assume $\frac{S}{x} = \frac{N}{x-n} + \frac{M}{x-m} + \frac{R}{x-r}$

$$= \frac{N.(x-m).(x-r) + M\,(x-n).(x-r) + R.(x-m).(x-n)}{x^3 - fx^2 + gx - e}$$

$\therefore\ Ax^2 - (B - fA).x + (C - fB + gA) = N.(x-m).(x-r) + M.(x-n).(x-r) + R.(x-m).(x-n)$, both of which sides rising to the same powers of x, N, M, R, are independent of x, and may be determined thus,

Let $x = n, m, r$, successively, and the corresponding results are

$$\frac{An^2 - (B - fA).n + C - fB + gA}{(n-m).(n-r)} = N$$

$$\frac{Am^2 - (B - fA)\,m + C - fB + gA}{(m-n).(m-r)} = M$$

$$\frac{Ar^2 - (B - fA)\,r + C - fB + gA}{(r-n).(r-m(} = R$$

$$\therefore\ \frac{S}{x} = \frac{N}{x-n} + \frac{M}{x-m} + \frac{R}{x-r}$$

Now, since y may be any number, let it $= 1$, $\therefore\ x = \frac{1}{y} = 1$

And $S = \frac{N}{1-n} + \frac{M}{1-m} + \frac{R}{1-r}$

$$= \left\{\begin{matrix} N + Nn + Nn^2 + Nn^3 + \ldots\ldots \infty \\ M + Mm + Mm^2 + Mm^3 + \ldots\ldots \infty \\ R + Rr + Rr^2 + Rr^3 + \ldots\ldots \infty \end{matrix}\right\} \text{(by division)}$$

which being geometric series, whose first terms are N, M, R, and common ratios n, m, r respectively, the problem is solved.

The same process extended, will serve to demonstrate the singular property of recurring series, that if the scale of relation be

$$f_1 + f_2 + f_3 + f_4 + \ldots f_n$$

the series may be decomposed into (n) geometric series, whose common ratios are the several roots of the equation (provided those roots are unequal)

$$x^n + f_1 x^{n-1} + f_2 x^{n-2} + \ldots f_n = 0$$

N.B. The use of the successive powers of arbitrary quantity (y) is precisely of the same nature with that of (t) in the Theory of Generating Functions. It merely serves to indicate the general term.

672. By the Theorem we have

$$u' = f(x+h) = fx + \frac{du}{dx}h + \frac{d^2u}{dx^2}\frac{h^2}{1.2} + \&c.$$

Let $x = \cos. u$

Then $u = \cos.^{-1}x$

And $u' = \cos.^{-1}(x+h)$

Also $\dfrac{du}{dx} = \dfrac{-1}{\sqrt{1-x^2}}$

$$\frac{d^2u}{dx^2} = \frac{-x}{(1-x^2)^{\frac{3}{2}}}$$

$$\frac{d^3u}{dx^3} = \frac{-1}{(1-x^2)^{\frac{3}{2}}} - \frac{3x^2}{(1-x^2)^{\frac{5}{2}}} = \frac{2}{(1-x^2)^{\frac{3}{2}}} - \frac{3}{(1-x^2)^{\frac{5}{2}}}$$

&c. = &c.

$$\therefore \cos.^{-1}(x+h) = \cos.^{-1}x - \frac{h}{\sqrt{1-x^2}} - \frac{x}{(1-x^2)^{\frac{3}{2}}}\cdot\frac{h^2}{1.2}$$

$$+ \left\{\frac{2}{(1-x^2)^{\frac{3}{2}}} - \frac{3}{(1-x^2)^{\frac{5}{2}}}\right\} \times \frac{h^3}{1.2.3} \pm \&c.$$

Now put $\left.\begin{array}{l} x = 0 \\ \text{and } h = x \end{array}\right\}$ successively.

Then, since $\cos.^{-1}x = (2p+1).\frac{\pi}{2}$. &c. on that supposition

we have $\cos.^{-1}x = (2p+1).\dfrac{\pi}{2} - x - \dfrac{x^3}{1.2.3} - \dfrac{3.x^5}{2.4.5} - \dfrac{3.5x^7}{2.4.6.7} - \&c.$

673. To expand θ in terms of its tangent we put

$\tan. \theta = x$

Then $\theta = \tan.^{-1}x = u$, and $u' = \tan.^{-1}(x+h)$

$$\text{And } u' = u + \frac{du}{dx}.h + \frac{d^2u}{dx^2}\cdot\frac{h^2}{1.2} + \frac{d^3u}{dx^3}\cdot\frac{h^3}{1.2.3} + \&c.$$

Now $\dfrac{du}{dx} = \dfrac{d\theta}{dx} = \dfrac{1}{1+x^2}$

$$\frac{d^2u}{dx^2} = \frac{-2x}{(1+x^2)^2}$$

$$\frac{d^3u}{dx^3} = \frac{6x^2-2}{(1+x^2)^3}$$

$$\frac{d^4u}{dx^4} = \frac{24x}{(1+x^2)^3} - \frac{48x^3}{(1+x^2)^4}$$

$$\frac{d^5u}{dx^5} = \frac{24}{(1+x^2)^3} - \frac{288x^2}{(1+x^2)^4} + \frac{384x^4}{(1+x^2)^5}$$

&c. = &c. Make $x = 0$

Then $u = \tan.^{-1} 0 = p\pi$, and $u' = \tan.^{-1} h$

$$\frac{du}{dx} = 1$$

$$\frac{d^2u}{dx^2} = 0$$

$$\frac{d^3u}{dx^3} = -2$$

$$\frac{d^4u}{dx^4} = 0$$

$$\frac{d^5u}{dx^5} = 2.3.4$$

&c. = &c.

$\therefore \tan.^{-1}h = p\pi + h - \frac{h^3}{3} + \frac{h^5}{5} - \frac{h^7}{7} + \ldots$ (p being any integer.)

Put $h = \tan.\ \theta$. Then $\tan.^{-1} h = \tan.^{-1} (\tan.\ \theta) = \theta$.

$$\text{And } \theta = p\pi + \tan.\ \theta - \frac{\tan.^3 \theta}{3} + \frac{\tan.^5 \theta}{5} - \text{\&c.}$$

674. *Maclaurin's* Theorem is

$f(x) = f_0 x + \frac{d_0 (fx)}{dx} x + \frac{d_0^2 (fx)}{dx^2} \cdot \frac{x^2}{1.2} + \frac{d_0^3 (fx)}{dx^3} \cdot \frac{x^3}{1.2.3}$ + &c., the zero indicating the values of the coefficients when $x = 0$

Here $fx = (1+x)^{\frac{1}{2}}$ $\therefore f_0 x = 1$

$$\frac{d.fx}{dx} = \frac{1}{(1+x)^{\frac{1}{2}}} \quad \therefore \frac{d_0 fx}{dx} = 1$$

$$\frac{d.^2fx}{dx^2} = \frac{-1}{2(1+x)^{\frac{3}{2}}}, \quad \therefore \frac{d_0^2 fx}{dx^2} = -\frac{1}{2}$$

$$\frac{d^3 fx}{dx^3} = \frac{+3}{4.(1+x)^{\frac{5}{2}}}, \therefore \frac{d_0^3 . fx}{dx^3} = \frac{3}{2^2}$$

$$\frac{d^4 fx}{dx^4} = \frac{-3.5}{2.^3(1+x)^{\frac{7}{2}}}, \therefore \frac{d_0^4 . fx}{dx^4} = \frac{-3.5}{2^3}$$

$$\frac{d^5 fx}{dx^5} = \frac{3.5.7}{2^4.(1+x)^{\frac{9}{2}}}, \therefore \frac{d_0^5 . fx}{dx^5} = \frac{3.5.7}{2^4}$$

$$\&c. = \&c.$$

$$\frac{d^n fx}{dx^n} = \frac{\pm 3.5.7...(2n-3)}{2^{n-1}.(1+x)^{\frac{2n-1}{2}}}, \therefore \frac{d_0{}^n fx}{dx^n} = \frac{\pm 3.5.....(2n-3)}{2^{n-1}}$$

Hence $(1+x)^{\frac{1}{2}} = 1 + \frac{x}{2} - \frac{1}{2} \cdot \frac{x^2}{1.2} + \frac{3}{2^2} \cdot \frac{x^3}{1.2.3} - \frac{3.5}{2^3} \times \frac{x^4}{1....4} + ... \pm \frac{3.5..(2n-3)}{2^{n-1}} \times \frac{x^n}{1.2....n} \mp$ &c., which may also be deduced by the Binomial Theorem.

675. This is called *Colson's* Theorem.

$$(a+b)^n = \frac{a^n}{\left(\frac{a}{a+b}\right)^n} = \frac{a^n}{\left\{1+\left(-1+\frac{a}{a+b}\right)\right\}^n} = \frac{a^n}{\left(1-\frac{b}{a+b}\right)^n}$$

$$= a^n . \left(1 - \frac{b}{a+b}\right)^{-n}$$

$$= a^n . \left\{1 + n . \frac{b}{a+b} + . n . \frac{n+1}{2} . \frac{b^2}{(a+b)^2} + \&c. \text{ to } \infty\right\}$$ by the Binomial Theorem.

Again, put $S = \frac{1}{3} + \frac{3}{2} \cdot \frac{1}{3^2} + \frac{3}{2} \cdot \frac{4}{3} \cdot \frac{1}{3^3} + ... \infty$

$$= \frac{1}{3} \cdot \left\{1 + \frac{3}{2} \cdot \frac{1}{3} + \frac{3}{2} \cdot \frac{4}{3} \cdot \frac{1}{3^2} + ... \infty\right\}$$

which being compared with the preceding form, we have

$$\left.\begin{array}{l} a^n = \frac{1}{3} \\ n = 3 \\ \text{and } \frac{b}{a+b} = \frac{1}{3} \end{array}\right\} \quad \begin{array}{l} \therefore a = \left(\frac{1}{3}\right)^{\frac{1}{3}} = \frac{1}{3^{\frac{1}{3}}} \\ \\ \text{Also } b = \frac{a}{2} = \frac{1}{2 \times 3^{\frac{1}{3}}} \end{array}$$ Wrong.

×2, then $2S = 2 \cdot \frac{1}{3} + \frac{2.3}{1.2} \cdot \frac{1}{3^2} + \frac{2.3.4}{1.2.3} \cdot \frac{1}{3^3} + \&c$

$1 + 2S = 1 + 2 \cdot \frac{1}{3} + \&c$

$= \left(1 - \frac{1}{3}\right)^{-2} = \left(\frac{2}{3}\right)^{-2} = \left(\frac{3}{2}\right)^2 = \frac{9}{4} \quad \therefore S = \frac{5}{8}$

$$\therefore S = (a+b)^n = \left(\frac{1}{3^{\frac{1}{3}}} + \frac{1}{2 \times 3^{\frac{1}{3}}}\right)^3 = \left(\frac{3}{2 \times 3^{\frac{1}{3}}}\right)^3$$

$$= \frac{3}{2}.$$

Wrong
u foot 1 of p 491

A great variety of series may be summed by this Theorem.

676. If we have two equations

$$\left.\begin{array}{l} u = z + x\phi u \\ v = fu \end{array}\right\} z \text{ being independent of the variation of } x, \text{ and}$$

f, ϕ denoting given functions of u, we may, by the Theorem of *Lagrange*, express v in terms of z and x generally thus

$$v = fz + \phi z . \frac{d.fz}{dz} . \frac{x}{1} + d \frac{\left\{(\phi z)^2 . \frac{d.fz}{dz}\right\}}{dz} . \frac{x^2}{1.2}$$

$$+ d^2 . \left\{(\phi z)^3 . \frac{dfz}{dz}\right\} \times \frac{x^3}{1.2.3 dz} + \&c.$$

Now in the question before us, we have

$$\left.\begin{array}{r} u = u \\ \phi u = \sin. u \\ x = e \\ v = fu = u \\ \therefore fz = z = nt \\ \text{and } \phi z = \sin. z = \sin. nt \end{array}\right\} \text{Hence by substitution we get}$$

$$u = nt + e \sin. nt. + \frac{e^2}{1.2} . \frac{d . \sin.^2 nt}{d(nt)} + \frac{e^3}{1.2.3} . \frac{d^2 . \sin.^3 nt}{(d.nt)^2} + \&c.$$

$$\text{Now } \frac{d. \sin.^2 nt}{d.nt} = 2 \sin. nt. \cos. nt. = \sin. 2nt$$

$$\frac{d^2 . \sin.^3 nt}{(d.nt)^2} = 6 \sin. nt. \cos.^2 nt - 3 \sin.^3 nt$$

$$= 6 \sin. nt - 9 \sin.^3 nt$$

$$= 6 \sin. nt - \frac{27}{4} \sin. nt + \frac{9}{4} \sin. 3nt$$

$$= \frac{9}{4} \sin.^3 nt - \frac{3}{4} . \sin. nt.$$

$\therefore u = nt + e \sin. nt + \frac{e^2}{1.2 \times 2}. 2 \sin. nt + \frac{e^3}{1.2.3 \times 2^2} \times$
$(3^2 \sin. 3nt - 3 \sin. nt) + \&c.$

For the proof of the Theorem the Reader may consult the Translation of *Lacroix*, p. 633, or the new edition of *Simpson*.

This Problem is of the greatest use in Astronomy. If $nt =$ the mean anomaly of a planet, e = eccentricity, then u will be the eccentric anomaly expressed in terms of the mean. See *Woodhouse's* Ast. Vol. ii. and *Laplace*, Mech. Cél. Lib. II.

677. We will first expand the fraction into three geometric series, thus

Let m, n, r be roots of the equation

$1 - \alpha x - \beta x^2 - \gamma x^3 = 0$

Then assuming $\frac{a+bx+cx^2}{(1-mx).(1-nx).(1-rx)} = \frac{M}{1-mx} + \frac{N}{1-nx}$
$+ \frac{R}{1-rx}$ we get

$a+bx+cx^2 = M.(1-nx)(1-rx) + N.(1-mx).(1-rx)$
$+ R.(1-mx).(1-nx)$ in which substituting $\frac{1}{m}, \frac{1}{n}, \frac{1}{r}$

successively for x, since M, N, R are independent of x, we get

$$a + \frac{b}{m} + \frac{c}{m^2} = M.\left(1 - \frac{n}{m}\right).\left(1 - \frac{r}{m}\right)$$

$$a + \frac{b}{n} + \frac{c}{n^2} = N.\left(1 - \frac{m}{n}\right).\left(1 - \frac{r}{n}\right)$$

$$a + \frac{b}{r} + \frac{C}{r^2} = R.\left(1 - \frac{r}{m}\right).\left(1 - \frac{r}{n}\right)$$

Hence $M = \frac{am^2+bm+c}{(m-n).(m-r)}$

$$N = \frac{an^2+bn+c}{(n-m).(n-r)}$$

$$R = \frac{ar^2+br+c}{(r-m).(r-n)}$$

$$\text{Now}\ \frac{M}{1-mx} = M.(1 + mx + m^2x^2 + m^3\ x^3 \ldots m^p\ x^p + \ldots.)$$

$$\frac{N}{1-nx} = N.(1 + nx + n^2x^2 + n^3\ x^3 \ldots n^px^p + \ldots.)$$

$$\frac{R}{1-rx} = R.(1 + rx + r^2x^2 + r^3x^3 \ldots r^px^p + \ldots)$$

Hence then the p^{th} term of the expansion is

$$T_p = (Mm^{p-1} + Nn^{p-1} + Rr^{p-1})\ x^{p-1}$$

This method which the Reader will perceive is related to the subject of recurring series, and the Integration of Finite Differences, will apply in the case of the degree of the Numerator being less than that of the Denominator.

If this be not the case M, N, R will not be independent of x, a circumstance which is essential to the legitimacy of the above process.

The method universally applicable is this:

$$\text{Assume}\ \frac{a + bx + cx^2}{1 - \alpha x - \beta x^2 - \gamma x^3} = A_1 + A_2x + A_3x^2 + \ldots. A_px^p + \ldots.$$

Hence,

$$\begin{aligned} a+bx+cx^2 = A_1 + A_2x + A_3\ x^2 &+ A_4\ x^3 + \ldots A_p\ x^p + \ldots. \\ - \alpha\ A_1x - \alpha A_2x^2 &- \alpha A_3x^3 + \ldots - \alpha A_{p-1}x^p - \ldots. \\ - \beta A_1x^2 &- \beta A_2x^3 + \ldots - \beta A_{p-2}\ x^p - \ldots. \\ &- \gamma\ A_1x^3 + \ldots - \gamma A_{p-3}\ x^p - \ldots. \end{aligned}$$

$\therefore$ Equating coefficients we have

$$\begin{aligned} A_1 &= a \\ A_2 &= b + \alpha\ A_1 \\ A_3 &= c + \alpha\ A_2 + \beta\ A_1 \\ A_4 &= \alpha\ A_3 + \beta\ A_2 + \gamma\ A_1 \\ A_5 &= \alpha\ A_4 + \beta\ A_3 + \gamma\ A_2 \\ \&c. &= \&c. \\ A_p &= \alpha\ A_{p-1} + \beta\ A_{p-2} + \gamma\ A_{p-3} \end{aligned}$$

exhibiting the series in its recurring form. It is easy from the above equations to find A_p and $\therefore$ the General Term.

678. Put $2\cos.\theta = x + \frac{1}{x}$ Then we have

$$(a^2 - ab.\,2\cos.\theta + b^2)^{-2s} = \left\{a^2 + b^2 - ab.\left(x + \frac{1}{x}\right)\right\}^{-2s}$$

$$= (a^2 + b^2)^{-2s}\left\{1 - \frac{ab}{a^2 + b^2}.\left(x + \frac{1}{x}\right)\right\}^{-2s} = (a^2 +$$

$$b^2)^{-2s}\left\{1 + \frac{2s\,ab}{a^2 + b^2}\left(x + \frac{1}{x}\right) + \frac{2s\,.\,\overline{2s+1}}{2}.\frac{a^2b^2}{(a^2+b^2)^2}\times\right.$$

$$\left(x + \frac{1}{x}\right)^2 + \frac{2s.(2s+1).(2s+2)}{2.3}.\frac{a^3b^3}{(a^2+b^2)^3}.\left(x + \frac{1}{x}\right)^3 +$$

&c. to ∞ }

Now $x + \frac{1}{x} = 2\cos.\theta$

$$\left(x + \frac{1}{x}\right)^2 = x^2 + \frac{1}{x^2} + 2 = 2\cos.2\theta + 2$$

$$\left(x + \frac{1}{x}\right)^3 = x^3 + \frac{1}{x^3} + 3.\left(x + \frac{1}{x}\right) = 2\cos.3\theta + 6\cos.\theta$$

Similarly $\left(x + \frac{1}{x}\right)^4 = \cos.4\theta + 8\cos.2\theta + 6$

$$x + \frac{1}{x}\Big)^5 = 2\cos.5\theta + 10.\cos.3\theta + 20.\cos.\theta$$

&c. = &c.

$$\left(x+\frac{1}{x}\right)^{2p} = 2.\cos.2p\,\theta + 2p.\,2\cos.\overline{2p - 2}\,.\,\theta$$

$$+\frac{2p.(2p-1)}{2}\times 2\cos.\overline{2p-4}\theta + \&c. + \frac{2p.(2p-1).(2p-2)\ldots.\overline{p-1}}{2.3\ldots.p}$$

$$\text{or} + \frac{2^p.1.3.5\ldots.\overline{2p-1}}{1.2\ldots p}$$

$$\left(x+\frac{1}{x}\right)^{2p+1} = 2\cos.(2p+1)\theta + (2p+1)\times 2\cos.(2p-1)\theta$$

$$+ (2p+1).\frac{2p}{2}\times 2\cos.(2p-3)\theta + \&c. + \frac{(2p+1).2p.(2p-1)\ldots.p+1}{1.2\ldots p}.$$

$$2\cos.\theta \text{ or} + \frac{2^p.1.3.5\ldots.\overline{2p+1}}{1.2\ldots p}.\,2\cos.\theta$$

&c. = &c.

Hence, if we put $\frac{ab}{a^2+b^2} = C$, $F_1 = 2s$, $F_2 = 2s.\frac{2s+1}{2}$, $F_3 =$ &c. and collecting the coefficients of the cosines of like multiples of θ, we get the expression

$$=(a^2+b^2)^{-2s}\times\left\{\begin{array}{l} 1+2F_2C^2+\frac{4.3}{2}.F_4C^4+\frac{6.5.4}{2.3}.F_6C^6+\dots\dots \\ +(F_1C+3F_3.C^3+\frac{5.4}{2}.F_5C^5+\frac{7.6.5}{1.2.3}.F_7C^7+\dots\dots) \\ \qquad\times\ 2\cos.\theta \\ +(F_2C^2+4.F_4C^4+\frac{6\,.\,5}{2}F_6C^6+\frac{8.7.6}{2.3}.F_8C^8+\dots\dots) \\ \qquad\times\ 2\cos.2\,\theta \\ +(F_3C^3+5.F_5C^5+\frac{7\,.\,6}{2}F_7C^7+\frac{9.8.7}{2\,.\,3}.F_9C^9+\dots) \\ \qquad\times\ 2\cos.3\,\theta \\ +\ \&c.\quad \&c.\quad \&c.\quad \&c. \end{array}\right.$$

the law of which is very evident. See the *Translation of Lacroix*, p. 672, &c.

This function may also be expanded by *Lagrange's* Theorem. See *Lagrange*, *Resolution des Equations Numériques*, Note XI.

679. Since cos. θ is the only variable in the expression

Assume $\frac{a.(1-e^2)}{1+e\cos.\theta} = A + A_1\cos.\theta + A_2\cos.^2\theta + \dots A_n\cos.^n\theta \dots$

$$\text{Then } a.(1-e^2) = \left\{\begin{array}{l} A + A_1\cos.\theta + A_2\cos.^2\theta + \dots\dots A_n\cos.^n\theta + \dots \\ \quad + eA\cos.\theta + eA_1\cos.^2\theta + \dots\dots eA_{n-1}.\cos.^n\theta + \dots \end{array}\right.$$

$$\begin{aligned} \therefore\ A &= a.(1-e^2) \\ A_1 &= -eA \\ A_2 &= -eA_1 = e^2A \\ A_3 &= -eA_2 = -e^3A \\ \&c. &= \&c. \\ A_n &= \pm e^nA \end{aligned}$$

Hence $\frac{r}{a.(1-e^2)} = 1 - e\cos.\theta + e^2.\cos.^2\theta - e^3.\cos.^3\theta + \ldots \pm e^n\cos.^n\theta \mp$ &c.

Now put $2\cos.\theta = x + \frac{1}{x}$

Then $2^2\cos.^2\theta = x^2 + \frac{1}{x^2} + 2 = 2\cos.2\theta + 2$

$$2^3.\cos.^3\theta = x^3 + \frac{1}{x^3} + 3\left(x + \frac{1}{x}\right) = 2\cos.3\theta + 3.2\cos.\theta$$

$$2^4\cos.^4\theta = 2\cos.4\theta + 4.2\cos.2\theta + 6$$

$$2^5.\cos.^5\theta = 2\cos.5\theta + 5.2\cos.3\theta + 10.2\cos.\theta$$

$$2^6.\cos.^6\theta = 2\cos.6\theta + 6.2.\cos.4\theta + 15.2\cos.2\theta + 20$$

&c. = &c. (See 678.)

Hence by substitution, and due arrangement, we get

$$\frac{r}{a(1-e^2)} = 1 + 2.\left(\frac{e}{2}\right)^2 + \frac{4.3}{2}.\left(\frac{e}{2}\right)^4 + \frac{6.5.4}{2.3}\left(\frac{e}{2}\right)^6 + \ldots\infty$$

$$-e.\cos.\theta\left\{1 + 3.\left(\frac{e}{2}\right)^2 + \frac{5.4}{2}.\left(\frac{e}{2}\right)^4 + \frac{7.6.5}{2.3}.\left(\frac{e}{2}\right)^6 + \ldots\infty\right\}$$

$$+\frac{e^2}{2}.\cos.2\theta\left\{1 + 4.\left(\frac{e}{2}\right)^2 + \frac{6.5}{2}.\left(\frac{e}{2}\right)^4 + \frac{8.7.6}{2.3}.\left(\frac{e}{2}\right)^6 + \ldots\infty\right\}$$

$$-\frac{e^3}{2^2}.\cos.3\theta\left\{1 + 5.\left(\frac{e}{2}\right)^2 + \frac{7.6}{2}.\left(\frac{e}{2}\right)^4 + \frac{9.8.7}{2.3}.\left(\frac{e}{2}\right)^6 + \ldots\infty\right\}$$

+ &c. &c. whose coefficients are summable in finite terms by Problem 776.

680. It is universally known that

$$a^x = 1 + la.x + la.^2\frac{x^2}{1.2} + la.^3\frac{x^3}{1.2.3} + \ldots.(a)$$

Put $lx = y$

Then $\frac{dx}{x} = dy$, $\therefore \frac{dx}{dy} = x$

Assume $x = A + By + Cy + Dy^3 + \ldots.$

Then $\frac{dx}{dy} = B + 2Cy + 3Dy^2 + 4Ey^3 + \ldots..$

Hence $A + By + Cy^2 + \ldots = B + 2Cy + 3Dy^2 + \ldots$

$\therefore B = A$

$$C = \frac{B}{2} = \frac{A}{2}$$

$$D = \frac{C}{3} = \frac{A}{2.3}$$

$$E = \frac{D}{4} = \frac{A}{2.3.4}$$

&c. = &c. = &c.

And because when $y = 0$, $x = 1$, $\therefore A = x = 1$

Hence $x = 1 + y + \dfrac{y^2}{2} + \dfrac{y^3}{2.3} + \dfrac{y^4}{2.3.4} + \ldots$ which might

however have been deduced from (a).

$$\text{Hence } x^2 = 1 + y + 2y^2 + \frac{4}{3}y^3 + \frac{17y^4}{3.4} + \&c.$$

$$x^3 = 1 + 3y + \frac{9}{2}y^2 + \frac{9}{2}y^3 + \frac{27}{2.4}y^4 + \&c.$$

$$x^4 = 1 + 4y + 8y^2 + \frac{32}{3}y^3 + \frac{73}{2.3}y^4 + \&c.$$

&c. = &c.

$\therefore$ substituting in (a) and arranging the terms according to their compound dimensions we get

$$a^x = 1 + la + y.la + \frac{y^2}{2}.la + \frac{y^3}{2.3}la + \frac{y^4}{2.3.4}la + \ldots$$

$$+ \frac{la^2}{2} + 2y.\frac{la^2}{2} + 2y^2.\frac{la^2}{2} + \frac{4y^3}{3}.\frac{la^2}{2} + \ldots$$

$$+ \frac{la^3}{2.3} + 3y.\frac{la^3}{2.3} + \frac{9y^2}{2}.\frac{la^3}{2.3} + \ldots$$

$$+ \frac{la^4}{2.3.4} + 4y.\frac{la^4}{2.3.4} + \ldots$$

$$+ \frac{la^5}{2.3.4.5} + \ldots$$

$\therefore$ collecting terms reduced to a common denominator

$$a^x = 1 + \frac{2yla + la^2}{2} + \frac{3y^2la + 6yla^2 + la^3}{2.3}$$
$$+ \frac{4y^3la + 2.3.4y^2la^2 + 3.4yla^3 + la^4}{2.3.4} + \&c. \&c.$$

Now since $y = lx = \dfrac{X}{m}$ and $la = \dfrac{A}{m}$ $\Big\}$ by the common rule, we finally obtain

$$a^x = 1 + \frac{2XA + A^2}{2m^2} + \frac{3X^2A + 2.3.XA^2 + A^3}{2.3.m^3}$$
$$+ \frac{4X^3A + 2.3.4.X^2A^2 + 3.4XA^3 + A^4}{2.3.4.m^4} + \&c.$$

THE INVERSE METHOD OF SERIES.

681. Let $\Sigma = \frac{2}{1.3} \times \frac{1}{3} + \frac{3}{3.5} \times \frac{1}{3^2} + \frac{4}{5.7} \times \frac{1}{3^3} + \ldots\infty$

Assume $S = x^{\frac{1}{2}} + \frac{x^{\frac{1}{2}+1}}{3} + \frac{x^{\frac{1}{2}+2}}{5} . + \frac{x^{\frac{1}{2}+3}}{7} + \ldots\infty$

Then $\frac{2dS}{dx} = x^{-\frac{1}{2}} + x^{\frac{1}{2}} + x^{\frac{1}{2}+1} + x^{\frac{1}{2}+2} + \ldots\infty$

$$= \frac{x^{-\frac{1}{2}}}{1-x}$$

$$\therefore S = \frac{1}{2} . \int \frac{x^{-\frac{1}{2}} dx}{1-x}$$

Again, put $S_1 = \int \frac{dx . S}{2} = \int \frac{x^{\frac{1}{2}} dx}{2} + \int \frac{x^{\frac{3}{2}} dx}{2.3} + \int \frac{x^{\frac{5}{2}} dx}{2.5} +$ &c.

$$= \frac{x^{\frac{3}{2}}}{1.3} + \frac{x^{\frac{5}{2}}}{3.5} + \frac{x^{\frac{7}{2}}}{5.7} + \ldots\infty$$

Then, to introduce the numerators, multiply by $x^{\frac{1}{2}}$, and we have

$$x^{\frac{1}{2}} S_1 = \frac{x^2}{1.3} + \frac{x^3}{3.5} + \frac{x^4}{5.7} + \ldots\infty$$

$$\therefore S_2 = \frac{d . x^{\frac{1}{2}} S_1}{dx} = \frac{2x}{1.3} + \frac{3x^2}{3.5} + \frac{4x^3}{5.7} + \ldots\infty$$

which will evidently $= \Sigma$ when $x = \frac{1}{3}$

Now $S_2 = \frac{d . x^{\frac{1}{2}} S_1}{dx} = \frac{1}{2} x^{-\frac{1}{2}} S_1 + \frac{x^{\frac{1}{2}} d S_1}{dx} = \frac{1}{2} . x^{-\frac{1}{2}} \times$

$$\int \frac{dxS}{2} + \frac{x^{\frac{1}{2}} S}{2} = \frac{1}{8} x^{-\frac{1}{2}} \int dx \int \frac{x^{-\frac{1}{2}} dx}{1-x} + \frac{x^{\frac{1}{2}}}{4} \int \frac{x^{-\frac{1}{2}} dx}{1-x}$$

But $\int dx \int \frac{x^{-\frac{1}{2}}dx}{1-x} = x \int \frac{x^{-\frac{1}{2}}dx}{1-x} - \int \frac{x^{\frac{1}{2}}dx}{1-x}$

$$\therefore \; S_2 = \frac{3}{8}\, x^{\frac{1}{2}} \int \frac{x^{-\frac{1}{2}}dx}{1-x} - \frac{1}{8}\, x^{-\frac{1}{2}} \int \frac{x^{\frac{1}{2}}dx}{1-x}$$

Again, put $x^{\frac{1}{2}} = u$

Then $\frac{x^{\frac{1}{2}}dx}{1-x} = \frac{2u^2\,du}{1-u^2} = \frac{2du}{1-u^2} - 2du$

And $\frac{x^{-\frac{1}{2}}dx}{1-x} = \frac{2du}{1-u^2}$

Hence $S_2 = \frac{3}{8} \times x^{\frac{1}{2}} \int \frac{2du}{1-u^2} - \frac{1}{8}\, x^{-\frac{1}{2}} \int \frac{2du}{1-u^2} + \frac{1}{4}$

$$= \frac{1}{8}\,(3x^{\frac{1}{2}} - x^{-\frac{1}{2}})\,.\,l.\,\frac{1+u}{1-u} + \frac{1}{4}$$

$$= \frac{1}{8}.\left(3x^{\frac{1}{2}} - \frac{1}{x^{\frac{1}{2}}}\right). \times l.\frac{1+\sqrt{x}}{1-\sqrt{x}} + \frac{1}{4}$$

Put $x = \frac{1}{3}$

Then $\Sigma = \frac{1}{8} \times \left(\frac{3}{\sqrt{3}} - \sqrt{3}\right).\,l.\,\frac{1+\sqrt{\frac{1}{3}}}{1-\sqrt{\frac{1}{3}}} + \frac{1}{4}$

$$= \frac{1}{8} \times (\sqrt{3} - \sqrt{3}).l.\frac{1+\sqrt{\frac{1}{3}}}{1-\sqrt{\frac{1}{3}}} + \frac{1}{4} = \frac{1}{4}$$

It will be found also by beginning at the $(n+1)^{\text{th}}$ term instead of the first in the above process, and taking the difference of the two resulting sums that the sum of (n) terms of the given series will be

$$\frac{1}{4} - \frac{1}{4.(1+2n).3^n}$$

Demoivre's method would have applied with greater brevity in this case, but as the principle employed above is of considerable importance in the summation of series we shall take every opportunity of familiarizing the student with its application. It may also be resolved by the Inverse Method of Differences, or Increments.

Again, $S = 1.2.4 + 3.4.6 + 5.6.8 + \ldots\ n$ terms

Then $\Delta S = (n+1)^{\text{th}}$ term $= (2n+1).(2n+2).(2n+4)$

$$= 2n.(2n+2).(2n+4) + (2n+2).(2n+4)$$

$$\therefore S = \frac{(2n-2).2n.(2n+2).(2n+4)}{4\times 2} + \frac{2n.(2n+2).(2n+4)}{3\times 2} + C$$

$$= 2.(n-1)n.(n+1).(n+2) + \frac{4}{3}.n.(n+1).(n+2) + C$$

$$= \frac{n.(n+1).(n+2)}{3} \times (6n-2) \text{ (since } C = 0)$$

682. Let $S = x + x^2 + x^3 + \ldots \infty = \dfrac{x}{1-x}$

Then $\dfrac{dS}{dx} = 1 + 2x + 3x^2 + \ldots \infty = \dfrac{1}{1-x} + \dfrac{x}{(1-x)^2} = \dfrac{1}{(1-x)^2}$

Similarly we have $1 - 2x + 3x^2 + \ldots \infty = \dfrac{1}{(1+x)^2}$

$$\therefore \frac{1+2x+3x^2+\ldots\infty}{1-2x+3x^2-\ldots\infty} = \left(\frac{1+x}{1-x}\right)^2$$

683. Let $S = \dfrac{1}{1.3} + \dfrac{1}{2.4} + \dfrac{1}{3.5} + \ldots \infty$ and assume

$u = 1 + x + x^2 + \ldots \infty = \dfrac{1}{1-x}$

Then $udx = dx + xdx + x^2dx + \ldots \infty = \dfrac{dx}{1-x}$

$$\therefore \int udx = x + \frac{x^2}{2} + \frac{x^3}{3} + \ldots \infty = \int \frac{dx}{1-x} = -l.(1-x)$$

Again multiply by xdx

Then $xdx\int udx = x^2dx + \frac{x^3dx}{2} + \frac{x^4dx}{3} + \ldots.\infty = -\, xdx\, l.(1-x)$

$\therefore\ S_1 = \int xdx\int udx = \frac{x^3}{1.3} + \frac{x^4}{2.4} + \frac{x^5}{3.5} + \ldots\infty = -\int xdx l.(1-x)$

But $\int xdx.\, l.\,(1-x) = \frac{x^2}{2}\, l.\,(1-x) + \int\frac{x^2dx}{2(1-x)}$

$$= \frac{x^2}{2}.\, l.(1-x) - \frac{1}{2}l.(1-x) - \frac{x^2}{4} - \frac{x}{2} + C$$

Hence $S_1 = \frac{x^2}{4} + \frac{x}{2} + l.\,\left\{(1-x)^{\frac{x^2}{2}-\frac{1}{2}}\right\}$

Let $x = 1$. Then $l.\,(1-x)^{\frac{x^2}{2}-\frac{1}{2}} = l.\ 1 = 0$ &c.

And $S = \frac{1}{1.3} + \frac{1}{2.4} + \ldots = \frac{1}{4} + \frac{1}{2} = \frac{3}{4}.$

Otherwise.

The $(n+1)^{th}$ term is $\frac{1}{(n+1).(n+3)} = \Delta\ S$ (S being the sum to n terms.)

$$\therefore\ \Delta\ S = \frac{n+2}{(n+1).(n+2).(n+3)} = \frac{1}{(n+2).(n+3)}$$
$$+\ \frac{1}{(n+1).(n+2).(n+3)}$$

$$\therefore\ \Sigma.\ \Delta\ S = S = -\frac{1}{n+2} - \frac{1}{2.(n+1).(n+2)} + C$$

Let $n = 1$

Then $C = \frac{1}{2} + \frac{1}{3} + \frac{1}{12} = \frac{3}{4}$

$$\therefore\ S = \frac{3}{4} - \frac{2n+3}{2.(n+1)(n+2)}$$

To sum $1 + 2^2 + 3^2 + \ldots..\ n^2 = S$, we have the geometric series $x + x^2 + x^3 + \ldots..\ \infty = \frac{x}{1-x}$

$$\therefore 1 + 2x + 3x^2 + 4x^3 + \ldots\ldots \infty = \frac{1}{(1-x)^2}$$

$$\therefore x + 2x^2 + 3x^3 + 4x^4 + \ldots\ldots \infty = \frac{x}{(1-x)^2}$$

And again differentiating, we obtain

$$1 + 2^2x + 3^2x^2 + 4^2x^3 + \ldots \infty = \frac{1+x}{(1-x)^3} = s$$

Again, we have $x^{n+1} + x^{n+2} + \ldots\ldots \infty = \dfrac{x^{n+1}}{1-x}$

$$\therefore (n+1).\, x^n + (n+2).x^{n+1} + \ldots\infty = \frac{(n+1)x^n}{1-x} + \frac{x^{n+1}}{(1-x)^2}$$

$$\therefore (n+1)\, x^{n+1} + (n+2)\, x^{n+2} + \ldots \infty = \frac{(n+1).x^{n+1}}{1-x} + \frac{x^{n+2}}{(1-x)^2}$$

Hence, $(n+1)^2 x^n + (n+2)^2.\, x^{n+1} + \ldots\infty = \dfrac{(n+1)^2 x^n}{1-x}$

$$+ \frac{(n+1).x^{n+1}}{(1-x)^2} + \frac{(n+2).x^{n+1}}{(1-x)^2} + \frac{2x^{n+2}}{(1-x)^3} = s'$$

$$\therefore S' = s - s' = \frac{1+x}{(1-x)^3} - \frac{(n+1)^2.x^n}{1-x} - \frac{(2n+3)x^{n+1}}{(1-x)^2}$$

$$- \frac{2x^{n+2}}{1-x)^3} = \frac{1+x}{(1-x)^3} - \frac{x^n.\{(n+1)^2 + (1-2n^2-2n)x + n^2x^2\}}{(1-x)^3}$$

Hence then we get the series

$$S' = 1 + 2^2x + 3^2x^2 + \ldots. n^2x^{n-1} =$$

$\dfrac{1 + x - x^n\{(n+1)^2 + (1-2n.\overline{n+1})x + n^2x^2\}}{(1-x)^3}$ which gives *immediately* the value for every possible case, except the one in the Problem.

In that series we have $x = 1$, and $S =$ a *vanishing fraction*, which we will investigate by putting $x = y + 1$ &c.

$$\text{Then } S' = \frac{2 + y - (1+y)^n.\{2 + (1-2n)y + n^2y^2\}}{y^3}$$

$$= \frac{2+y-(1+ny+n.\frac{n-1}{2}y^2+n.\frac{n-1}{2}.\frac{n-2}{3}y^3+R.y^4)\{2+(1-2n)y+n^2y^2\}}{y^3}$$

$$=\frac{1}{y^3}\times\left\{2+y-\left(2+\begin{array}{c}\overline{1-2n}.y+n^2y^2+n.\frac{n-1}{2}.(1-2n)y^3\\ +\ 2n.y+n.(1-2n)y^2+2n\frac{n-1}{2}.\frac{n-2}{3}y^3+R'y^4\\ +\ n.(n-1)y^2+n^3y^3\end{array}\right)\right\}$$

$$= -\left\{n.\frac{n-1}{2}.(1-2n)+2n.\frac{n-1}{2}.\frac{n-2}{3}+n^3\right\}-R'y$$

$$= \frac{n^3}{3}+\frac{n^2}{\cancel{3}}-\frac{n}{6}-R'y$$ 2

Put $x = 1$, $\therefore\ y = 0$, and $R'y = 0$

Then $S = 1 + 2^2 + 3^2 + \ldots.\ n^2 = \frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6}$

Otherwise

$$\Delta\ S = (n+1)^2 = n.(n+1)+(n+1)$$

$$\therefore\ S = \frac{(n-1).n.(n+1)}{3\Delta n} + \frac{n.(n+1)}{2\Delta n} + C$$

$$= \frac{n.(n^2-1)}{3} + \frac{n.(n+1)}{2} = \frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6}$$ the same as before.

684. $S = 1.3^2 + 3.5^2 + 5.7^2 + \ldots.. (2n-1).(2n+1)^2$

Hence $\Delta\ S = (2n+1).(2n+3)^2 = (2n+1).(2n+3)\times(2n-1+4)$

$$= (2n-1).(2n+1)(2n+3)+4.(2n+1).(2n+3)$$

$$\therefore\ S = \frac{(2n-3).(2n-1)(2n+1)(2n+3)}{4\Delta T} + \frac{4.(2n-1).(2n+1).(2n+3)}{3\Delta T} + C$$

$$= \frac{(4n^2-9).(4n^2-1)}{8} + \frac{2.(4n^2-1).(2n+3)}{3} + C$$

T denoting any factor whatever, and $\therefore \Delta T = 2$.

Let now $n = 1$

$$\text{Then } C = 9 + \frac{15}{8} - 10 = \frac{7}{8}$$

$$\therefore S = \frac{(4n^2-9)(4n^2-1)}{8} + \frac{2.(4n^2-1).(2n+3)}{3} + \frac{7}{8}.$$

$$\text{Again } S = \frac{1}{1.2} - \frac{1}{2.4} + \frac{1}{3.6} - \frac{1}{4.8} + \ldots\ldots \infty$$

$$= \frac{1}{2}.\left\{\frac{1}{1} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots\ldots \infty\right\}$$

$$\text{Take } dx - xdx + x^2dx - \ldots\ldots \infty = \frac{dx}{1+x}$$

$$\text{Then } x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots\ldots \infty = \int \frac{dx}{1+x}$$

$$\therefore dx - \frac{xdx}{2} + \frac{x^2dx}{3} - \ldots\ldots \infty = \frac{dx}{x}\int\frac{dx}{1+x}$$

$$\therefore x - \frac{x^2}{2^2} + \frac{x^3}{3^2} - \ldots\ldots \infty = \int\frac{dx}{x}\int\frac{dx}{1+x}$$

$\therefore S = \frac{1}{2}.\int\frac{dx}{x}\int\frac{dx}{1+x}$, the integral being taken between $x = 0$ and $x = 1$.

This value may be elegantly exhibited by means of hyperbolic areas.

Let AMm (fig. 56) be an equilateral hyperbola whose asymptotes, $\perp$ to one another, are CR, CT. Take CB $=$ BA $= 1$, and PM, CP, any other co-ordinates, A being the vertex. Put PB $= x$, and PM $= y$,

Then by the property of the curve we have

$$(1 + x) \times y = CB \times AB = 1$$

$$\therefore y = \frac{1}{1+x}$$

$$\therefore\ d \text{ Area CPMA} = y dx = \frac{dx}{1+x}$$

$$\therefore\ \text{CPMA} = \int \frac{dx}{1+x}$$

Now let PM′ be always taken $= \dfrac{\text{CPMA}}{x}$ tracing out the curve A′M′

Then d . Area CPM′A′ $= dx . \dfrac{\text{CPMA}}{x} = \dfrac{dx}{x} \displaystyle\int \frac{dx}{1+x}$

$\therefore$ Area CPM′A′ $= \displaystyle\int \frac{dx}{x} \int \frac{dx}{1+x}$ and the finite portion of it BPM′A′ (BP being $= 1$) will be the value of the $\displaystyle\int \frac{dx}{x} \int \frac{dx}{1+x}$ taken between $x = 0$ and $= 1$

$$\therefore\ S = \frac{1}{2} . (\text{Area BPM′A′})$$

Otherwise.

Put $\sin. \theta \left(= \theta - \dfrac{\theta^3}{2 . 3} + \dfrac{\theta^5}{2 . 3 . 4 . 5} - \ldots\ldots \infty\right) = 0$

$\therefore\ 1 - \dfrac{\theta^2}{2 . 3} + \dfrac{\theta^4}{2 . 3 . 4 . 5} - \ldots\ldots \infty = 0$ which will be satisfied if we put for θ, $\pm \pi$, $\pm 2\pi$, ... &c. ($\because$ sin. $0 = 0$, sin. $\pi = 0$ &c.)

Hence then

$(\theta^2 - \pi^2) . (\theta^2 - 4\pi^2) (\theta^2 - 9\pi^2) \ldots\ldots = 1 - \dfrac{\theta^2}{2 . 3} + \dfrac{\theta^4}{2 . 3 . 4 . 5} -$ &c. is an equation whose roots are π^2, $4\pi^2$ &c.

$\therefore\ \dfrac{1}{2 . 3} = \dfrac{\text{last coefficient but one}}{\text{last coefficient}} =$ sum of reciprocals of the roots $= \dfrac{1}{\pi^2} + \dfrac{1}{4\pi^2} + \dfrac{1}{9\pi^2} + \ldots\ldots \infty$

$$\therefore\ 1 + \frac{1}{2^2} + \frac{1}{3^2} + \ldots\ldots \infty = \frac{\pi^2}{6}$$

$$\text{Hence } \frac{2}{2^2} + \frac{2}{4^2} + \frac{2}{6^2} + \ldots\ldots \infty = \frac{\pi^2}{12}$$

$$\therefore\ 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots\ldots \infty = \frac{\pi^2}{12}$$

$$\therefore\ S = \frac{\pi^2}{24}$$

$$\text{Again } S = \frac{1}{1.2.4} - \frac{1}{2.3.5} + \frac{1}{3.4.6} - \ldots\ldots \infty$$

Here we have

$$dx - xdx + x^2dx - x^3dx + \ldots\ldots \infty = \frac{dx}{1+x}$$

$$\therefore\ x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots\ldots \infty = \int\frac{dx}{1+x}$$

$$\therefore\ xdx - \frac{x^2dx}{2} + \frac{x^3dx}{3} - \ldots\ldots \infty = dx\int\frac{dx}{1+x}$$

$$\therefore\ \frac{x^2}{1.2} - \frac{x^3}{2.3} + \frac{x^4}{3.4} - \ldots\ldots \infty = \int dx \int\frac{dx}{1+x}$$

$$\text{Also } \frac{x^3dx}{1.2} - \frac{x^4dx}{2.3} + \frac{x^5dx}{3.4} - \ldots\ldots \infty = xdx\int dx\int\frac{dx}{1+x}$$

$$\therefore\ \frac{x^4}{1.2.4} - \frac{x^5}{2.3.5} + \frac{x^6}{3.4.6} - \ldots\ldots \infty = \int xdx\int dx\int\frac{dx}{1+x}$$

$$\text{But } \int dx\int\frac{dx}{1+x} = x\,.\,l\,(1+x) - \int\frac{xdx}{1+x}$$

$$= (x+1)\,.\,l.\,(1+x) - x$$

$$\therefore\ \int xdx\int dx\int\frac{dx}{1+x} = \int x\,.\,\overline{x+1}\,dx\,l\,(1+x) - \int x^2dx$$

$$= \left(\frac{x^3}{3} + \frac{x^2}{2}\right) l(1+x) - \frac{x^3}{3} - \int\frac{\left(\frac{x^3}{3} + \frac{x^2}{2}\right)dx}{1+x}$$

$$= \left(\frac{x^3}{3} + \frac{x^2}{2} - \frac{1}{6}\right).\,l(1+x) - \frac{4x^3}{9} - \frac{x^2}{12} + \frac{x}{6}$$ there being no correction.

Put $x = 1$. Then $l.(1 + x) = l.2$

and $S = \frac{2}{3}.l.2 - \frac{19}{36}$

685. $$S = na + n.\frac{n-1}{2}d_1 + n.\frac{n-1}{2}.\frac{n-2}{3}d_2 + \ldots\ldots$$

where $n =$ the number of terms summed, a the first term, d_1, the first term of the first difference, d_2 the first term of second difference, &c. = &c. until we arrive at $d_m = 0$. See *Wood's Algebra.*

Here 1, 8, 27, 64, 125

7, 19, 37, 61

12, 18, 24

6, 6

0

$$\therefore a = 1, d_1 = 7, d_2 = 12, d_3 = 6, d_4 = 0$$

$$\therefore S = n + 7n.\frac{n-1}{2} + 12.n.\frac{n-1}{2}.\frac{n-2}{3} + 6.n \times \frac{n-1}{2}.\frac{n-2}{3}.\frac{n-3}{4}$$

$$= \left(n.\frac{n+1}{2}\right)^2$$

Again, put $S = 1.2 + 2.3 + 3.4 + \ldots. n.(n+1)$

Then $\Delta S = (n+1).(n+2)$

$$\therefore S = \frac{n.(n+1).(n+2)}{3\Delta n} + C = \frac{n.(n+1).(n+2)}{3}$$

Again, put $S = \frac{1}{1.2} + \frac{1}{2.3} + \ldots.\frac{1}{n.(n+1)}$

Then $\Delta S = \frac{1}{(n+1).(n+2)}$

$$\therefore S = -\frac{1}{n+1} + C = 1 - \frac{1}{n+1} = \frac{n}{n+1}$$

686. $S = 1 + 3 + 5 + 7 + \ldots\ldots 2n - 1$ (a common arithmetic series)

$$\therefore \Delta S = 2n + 1$$

$$\text{and } S = \frac{(2n-1).(2n+1)}{2 \times 2} + C = \frac{4n^2-1}{4} + \frac{1}{4} = n^2$$

Again, $S = 3 - 1 + \frac{1}{3} - \frac{1}{9} + \ldots . \infty$ which is a geometric series, whose first term is 3, and common ratio $-\frac{1}{3}$

$$\therefore S = \frac{3}{1 + \frac{1}{3}} = \frac{9}{4} \text{ by the common rule.}$$

$$\text{Again } S = \frac{1}{1.2.3} + \frac{1}{2.3.4} + \frac{1}{3.4.5} + \ldots . \infty$$

Hence $\Delta S = \frac{1}{(n+1).(n+2).(n+3)}$ (S being the sum to n terms)

$$\therefore S = -\frac{1}{2(n+1).(n+2)} + C = \frac{1}{4} - \frac{1}{2(n+1).(n+2)}$$

Let $n = \infty$

Then $S = \frac{1}{4}$

687. $S = \frac{10}{1.2.3.4} + \frac{14}{2.3.4.5} + \ldots . \frac{6+4n}{n.(n+1).(n+2).(n+3)}$

$$\text{Here } \Delta S = \frac{6 + 4(n+1)}{(n+1).(n+2).(n+3).(n+4)}$$

$$= \frac{6}{(n+1)(n+2).(n+3)(n+4)} + \frac{4}{(n+2).(n+3).(n+4)}$$

$$\therefore S = C - \frac{6}{3.(n+1).(n+2).(n+3)} - \frac{4}{2.(n+2).(n+3)}$$

$$= C - \frac{2}{(n+1).(n+3)} = \frac{2}{3} - \frac{2}{(n+1).(n+3)}$$

which is the sum to n terms.

Let $n = \infty$

Then $S = \frac{2}{3}$.

Again,

$$S = \frac{5}{1.2.3} \times \frac{1}{2^2} + \frac{6}{2.3.4} \times \frac{1}{2^3} + \ldots\ldots \infty$$

Take $dx + xdx + x^2dx + \ldots\ldots \infty = \frac{dx}{1-x}$

$$\therefore x + \frac{x^2}{2} + \frac{x^3}{3} + \ldots\ldots \infty = \int \frac{dx}{1-x}$$

Similarly $\frac{x^2}{1.2} + \frac{x^3}{2.3} + \frac{x^4}{3.4} + \ldots \infty = \int dx \int \frac{dx}{1-x}$

and $\frac{x^3}{1.2.3} + \frac{x^4}{2.3.4} + \frac{x^5}{3.4.5} + \ldots \infty = \int dx \int dx \int \frac{dx}{1-x}$

and $\frac{x^5}{1.2.3} + \frac{x^6}{2.3.4} + \frac{x^7}{3.4.5} + \ldots \infty = x^2 \int dx \int dx \int \frac{dx}{1-x}$

Hence $\frac{5x^4}{1.2.3} + \frac{6x^5}{2.3.4} + \frac{7x^6}{3.4.5} + \ldots \infty = 2x \int dx \int dx \int \frac{dx}{1-x} + x^2 \int dx \int \frac{dx}{1-x}$ by differentiation.

$$\therefore \frac{5x^4}{1.2.3} + \frac{6x^5}{2.3.4} + \frac{7x^6}{3.4.5} + \ldots \infty = 2x . \int dx \int dx \times \int \frac{dx}{1-x} + x^2 \int dx \int \frac{dx}{1-x} = 3x^2 \int dx \int \frac{dx}{1-x} - 2x . \int x dx \int \frac{dx}{1-x}$$

$$= \frac{5}{2} x^3 - x^2 + (3x^2 - x - 2x^3)\, l.(1-x)$$

there being no correction.

Let $x = \frac{1}{2}$ and multiply by 4.

Then $S = \frac{5}{1.2.3} \times \frac{1}{2^2} + \frac{6}{2.3.4} \times \frac{1}{2^3} + \ldots\ldots \infty =$
$\frac{5}{4} - 1 = \frac{1}{4}$

Again,

$$S = \frac{1}{1.3} - \frac{1}{3.5} + \frac{1}{5.7} - \ldots\ldots \infty$$

Take $dx - x^2dx + x^4dx - \ldots\ldots \infty = \frac{dx}{1+x^2}$

Then $x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots\ldots \infty = \int \frac{dx}{1+x^2}$

Again $x^2dx - \frac{x^4dx}{3} + \frac{x^6dx}{5} - \ldots\ldots \infty = xdx \int \frac{dx}{1+x^2}$

$\therefore \frac{x^3}{1.3} - \frac{x^5}{3.5} + \frac{x^7}{5.7} - \ldots\ldots \infty = \int xdx \int \frac{dx}{1+x^2}$

$$\text{Now } \int xdx \int \frac{dx}{1+x^2} = \frac{x^2}{2} \int \frac{dx}{1+x^2} - \int \frac{x^2dx}{2.(1+x^2)}$$
$$= \frac{x^2}{2} \int \frac{dx}{1+x^2} - \frac{x}{2} + \frac{1}{2} \int \frac{dx}{1+x^2}$$
$$= \frac{x^2+1}{2} . \tan.^{-1} x - \frac{x}{2} + C$$

$$\left.\begin{array}{l}\text{Let } x = 0 \\ \text{And } x = 1\end{array}\right\} \text{ Then } \tan.^{-1} x \left\{\begin{array}{l} = p\pi \\ = p\pi + \frac{\pi}{4} \end{array}\right\}$$

And $S = \frac{\pi}{4} - \frac{1}{2}$

688. $S = 1 - \frac{1}{2^3} + \frac{1}{2^5} - \frac{1}{2^7} + \ldots\ldots$

This series is geometric after the first term, the common ratio being $\left(-\frac{1}{2^2}\right)$.

Hence, by the common rule for such forms, we have

$$S = 1 - \frac{1}{2^3} . \left\{ 1 - \frac{1}{2^2} + \frac{1}{2^4} - \ldots\ldots \right\}$$

$$= 1 - \frac{1}{2^3} \cdot \frac{\left(-\frac{1}{2}\right)^{2n-2} - 1}{-\frac{1}{2^2} - 1} = 1 + \frac{1}{10} \cdot \left(\frac{1}{2^{2n-2}} - 1\right)$$

the value to (n) terms.

Let $n = \infty$

Then $S = 1 - \frac{1}{10} = \frac{9}{10}$.

Again $S = \frac{1}{1 \,.\, 5} + \frac{1}{2 \,.\, 6} + \frac{1}{3 \,.\, 7} + \ldots \frac{1}{n \,.\, (n+4)}$

$$\therefore \Delta S = \frac{1}{(n+1)\,(n+5)} = \frac{(n+2).(n+3).(n+4)}{(n+1).(n+2).(n+3)(n+4).(n+5)}$$

Now assume $(n+2) \,.\, (n+3) \,.\, (n+4) = A \,.\, (n+1).(n+2).(n+3) + B.(n+1) \,.\, (n+2) + C \,.\, (n+1) + D$, and equating coefficients of like powers of n when expanded, we get

$$A = 1,\ B = 3,\ C = 6,\ D = 6,$$

$$\therefore \Delta S = \frac{1}{(n+4) \,.\, (n+5)} + \frac{3}{(n+3).(n+4).(n+5)} + \frac{6}{(n+2).(n+3)\ldots(n+5)} + \frac{6}{(n+1)\ldots(n+5)}$$

$$\therefore S = C - \frac{1}{n+4} - \frac{3}{2\,(n+3) \,.\, (n+4)} - \frac{2}{(n+2)\ldots(n+4)} - \frac{3}{2\,(n+1)\ldots(n+4)}$$

$$= \frac{25}{48} - \frac{2n^3 + 15n^2 + 35n + 25}{2.(n+1) \,.\, (n+2) \,.\, (n+3) \,.\, (n+4).(n+5)}$$

Let $n = \infty$

Then $S = \frac{25}{48} - \frac{2\,\infty^3}{2 \,.\, \infty^5} = \frac{25}{48} - \frac{1}{\infty^2} = \frac{25}{48}$

Again,

$S = \frac{1}{1 \,.\, 3} + \frac{1}{3 \,.\, 7} + \ldots \infty$

Here take the expression

$$dx + x^2dx + x^4dx + \ldots\ldots \infty = \frac{dx}{1-x^2}$$

$$\therefore\ x + \frac{x^3}{3} + \frac{x^5}{5} + \ldots\ \infty\ \ldots = \int\frac{dx}{1-x^2}$$

Now multiply by $x^{-\frac{1}{2}}dx$, integrate, and divide by 2, and there results

$$\frac{x^{\frac{3}{2}}}{1\,.\,3} + \frac{x^{\frac{7}{2}}}{3\,.\,7} + \frac{x^{\frac{11}{2}}}{5.11} + \ldots\ \infty = \int\frac{x^{-\frac{1}{2}}dx}{2}\int\frac{dx}{1-x^2}$$

$$= x^{\frac{1}{2}}\int\frac{dx}{1-x^2} - \int\frac{x^{\frac{1}{2}}dx}{1-x^2}$$

Put $x^{\frac{1}{2}} = u$. Then

$$\int\frac{x^{\frac{1}{2}}dx}{1-x^2} = \int\frac{2u^2du}{1-u^4} = \int\frac{du}{1-u^2} - \int\frac{du}{1+u^2}$$

$$= \frac{1}{2}\,.\,l.\,\frac{1+u}{1-u} - \tan.^{-1}u.$$

Hence $\frac{x^{\frac{3}{2}}}{1\,.\,3} + \frac{x^{\frac{7}{2}}}{3\,.\,7} + \ldots\ \infty = \frac{x^{\frac{1}{2}}}{2}\,.\,l.\,\frac{1+x}{1-x} -$

$\frac{1}{2}\,.\,l.\,\frac{1+\sqrt{x}}{1-\sqrt{x}} + \tan.^{-1}\sqrt{x}$,

which being taken between $x = 0$ and 1, we have

$$\frac{1}{1\,.\,3} + \frac{1}{3\,.\,7} + \frac{1}{5.11} + \ldots\ \infty = l.\,\frac{\sqrt{1+1}}{1+1} + \frac{\pi}{4} = \frac{\pi}{4}$$

$$- \frac{1}{2}\,l.2.$$

689. $S = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots$

We have $\cos.\,\theta = 1 - \frac{\theta^2}{2} + \frac{\theta^4}{2\,.\,3\,.\,4} - \frac{\theta^6}{2.3.4.5.6} + \&c.$

whose factors are $\left(\theta \pm \frac{\pi}{2}\right), \left(\theta \pm \frac{3\pi}{2}\right), \left(\theta \pm \frac{5\pi}{2}\right)$ &c., because $\pm$

$\frac{\pi}{2}$, $\pm\ \frac{3\pi}{2}$ &c. being substituted for θ, the equation

$$\text{Cos. } \theta = 0$$

will always be verified.

Hence then we have

$$\left(\theta + \frac{\pi}{2}\right) . \left(\theta - \frac{\pi}{2}\right) . \left(\theta + \frac{3\pi}{2}\right) . \left(\theta - \frac{3\pi}{2}\right) \times \&c. =$$

$$1 - \frac{\theta^2}{2} + \frac{\theta^4}{2 . 3 . 4} - \&c.$$

$$\therefore \left\{ \theta^2 - \left(\frac{\pi}{2}\right)^2 \right\} . \left\{ \theta^2 - 3^2 . \left(\frac{\pi}{2}\right)^2 \right\} . \left\{ \theta^2 - 5^2 . \left(\frac{\pi}{2}\right)^2 \right\} .$$

$$\&c. = 1 - \frac{\theta^2}{2} + \frac{\theta^4}{2 . 3 . 4} - \&c.$$

Now, from the theory of Equations, $\frac{\text{the coefficient of } (\theta^2)^1}{\text{coefficient of } (\theta^2)^0} =$ sum of the reciprocals of the roots.

$$\therefore \frac{1}{\left(\frac{\pi}{2}\right)^2} + \frac{1}{3^2 . \left(\frac{\pi}{2}\right)^2} + \frac{1}{5^2 . \left(\frac{\pi}{2}\right)^2} + \ldots . \infty = \frac{\frac{1}{2}}{1} = \frac{1}{2}$$

$$\therefore \frac{1}{1} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\ldots \infty = \frac{\pi^2}{8}$$

See (684).

690. This is a common Geometric Series, whose first being unity, and common ratio $\frac{1}{4}$ we have, by the rule,

$$S = \frac{ar^n - a}{r - 1} = \frac{1 \times (\frac{1}{4})^n - 1}{\frac{1}{4} - 1}$$

Hence $l. (1 - \frac{3}{4} S) = - nl4$, which will give S by means of the tables.

691. From the given equations we easily obtain, by successive substitutions

$$
\begin{aligned}
a &= a \\
b &= a + \Delta a \\
c &= a + 2\Delta a + \Delta^2 a \\
d &= a + 3\Delta a + 3\Delta^2 a + \Delta^3 a \\
e &= a + 4\Delta a + 6\Delta^2 a + 4\Delta^3 a + \Delta^4 a \\
f &= a + 5\Delta a + 10\Delta^2 a + 10\Delta^3 a + 5\Delta^4 a + \Delta^5 a \\
\&c. &= \&c.
\end{aligned}
$$

the coefficients of Δa, $\Delta^2 a$, $\Delta^3 a$ .... being the sum of the second, third, fourth, &c., terms of $(1+1)^2$, $(1+1)^3$, $(1+1)^4$, $(1+1)^5$, &c., expanded, respectively.

Hence $a + bx + cx^2 + dx^3 + \ldots\ldots \infty$

$$
\begin{aligned}
&= a \,.\, (1+x+x^2+x^3+ \ldots\ldots) \\
&+ x \,\Delta\, a\,(1+2x+3\,x^2+4\,x^3 + \ldots\ldots) \\
&+ \frac{x^2\Delta^2 a}{2}(2+3\,.\,2\,.\,x+4\,.\,3\,x^2+5\,.\,4\,.\,x^3 + \ldots\ldots) \\
&+ \frac{x^3\Delta^3 a}{2\,.\,3}(3.2 + 4.3.2x + 5.4.3x^2 + 6.5.4x^3 + \ldots\ldots) \\
&+ \frac{x^4\Delta^4 a}{2.3.4}(4.3.2+5.4.3.2x+6.5.4.3x^2+7.6.5.4x^3 + \ldots\ldots) \\
&+ \&c.
\end{aligned}
$$

Now each of these series is recurring, the respective scales of relation being 1, $2-1$, $3-3+1$, $4-6+4-1$, &c., and applying the common rule, it will be found that their sums are respectively

$$
\begin{aligned}
&\frac{1}{1-x} \\
&\frac{1}{1-2x+x^2}\left(=\frac{1}{(1-x)^2}\right) \\
&\frac{2}{1-3x+3x^2-x^3}\left(=\frac{2}{(1-x)^3}\right) \\
&\frac{2\,.\,3}{1-4x+6x^2-4x^3+x^4}\left(=\frac{2.3}{(1-x)^4}\right) \\
&\&c. \quad \&c.
\end{aligned}
$$

Hence then we finally get

$$
a+bx+cx^2+ \ldots\ldots \infty = \frac{a}{1-x} + \frac{x\Delta a}{(1-x)^2} + \frac{x^2\Delta^2 a}{(1-x)^3} + \ldots\ldots
$$

The above recurring series may be summed also by the form

$$\Sigma\,(u_m v^n) = u_m \frac{v^n}{v-1} - \Delta u_m \frac{v^{n+1}}{(v-1)^2} + \Delta^2 u_m \frac{v^{n+2}}{(v-1)^3} - \&c.$$

which it is easy to prove by the method of integrating by parts.

692. Let $P = (a+b).(a+b-1).(a+b-2)\ldots.(a+b-n+1)$, and since a is independent of b, assume

$P = A + A_1 b + A_2 b.(b-1) + A_3 . b(b-1)(b-2) + \ldots . A_n . \times b.(b-1)\ldots.(b-n+1)$, $A, A_1 \ldots A_n$ being functions of (a), at present undetermined. Then, denoting by $P_0, P_1, P_2 \ldots P_n$ the values of P corresponding to $b = 0, 1, 2, 3 \ldots . n$ respectively, we obtain

$$A = P_0$$
$$A_1 = P_1 - A = P_1 - P_0$$
$$2\,A_2 = P_2 - 2A_1 - A = P_2 - 2P_1 + P_0$$
$$3.2.A_3 = P_3 - 3.2A_2 - 3A_1 - A = P_3 - 3P_2 + 3P_1 - P_0$$
$$4.3.2.A_4 = P_4 - 4.3.2\,A_3 - 4.3A_2 - 4A_1 - A = P_4 - 4P_3 + 6P_2 - 4P_1 + P_0$$
$$\&c. = \&c.$$

Hence $A = a.(a-1).(a-2)\ldots.(a-n+1)$

$A_1 = (a+1).a-(a-1)\ldots.(a-n+2) - a(a-1) \times (a-2)\ldots.(a-n+1)$

$= a(a-1)\ldots.(a-n+2)\{a+1-\overline{a-n+1}\}$

$= n.a.(a-1)\ldots.(a-n+2)$

Also $2A_2 = (a+2).(a+1).a.(a-1)\ldots.(a-n+3) - 2.(a+1) \times a.(a-1)\ldots.(a-n+2) + a.(a-1)\ldots.(a-n+1) = a.(a-1)\ldots..(a-n+3)\,\{(a+2).(a+1) - 2(a+1).(a-n+2) + (a-n+2) \times (a-n+1)\} = a.(a-1)\ldots.(a-n+3).\{n^2-n\}$

$\therefore A = a.(a-1)\ldots.(a-n+1)$

$A_1 = n.a.(a-1)\ldots.(a-n+2)$

$A_2 = n.\dfrac{n-1}{2}a.(a-1)\ldots.(a-n+3)$ and by proceeding in the same manner A_3, A_4, &c., may be found. The law, however, is already manifest. We have, therefore,

$(a+b).(a+b-1)\ldots.(a+b-n+1) = a.(a-1)\ldots.(a-n+1) + nb \times a.(a-1)\ldots.(a-n+2)$

$$+ n.\frac{n-1}{2}.b.(b-1) \times a.(a-1)\ldots.(a-n+3)$$

$+$ &c.

By this theorem a certain class of series may be summed.

693. $$S = \frac{2}{1.2.3} - \frac{3}{2.3.4} + \frac{4}{3.4.5} - \ldots \infty$$

$$= \frac{1}{1.3} - \frac{1}{2.4} + \frac{1}{3.5} - \ldots \infty$$

Now $dx - xdx + x^2dx - x^3dx + x^4dx - \ldots \infty = \dfrac{dx}{1+x}$

$$\therefore x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots \infty = \int \frac{dx}{1+x}$$

Multiply by xdx and integrate; then

$$\frac{x^3}{1.3} - \frac{x^4}{2.4} + \frac{x^5}{3.5} - \ldots \infty = \int xdx \int \frac{dx}{1+x}$$

But $\displaystyle\int xdx \int \frac{dx}{1+x} = \frac{x^2}{2}.l.(1+x) - \frac{1}{2}\int \frac{x^2dx}{1+x}$

$$= \frac{x^2}{2}.l.(1+x) - \int \frac{xdx}{2} + \int \frac{dx}{2} - \int \frac{dx}{2(1+x)}$$

$$= \frac{x^2-1}{2}.l.(1+x) - \frac{x^2}{4} + \frac{x}{2}$$ which, between the limits of $x = 0$ and 1, gives

$$S = \frac{1-1}{2}.l.2 - \frac{1}{4} + \frac{1}{2} = \frac{1}{4}.$$

Again $$S = \frac{4}{1.2.5} + \frac{5}{2.3.6} + \frac{6}{3.4.7} + \ldots \frac{n+3}{n.(n+1).(n+4)}.$$

We have $$\Delta S = \frac{n+4}{(n+1).(n+2).(n+5)} = \frac{1}{(n+1).(n+2)}$$

$$- \frac{\quad}{(n+1)(n+2).(n+5)}$$

$$= \frac{1}{(n+1).(n+2)} - \frac{(n+3).(n+4)}{(n+1).(n+2).(n+3).(n+4)(n+5)}$$

Now $(n+3).(n+4). = (n+4-1).(n+5-1) = (n+4) \times (n+5) - (n+5) - (n+4) + 1 = (n+4).(n+5) - 2(n+5) + 2.$

$$\therefore \Delta S = \frac{1}{(n+1).(n+2)} - \frac{1}{(n+1).(n+2).(n+3)}$$
$$+ \frac{2}{(n+1)...(n+4)} - \frac{2}{(n+1)..(n+5)}$$
$$\therefore S = C - \frac{1}{n+1} + \frac{1}{2(n+1)(n+2)} - \frac{2}{3.(n+1)(n+2).(n+3)}$$
$$+ \frac{1}{2.(n+1)...(n+4)}$$
$$= C - \frac{6n^3+51n^2+139n+121}{6.(n+1)(n+2)....(n+4)}$$

Let $n = 1$, then $C = \frac{121}{144}$.

And $S = \frac{121}{144} - \frac{6n^3+51n^2+139n+121}{6.(n+1).(n+2).(n+3).(n+4)}$ which is the value of S to (n) terms.

Let $n = \infty$

Then S becomes $\frac{121}{144} - \frac{6\infty^3}{6\infty^4} = \frac{121}{144} - \frac{1}{\infty} = \frac{121}{144}$.

694. $S = \frac{1}{1} - \frac{1}{3^3} + \frac{1}{5^3} - ..... \infty$

Here we will take Euler's Series, (see 665.)

$$\frac{\theta}{2} = \sin.\theta - \frac{\sin.2\theta}{2} + \frac{\sin.3\theta}{3} - \frac{\sin.4\theta}{4} + ...$$

Then $\frac{\theta d\theta}{2} = d.\theta \sin.\theta - \frac{d\theta.\sin.2\theta}{2} + \frac{d\theta.\sin.3\theta}{3} + ...$ and integrating

$$\frac{\theta^2}{4} = -\cos.\theta + \frac{\cos.2\theta}{2^2} - \frac{\cos.3\theta}{3^2} + ... + C$$

Let $\theta = 0$

Then $C = 1 - \frac{1}{2^2} + \frac{1}{3^2} - .... \infty = \frac{\pi^2}{12} = \frac{Q^2}{3}$ (see 684)

$$\frac{\theta^2}{4} = \frac{Q^2}{3} - \cos.\theta + \frac{\cos.2\theta}{2^2} - \frac{\cos.3\theta}{3^2} + ...$$

Multiply by $d\theta$ and integrate. Then

$\frac{\theta^3}{12} = \frac{Q^2\theta}{3} - \sin.\theta + \frac{\sin.2\theta}{2^3} - \frac{\sin.3\theta}{3^3} + ...$ there being no correction.

Now let $\theta = \frac{\pi}{2} = Q$

Then $\frac{Q^3}{12} = \frac{Q^3}{8} - 1 + 0 + \frac{1}{3^3} + 0 - \frac{1}{5^3} + 0 - \&c.$

Hence then $S = 1 - \frac{1}{3^3} + \frac{1}{5^3} - \ldots\ \infty = \frac{Q^3}{8} - \frac{Q^3}{12} = \frac{3Q^3}{12} = \frac{Q^3}{4}$

695. $S = \frac{1}{1} + \frac{1}{4} + \frac{1}{10} + \ldots\ \infty$

Then $(n+1)^{\text{th}}$ term is $\dfrac{1}{\dfrac{4.5\ \ldots\ 3+n}{1.2\ \ldots\ldots\ n}} = \dfrac{1.2\ \ldots\ldots\ n}{4.5\ \ldots\ 3+n}$

$= 2 \times 3 \dfrac{4.5\ \ldots\ldots\ n}{4.5\ \ldots\ 3+n} = 2.3 \times \dfrac{1}{(n+1).(n+2).(n+3)} = \Delta S',$

S' being the sum to n terms.

$\therefore S' = C - \dfrac{3}{(n+1).(n+2)}$

$= \dfrac{3}{2} - \dfrac{3}{(n+1).(n+2)}$ the sum to n terms.

Let $n = \infty$

Then $S = \dfrac{3}{2} - \dfrac{3}{\infty^2} = \dfrac{3}{2}$

696. $\left.\begin{array}{l} \text{Let } S = \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots\ \infty \\ S' = \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \ldots\ \infty \end{array}\right\}$ which being geometric series whose first terms are $\frac{1}{4}, \frac{1}{3}$, and common ratios $\frac{1}{2}$ and $\frac{1}{3}$ respectively, we have

$S = \dfrac{\frac{1}{4}}{1 - \frac{1}{2}} = \dfrac{1}{2}$

And $S' = \dfrac{\frac{1}{3}}{1-\frac{1}{3}} = \dfrac{1}{2}$

$\therefore$ S′ is = S.

697. $S = \dfrac{1}{1.2} - \dfrac{3}{4.5} + \dfrac{5}{7.8} - \ldots \infty$

Let $s = \dfrac{x}{1.2} - \dfrac{3x^2}{4.5} + \dfrac{5x^3}{7.8} - \ldots \dfrac{(2n-1)\,x^n}{(3n-2).(3n-1)} \pm$ &c.

Then $px^r s = p.\dfrac{x^{r+1}}{1.2} - \dfrac{3px^{r+2}}{4.5} + \ldots \dfrac{(2n-1)\,px^{n+r}}{(3n-2).(3n-1)} \pm$ &c.

And $\dfrac{pd\,(x^r s)}{dx} = (r+1)\dfrac{px^r}{1.2} - \dfrac{3(pr+2p)x^{r+1}}{4.5} + \ldots$

$\dfrac{(2n-1)\,(pn+pr)x^{n+r-1}}{(3n-1)\,(3n-2)} \pm$ &c.

Put $pn = 3n$ and $pr = -2$

Then $pn + pr = 3n-2$, and $p = 3$, and $r = -\dfrac{2}{3}$

$\therefore \dfrac{3d\,(x^{-\frac{2}{3}}s)}{dx} = \dfrac{x^{-\frac{2}{3}}}{2} - \dfrac{3x^{\frac{1}{3}}}{5} + \dfrac{5x^{\frac{4}{3}}}{8} - \ldots \dfrac{(2n-1)\,x^{n-\frac{5}{3}}}{(3n-1)}$

Multiply by px^r and differentiate; then

$\dfrac{3p}{dx}d.\left\{\dfrac{x^r d(x^{-\frac{2}{3}}s)}{dx}\right\} = \dfrac{p.(r-\frac{2}{3})}{2}x^{r-\frac{5}{3}} - \dfrac{3.(pr+\frac{p}{3})}{5}x^{r-\frac{2}{3}} + \ldots$

$\dfrac{(2n-1)(pr+p.n-p\frac{5}{3})}{3n-1} \times x^{r+n-\frac{8}{3}} \pm \ldots$

Put $pn = 3n$ and $pr - p.\dfrac{5}{3} = -1$

Then $p = 3$ and $r = \dfrac{4}{p} = \dfrac{4}{3}$

Hence $\dfrac{9}{dx}d.\left\{\dfrac{x^{\frac{4}{3}}d(x^{-\frac{2}{3}}s)}{dx}\right\} = x^{-\frac{1}{3}} - 3\,x^{\frac{2}{3}} + 5\,x^{\frac{5}{3}} - \ldots$

$(2n-1)\,x^{n-\frac{4}{3}} \pm$ &c.

Again, multiply by $px^r dx$, and call the above expression s'.

Then $px^r s'dx = px^{r-\frac{1}{3}}dx - 3px^{r+\frac{2}{3}}dx + \dots (2n-1)px^{r+n-\frac{4}{3}} \pm \dots$

$\therefore \int px^r s'dx = \frac{p}{r+\frac{2}{3}}x^{r+\frac{2}{3}} - \frac{3p}{(r+\frac{5}{3})}x^{r+\frac{5}{3}} + \dots \frac{(2n-1)p}{r+n-\frac{1}{3}}x^{r+n-\frac{1}{3}} \pm \dots$

$$\text{Let } 2pn = n \text{ and } r - \frac{1}{3} = -p$$

$$\text{Then } p = \frac{1}{2} \text{ and } r = \frac{1}{3} - \frac{1}{2} = -\frac{1}{6}$$

$$\therefore \int \frac{s'}{2} x^{-\frac{1}{6}} dx = x^{\frac{1}{2}} - x^{\frac{3}{2}} + x^{\frac{5}{2}} - \dots\dots \infty = \frac{x^{\frac{1}{2}}}{1+x}$$

$$\text{But } s' = \frac{9}{dx} d.\left\{\frac{x^{\frac{4}{3}}d.(x^{-\frac{2}{3}}s)}{dx}\right\} = \frac{9}{dx}.d\left\{-\frac{2}{3}x^{-\frac{1}{3}}s + \frac{x^{\frac{2}{3}}ds}{dx}\right\}$$

$$= \frac{9}{dx}.\left\{\frac{2}{9}x^{-\frac{4}{3}}dx.s + \frac{x^{\frac{2}{3}}d^2s}{dx}\right\} = 2x^{-\frac{4}{3}}s + \frac{9x^{\frac{2}{3}}d^2s}{dx^2}.$$

$$\text{Hence } \frac{x^{\frac{1}{2}}}{1+x} = \int\left\{x^{-\frac{3}{2}}dx.s + \frac{9}{2}.\frac{x^{\frac{1}{2}}d^2s}{dx}\right\}$$

$$\therefore \frac{d^2s}{dx^2} + \frac{2}{9x^2}s = \frac{2}{9x^{\frac{1}{2}}dx}.d\frac{x^{\frac{1}{2}}}{1+x} = \frac{1}{9x^{\frac{1}{2}}}.\frac{x^{-\frac{1}{2}} - x^{\frac{1}{2}}}{(1+x)^2}$$

$$= \frac{1}{9}.\frac{1-x}{x.(1+x)^2}.$$

We have therefore reduced the series to the form of a differential equation, which being integrated by the method used in Linear Equations, or by approximation, will give s in terms of x, and $\therefore$ the value of S. Instead of conducting the Student through this tedium, we will exhibit a process by which Series may frequently be transformed into one or more of greater simplicity.

In the series before us, the general term is

$$\pm \frac{2n-1}{(3n-2).(3n-1)}$$

$$\text{Assume } \frac{2n-1}{(3n-2).(3n-1)} = \frac{A}{3n-2} + \frac{B}{3n-1}$$

Hence $A.(3n-1) + B.(3n-2) = 2n-1$

Let $n = \frac{2}{3}$ and $\frac{1}{3}$ successively; and we get

$$A = \frac{4}{3} - 1 = \frac{1}{3}, \text{ and } B = 1 - \frac{2}{3} = \frac{1}{3}$$

$$\therefore \pm \frac{2n-1}{(3n-2).(3n-1)} = \pm \frac{1}{3}\cdot \left(\frac{1}{3n-2} + \frac{1}{3n-1}\right)$$

$\therefore$ putting $n = 1, 2, 3$, &c. the series takes the form

$$3S = \frac{1}{1} - \frac{1}{4} + \frac{1}{7} - \frac{1}{10} - \dots\dots \infty$$

$$+ \frac{1}{2} - \frac{1}{5} + \frac{1}{8} - \frac{1}{11} - \dots\dots \infty$$

Now, by division,

$$\frac{dx}{1+x^3} = dx - x^3dx + x^6dx - \dots\dots \infty$$

And $\frac{xdx}{1+x^3} = xdx - x^4dx + x^7dx - \dots\dots \infty$

$$\therefore \int \frac{dx}{1+x^3} = x - \frac{x^4}{4} + \frac{x^7}{7} - \dots\dots \infty$$

And $\int \frac{xdx}{1+x^3} = \frac{x^2}{2} - \frac{x^5}{5} + \frac{x^8}{8} - \dots\dots \infty$

$\therefore S = \frac{1}{3}\cdot \int \frac{dx}{1+x^3} + \frac{1}{3}\cdot \int \frac{xdx}{1+x^3}$ between the limits of $x=0$ and 1, which $= \frac{1}{3} \times \left\{\frac{1}{3} l. \frac{1+x}{\sqrt{1-x+x^2}} + \frac{1}{\sqrt{3}} \tan.^{-1} \frac{x\sqrt{3}}{2-x}\right.$ $\left. - \frac{1}{3} l. \frac{1+x}{\sqrt{x^2-x+1}} + \frac{1}{\sqrt{3}} \tan.^{-1} \frac{x\sqrt{3}}{2-x}\right\} = \frac{2}{2\sqrt{3}} \frac{\tan.^{-1} x\sqrt{3}}{2-x}$.

Let $x = 0$, and 1 successively.

$$\text{Then } S = \frac{2}{3.\sqrt{3}}\cdot \left(2p\pi + \frac{\pi}{4}\right) - \frac{2}{3.\sqrt{3}}\cdot 2p\pi$$

$$= \frac{\pi}{6\sqrt{3}}.$$

To find the sum of $S = 2+6+14+30+62+126 + \dots n$ terms, we have $S = 2(1 + 3 + 7 + 15 + 63 + \dots 2^n - 1)$

$\therefore \Delta S = 2^{n+1} - 2.$

And since $\Delta a^x = a^{x+1} - a^x = a^x.(a-1)$

and $\therefore \Sigma a^x = \frac{a^x}{a-1} + C,$

$$S = \Sigma 2^{n+2} - \Sigma^2 = \frac{2^{n+2}}{2-1} - 2n + C = 2.(2^{n+1} - n) + C$$

$$= 2^{n+2} - 2n - 4 = 2.(2^{n+1} - n - 2)$$

698. Let $\sin. 3a + \sin. 5a + \sin. 7a + \dots \sin. (2m-1)a = S.$

Then $\Delta S = \sin. (2m+1)a$

$$\therefore S = \Sigma \sin. (2m+1)a = \frac{1}{2\sqrt{-1}} \Sigma \left(e^{(2m+1)a\sqrt{-1}} - e^{-(2m+1)a\sqrt{-1}}\right)$$

$$= \frac{1}{2\sqrt{-1}} \times \left\{\frac{e^{(2m+1)a\sqrt{-1}}}{e^{2a\sqrt{-1}} - 1} - \frac{e^{-(2m+1)a\sqrt{-1}}}{e^{-2a\sqrt{-1}} - 1}\right\} + C, 2a\sqrt{-1}$$

being $= \Delta (2m+1)a\sqrt{-1}.$

$$\therefore S = \frac{1}{2\sqrt{-1}} \times \frac{e^{(2m-1)a\sqrt{-1}} - e^{-(2m-1)a\sqrt{-1}} - e^{(2m+1)a\sqrt{-1}} + e^{-(2m+1)a\sqrt{-1}}}{2 - \left(e^{2a\sqrt{-1}} + e^{-2a\sqrt{-1}}\right)} + C$$

$$= \frac{\sin. (2m-1)a - \sin. (2m+1)a}{2 - 2\cos. 2a} + C$$

$$= \frac{\cos. 2ma. \sin. a}{\cos. 2a - 1} + C = -\frac{\cos. 2ma}{2\sin. a} + C$$

Let $m = 1$

Then $C = \frac{\cos. 2a}{2\sin. a}$

And $S = \frac{\cos. 2a - \cos. 2ma}{2\sin. a} = \frac{\sin. (m+1)a. \sin. (m-1)a}{\sin. a}$

Otherwise.

Let $S = \sin. a + \sin. (a+b) + \sin. (a+2b) + \dots \sin. (a + \overline{n-1}.b)$

Then $2\sin.\frac{b}{2}.\ S = 2\sin.\frac{b}{2}.\ \sin. a + 2\sin.\frac{b}{2}.\ \sin.(a+b)$

$+ \ldots. 2\sin.\frac{b}{2}.\ \sin.(a + \overline{n-1}.b)$

$$= \cos.\left(a-\frac{b}{2}\right) - \cos.\left(a+\frac{b}{2}\right)$$
$$+ \cos.\left(a+\frac{b}{2}\right) - \cos.\left(a+\frac{3b}{2}\right)$$
$$+ \cos.\left(a+\frac{3b}{2}\right) - \cos.\left(a+\frac{5b}{2}\right)$$
$$+ \text{\&c.}$$
$$+ \cos.\left(a+\frac{2n-3}{2}b\right) - \cos.\left(a+\frac{2n-1}{2}b\right)$$

by the form $\cos.(A - B) - \cos.(A + B) = 2\sin. A. \sin. B$

$$\text{Hence } S = \frac{\cos.\left(a-\frac{b}{2}\right) - \cos.\left(a+\frac{2n-1}{2}b\right)}{2\sin.\frac{b}{2}}$$

$$= \frac{\sin.\left(a+\frac{n-1}{2}b\right).\ \sin.\frac{n}{2}b}{\sin.\frac{b}{2}}$$

Substitute $3a$ and $2a$, and $m - 1$ for a, b, and n respectively, and the resulting value of S will be that stated in the problem.

The Series may also be summed by considering it a *recurring* series. (See 729.)

699. $S = \frac{1.2}{2.3} - \frac{2.3}{3.4} + \frac{3.4}{4.5} - ..\infty$, which may be reduced to $S = \frac{1}{3} - \frac{2}{4} + \frac{3}{5} - \frac{4}{6} + \ldots \infty$.

Now we have

$$x - x^2 + x^3 - x^4 + \ldots. \infty = \frac{x}{1+x}$$

$\therefore$ differentiating

$$1 - 2x + 3x^2 - 4x^3 + \ldots\ldots \infty = \frac{1}{1+x} - \frac{x}{(1+x)^2} = \frac{1}{(1+x)^2}$$

$$\therefore x^2dx - 2x^3dx + 3x^4dx - \ldots\ldots \infty = \frac{x^2dx}{(1+x)^2}$$

$$\therefore \frac{x^3}{3} - \frac{2x^4}{4} + \frac{3x^5}{5} - \ldots\ldots \infty = \int \frac{x^2dx}{(1+x)^2}$$

But it will easily be found that

$$\int \frac{x^2dx}{(1+x)^2} = \frac{(x^2-2)}{1+x} - 2\,l.\,(1+x) + C$$

$$= 2 + \frac{x^2-2}{1+x} - 2\,l.\,(1+x)$$

Let $x = 1$

Then $S = \frac{1}{3} - \frac{2}{4} + \ldots\ldots = \int \frac{x^2dx}{(1+x)^2}$, taken between $x = 0$ and 1,

$$= 2 - \frac{1}{2} - 2.\,l.\,2 = \frac{3}{2} - 2\,l.\,2$$

Again $S = \frac{10}{1.2.3.4} + \frac{14}{2.3.4.5} + \ldots\ldots \infty$

Let $S' = \frac{10}{1.2.3.4} + \frac{14}{2.3.4.5} + \ldots\ldots \frac{6+4n}{n.(n+1)\,(n+2)\,.\,(n+3)}$

Then $\Delta S' = \frac{6+4(n+1)}{(n+1).(n+2).(n+3).(n+4)} = \frac{6}{(n+1)\ldots(n+4)}$

$$+ \frac{4}{(n+2)\,(n+3)\,(n+4)}$$

$$\therefore S' = C - \frac{6}{3.(n+1).(n+2).(n+3)} - \frac{4}{2.(n+2).(n+3)}$$

$$= C - \frac{2}{(n+1).(n+3)}$$

Let $n = 1$

Then $C = \frac{10}{1.2.3.4} + \frac{1}{4} = \frac{16}{24} = \frac{2}{3}$.

$$\therefore S' = \frac{2}{3} - \frac{2}{(n+1).(n+3)}$$

Let $n = \infty$

Then we get

$$S = \frac{2}{3}.$$

700. $$S = \frac{1}{1.5} - \frac{1}{2.6} + \frac{1}{3.7} - \dots \frac{1}{n.(n+4)}$$

Assume $\frac{1}{1} - \frac{1}{2} + \frac{1}{3} - \dots \pm \frac{1}{n-1} \mp \frac{1}{n} = S'$

Then $\frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \dots \pm \frac{1}{n+3} \mp \frac{1}{n+4} = S' + \frac{1}{4}$

$$-\frac{1}{3} + \frac{1}{2} - 1 \pm \left(\frac{1}{n+1} - \frac{1}{n+2} + \frac{1}{n+3} - \frac{1}{n+4}\right) = S'$$

$$- \frac{7}{12} \pm \frac{2}{(n+1)\dots(n+4)} \times (n^2 + 5n + 7)$$

Hence by subtraction we obtain

$$\frac{4}{1.5} - \frac{4}{2.6} + \frac{4}{3.7} - \dots \pm \frac{4}{(n+1)\dots(n+3)} \mp \frac{4}{n.(n+4)}$$

$$= \frac{7}{12} \mp \frac{2}{(n+1)\dots(n+4)} (n^2 + 5n + 7)$$

And $\therefore$

$$S = \frac{7}{48} \mp \frac{n^2 + 5n + 7}{2.(n+1).(n+2).(n+3).(n+4)}$$ according as n is even or odd.

The Differential Calculus will apply in this case, but not with equal facility.

Again, let $S = \frac{1}{1.2^2.3^2} + \frac{1}{2.3^2.4^2} + \dots \frac{1}{n.(n+1)^2(n+2)^2}$.

And assume $\frac{1}{n.(n+1)^2(n+2)^2} = \frac{A}{n} + \frac{P}{(n+1)^2} + \frac{Q}{n+1}$

$$+ \frac{P_1}{(n+2)^2} + \frac{Q_2}{n+2}.$$

Then, reducing to a common denominator and equating coefficients of the same powers of (n) on both sides the equation, we obtain

$$A = \frac{1}{4}, P = -1, Q = 1, P_1 = -\frac{1}{2}, Q_1 = -\frac{5}{4}$$

$$\therefore \frac{1}{n.(n+1)^2.(n+2)^2} = \frac{1}{4n} - \frac{1}{(n+1)^2} + \frac{1}{n+1} - \frac{1}{2.(n+2)^2} - \frac{5}{4.(n+2)}$$

$$= \frac{3n+1}{2n.(n+1).(n+2)} - \frac{1}{(n+1)^2} - \frac{1}{2.(n+2)^2}$$

Hence $\Delta S = \frac{3n+4}{2.(n+1).(n+2).(n+3)} - \frac{1}{(n+2)^2} - \frac{1}{2.(n+3)^2}$

$$= \frac{3}{2}.\frac{1}{(n+2).(n+3)} + \frac{1}{2.(n+1).(n+2).(n+3)} - \frac{1}{(n+2)^2} - \frac{1}{2(n+3)^2}$$

$$\therefore S = C - \frac{3}{2}.\frac{1}{n+2} - \frac{1}{4.(n+1).(n+2)} - \Sigma\frac{1}{(n+2)^2} - \Sigma\frac{1}{2.(n+3)^2}$$

Now $S\frac{1}{(n+2)^2} = \frac{1}{2^2} + \frac{1}{3^2} + \ldots\frac{1}{(n+1)^2} = N$, by supposition.

$$\therefore S\frac{1}{2.(n+3)^2} = \frac{1}{2}\left\{\frac{1}{3^2} + \frac{1}{4^2} + \ldots\frac{1}{(n+2)^2}\right\} = \frac{N}{2} - \frac{1}{8} + \frac{1}{2.(n+2)^2}$$

$$\therefore S = C - \frac{3}{2}.\frac{1}{n+2} - \frac{1}{4.(n+1).(n+2)} - \frac{3N}{2} + \frac{1}{8} - \frac{1}{2.(n+2)^2}$$

$$= C + \frac{1}{8} - \frac{6n^2+21n+16}{4(n+1).(n+2)^2} - \frac{3N}{2}$$

Let now $n = 1$. Then $N = \frac{1}{2^2}$ and $S = \frac{1}{1.2^2.3^2}$

Hence $C + \frac{1}{8} = 1$.

$\therefore S = 1 - \frac{6n^2+21n+16}{4.(n+1).(n+2)^2} - \frac{3N}{2}$ in which N is given by the Problem.

When $n = \infty$ we have,

$$N = \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots\infty = \frac{\pi^2}{6} - 1 \text{ (See 702)}$$

$$\text{And } S = 1 - \frac{\pi^2}{4} + \frac{3}{2} = \frac{5}{2} - \frac{\pi^2}{4}.$$

701. Let $S = \dfrac{5.6}{1.2.3.4} + \dfrac{6.7}{2.3.4.5} + \ldots \dfrac{(n+4)(n+5)}{n.(n+1)(n+2)(n+3)}$

$$\text{Then } \Delta\, S = \frac{(n+5).(n+6)}{(n+1)\ldots(n+4)}$$

Now $(n+5).(n+6) = n^2+11n+30 = (n+3).(n+4)+4(n+4)+2$

$$\therefore\ \Delta\, S = \frac{1}{(n+1).(n+2)} + \frac{4}{(n+1)..(n+3)} + \frac{2}{(n+1)\ldots(n+4)}$$

$$\therefore\ S = C - \frac{1}{n+1} - \frac{2}{(n+1)..(n+2)} - \frac{2}{3.(n+1).(n+2).(n+3)}$$

$$= C - \frac{3n^2+21n+38}{3.(n+1)(n+2).(n+3)}$$

$$= \frac{19}{9} - \frac{3n^2+21n+38}{3.(n+1)(n+2).(n+3)}.$$

702. To sum $\dfrac{1}{1.3} - \dfrac{1}{2.4} + \ldots - \dfrac{1}{n.(n+2)} = S.$

$$\text{Let } \frac{1}{1} - \frac{1}{2} + \frac{1}{3} - \ldots - \frac{1}{n} = s$$

Then $\dfrac{1}{3} - \dfrac{1}{4} + \dfrac{1}{5} - \ldots - \dfrac{1}{n+2} = s + \dfrac{1}{2} - 1 + \dfrac{1}{n+1}$
$- \dfrac{1}{n+2}$, and by subtraction

$$\frac{2}{1.3} - \frac{2}{2.4} + \ldots \frac{2}{n.(n+2)} = \frac{1}{2} - \frac{1}{(n+1).(n+2)}$$

$$\therefore\ S = \frac{1}{4} - \frac{1}{2.(n+1).(n+2)}.$$

703. To find $S = \dfrac{1}{1.2.4} + \dfrac{1}{2.3.5} + \ldots \dfrac{1}{n.(n+1).(n+3)}.$

$$\Delta S = \frac{1}{(n+1).(n+2).(n+4)} = \frac{n+3}{(n+1).(n+2).(n+3).(n+4)}$$

$$= \frac{1}{(n+1).(n+2).(n+3)} - \frac{1}{(n+1).(n+2).(n+3).(n+4)}$$

$$\therefore S = C - \frac{1}{2(n+1).(n+2)} + \frac{1}{3(n+1).(n+2).(n+3)}$$

$$= \frac{7}{36} - \frac{1}{2.(n+1).(n+2)} + \frac{1}{3.(n+1).(n+2).(n+3)}$$

704. Sum $1 - \frac{1}{2} + \frac{1}{4} - \ldots$ to n terms and ∞.

The series being geometric (common ratio $= -\frac{1}{2}$) we have by the ordinary rule

$$S = \frac{1 \times \left(-\frac{1}{2}\right)^n - 1}{-\frac{1}{2} - 1} = \frac{(\mp \frac{1}{2^n} - 1) \times 2}{-3} = \frac{2^n \pm 1}{3 \times 2^{n-1}}.$$

Let $n = \infty$

$$\text{Then } S = \frac{2}{3} \pm \frac{1}{3 \times 2^{\infty - 1}}$$

$$= \frac{2}{3}.$$

705. $S = \frac{1}{1.5} + \frac{1}{3.7} + \frac{1}{5.9} + \ldots \frac{1}{(2n-1) \times (2n+3)}.$

$$\text{Here } \Delta S = \frac{1}{(2n+1).(2n+5)} = \frac{2n+3}{(2n+1).(2n+3).(2n+5)}$$

$$= \frac{1}{(2n+1).(2n+3)} - \frac{2}{(2n+1).(2n+3).(2n+5)}$$

$$\therefore S = C - \frac{1}{2(2n+1)} + \frac{1}{2.(2n+1).(2n+3)}$$

$$= C - \frac{n+1}{(2n+1).(2n+3)} = \frac{1}{3} - \frac{n+1}{(2n+1).(2n+3)}.$$

Let $n = \infty$

$$\text{Then } S = \frac{1}{3} - \frac{\infty}{2\infty \times 2\infty} = \frac{1}{3}.$$

Again, to sum $\frac{1}{1.2} + \frac{1}{3.4} + \ldots\ldots \infty$ we have, by division,

$$dx - xdx + x^2dx - x^3dx + \ldots\ldots \infty = \frac{dx}{1+x}$$

$$\therefore x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots\ldots \infty = l.(1+x)$$

which, being taken between $x = 0$ and 1, becomes

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \ldots\ldots \infty = l.\,2$$

And, collecting the terms by pairs, we get

$$\frac{1}{1.2} + \frac{1}{3.4} + \frac{1}{5.6} + \ldots\ldots \infty = l.\,2$$

$$\text{Again, } S = \frac{1}{1.2.5} + \frac{1}{2.3.7} + \ldots\ldots \infty.$$

Here we take

$$dx + xdx + x^2dx + \ldots\ldots \infty = \frac{dx}{1-x}$$

$$\therefore x + \frac{x^2}{2} + \frac{x^3}{3} + \ldots\ldots \infty = \int \frac{dx}{1-x}$$

Multiply by dx and again integrate

$$\text{Then } \frac{x^2}{2} + \frac{x^3}{2.3} + \frac{x^4}{3.4} + \ldots\ldots \infty = \int dx \int \frac{dx}{1-x}$$

Multiply by $\frac{dx}{x^{\frac{1}{2}}}$ and integrate

$$\text{Then } \frac{x^{\frac{5}{2}}}{1.2.5} + \frac{x^{\frac{7}{2}}}{2.3.7} + \frac{x^{\frac{9}{2}}}{3.4.9} + \ldots\ldots \infty = \frac{1}{2}\int \frac{dx}{\sqrt{x}} \int dx \int \frac{dx}{1-x}$$

Now, in order to take this integral between the values 0 and 1 of x, for the sake of brevity, we will make x, or any positive power of x, whenever it occurs without the integral sign, equal to unity; because such a supposition will evidently not affect the limit of $x = 0$.

By the form $\int udv = vu - \int vdu$,

$$\int \frac{1}{2} \frac{dx}{\sqrt{x}} \int dx \int \frac{dx}{1-x} = \sqrt{x} \int dx \int \frac{dx}{1-x} - \int \sqrt{x}\, dx \int \frac{dx}{1-x}$$

$$= x \int \frac{dx}{1-x} - \int \frac{xdx}{1-x} - \frac{2}{3} x^{\frac{3}{2}} \int \frac{dx}{1-x} + \frac{2}{3} \int \frac{x^{\frac{3}{2}}dx}{1-x}$$

$$= \frac{2}{3}.\ l(1-x) + \frac{x}{3} + \frac{2}{3}.\ \int \frac{x^{\frac{1}{2}}dx}{1-x}$$

$$= \frac{2}{3}.\ l.(1-x) + \frac{2}{3}.\ l.\ \frac{1+\sqrt{x}}{1-\sqrt{x}} - \frac{x}{3} - \frac{4}{3}\sqrt{x}$$

$$= \frac{4}{3}.l.(1+\sqrt{x}) - \frac{x}{3} - \frac{4}{3}\sqrt{x} \text{ by logarithms.}$$

Put $x = 0$. Then the integral $= \frac{4}{3} l.(1) = 0$.

Let $x = 1$. Then it $= \frac{4}{3}.\ l.\ 2 - \frac{1}{3} - \frac{4}{3} = \frac{4}{3}.l\,2 - \frac{5}{3}$

Hence $\frac{1}{1.2.5} + \frac{1}{2.3.7} + \ldots.\ \infty = \frac{4}{3}.l.2 - \frac{5}{3}$

Again, $S = 1 - \frac{1}{3.3} + \frac{1}{5.3^2} - \ldots.\ \infty$.

We have, by division,

$$\frac{dx}{x^{\frac{1}{2}}} - x^{\frac{1}{2}}dx + x^{\frac{3}{2}}dx - \ldots.\ \infty = \frac{dx}{\sqrt{x}.(1+x)}$$

$$\therefore 1 - \frac{x}{3} + \frac{x^2}{5} - \ldots.\ \infty = \frac{1}{2\sqrt{x}}. \int \frac{dx}{\sqrt{x}.(1+x)}$$

Now, put $x = u^2$

Then $\int \frac{dx}{\sqrt{x}.(1+x)} = 2\int \frac{du}{1+u^2} = 2 \tan.^{-1} u = 2 \tan.^{-1} \sqrt{x}$

$\therefore$ taking the integral between $x = 0$ and $\frac{1}{3}$, we have

$$1 - \frac{1}{3.3} + \frac{1}{5.3^2} - \frac{1}{7.3^3} + \ldots. = \sqrt{3}.\tan.^{-1}\frac{1}{\sqrt{3}}$$

$$\therefore S = \sqrt{3} \times 30^\circ = \sqrt{3}.\ \frac{\pi}{6} = \frac{\pi}{2\sqrt{3}}.$$

706. $S = 12 + 4 + \frac{4}{3} + \frac{4}{3^2} + \ldots. \infty.$

This is a geometric series, whose common ratio is $\frac{1}{3}$.

$$\therefore S = \frac{a}{1-r} = \frac{12}{1-\frac{1}{3}} = \frac{36}{2} = 18.$$

Again $S = \frac{1}{1.2} + \frac{1}{2.3} + \ldots. \infty.$ See 685.

707. To sum $\frac{1}{3} + \frac{1.2}{3.4} + \frac{1.2.3}{3.4.5} + \ldots. \infty.$

We first transform it to its equivalent

$$\frac{2}{2.3} + \frac{2}{3.4} + \frac{2}{4.5} + \ldots. \frac{2}{(n+1).(n+2)} + \ldots. \infty$$

Let $S = \frac{2}{2.3} + \frac{2}{3.4} + \ldots. \frac{2}{(n+1).(n+2)}$

Then $\Delta S = \frac{2}{(n+2).(n+3)}$

And $S = C - \frac{2}{n+2} = 1 - \frac{2}{n+2} = \frac{n}{n+2}.$

Let $n = \infty$

Then S becomes $= \frac{1}{3} + \frac{1.2}{3.4} + \ldots. \infty = \frac{\infty}{\infty+2} = \frac{\infty}{\infty} = 1.$

To sum $\frac{1}{1.5} + \frac{1}{3.7} + \ldots. n$ terms. See 705.

708. To sum $S = 1 + 3 + 6 + 10 + 15 + \ldots. \frac{m.(m+1)}{2}$,

which are figurates of the second order, we have

$$\Delta S = \frac{(m+1).(m+2)}{2}$$

$$\therefore S = \frac{m.(m+1).(m+2)}{2.3} + C = \frac{m.(m+1).(m+2)}{2.3}$$

709. To sum $\frac{1}{3} - \frac{1}{3.2} + \frac{1}{3.2^2} - \ldots \infty$.

This being a geometric series, whose first term is $\frac{1}{3}$, and common ratio $\left(-\frac{1}{2}\right)$ we have

$$S = \frac{a}{1-r} = \frac{\frac{1}{3}}{1+\frac{1}{2}} = \frac{2}{9}.$$

To sum $1 + 2^2 + \ldots n^2$, see 683.

710. To sum $\frac{2}{10} - \frac{2}{100} + \frac{2}{1000} - \ldots \infty$.

The series being geometric (common ratio $= -\frac{1}{10}$), we have

$$S = \frac{a}{1-r} = \frac{\frac{2}{10}}{1+\frac{1}{10}} = \frac{2}{11}$$

Again, $5 + 7 + 9 + 11 \ldots$ 50 terms is an arithmetic series, whose common difference is 2. $\therefore$ 50[th] term $= 5 + 49 \times 2 = 103$.

$$\left.\begin{array}{l}\text{Put } 5 + 7 + 9 + \ldots 101 + 103 = S \\ \text{Then } 103 + 101 + \ldots 7 + 5 = S\end{array}\right\} \text{and by addition}$$

$$50 \times 108 = 2S,$$

$$\therefore S = 25 \times 108 = 1080$$

711. To sum $\frac{1}{2} - \frac{1}{2.2^3} + \frac{1}{2.2^6} - \ldots \infty$ which is a geometric series, whose common ratio is $-\frac{1}{2^3}$, we have

$$S = \frac{a}{1-r} = \frac{\frac{1}{2}}{1+\frac{1}{2^3}} = \frac{2^2}{9} = \left(\frac{2}{3}\right)^2.$$

Again, to sum $\frac{2}{1.3.4} + \frac{3}{2.4.5} + \ldots \frac{n+1}{n.(n+2).(n+3)}$, we have

$$\Delta\ S = \frac{n+2}{(n+1).(n+3).(n+4)} = \frac{(n+2)^2}{(n+1).(n+2).(n+3).(n+4)}$$

But $(n+2)^2 = (n+2).(n+1) + (n+2) = (n+1).(n+2) + (n+1) + 1$

$$\therefore\ \Delta\ S = \frac{1}{(n+3).(n+4)} + \frac{1}{(n+2).(n+3).(n+4)}$$

$$+ \frac{1}{(n+1).(n+2).(n+3).(n+4)}$$

$$\therefore S = C - \frac{1}{n+3} - \frac{1}{2.(n+2).(n+3)} - \frac{1}{3.(n+1).(n+2).(n+3)}$$

$$= C - \frac{6n^2+21n+17}{6.(n+1).(n+2).(n+3)} = \frac{17}{36} - \frac{6n^2+21n+17}{6.(n+1).(n+2).(n+3)}.$$

Again, to sum $1^2 + 4^2 + 7^2 + 10^2 + \ldots (3n-2)^2$, we have

$$\Delta\ S = (3n+1)^2 = (3n+1).(3n+4) - 3.(3n+1)$$

$$\therefore\ S = \frac{(3n-2).(3n+1).(3n+4)}{3\times 3} - 3.\frac{(3n-2).(3n+1)}{2\times 3} + C$$

$$= \frac{(3n-2).(3n+1).(6n-1)}{18} + C$$

$$= \frac{(3n-2).(3n+1).(6n-1)}{18} - \frac{1}{9}.$$

712. To sum $\frac{5}{1.2}\times\frac{1}{3} + \frac{7}{2.3}\times\frac{1}{9} + \frac{9}{3.4}\times\frac{1}{27} + \ldots n$ terms.

In this example we will adopt *Demoivre's* method of Multiplication.

Assume $1 + \frac{x}{2} + \frac{x^2}{3} + \ldots \frac{x^{n-1}}{n} + \frac{x^n}{n+1} = S$

And multiply by $ax - b$.

Then we have

$$\frac{(2a-b)x}{1.2} + \frac{3a-2b}{2.3}x^2 + \ldots \frac{(n+1)a - nb}{n.(n+1)}x^n = (ax-b) \times$$

$$S - \frac{ax^{n+1}}{n+1} + b.$$

Put $2a - b = 5$

And $a - b$ (= the common difference of the numerators) = 2

Hence $b = 1$
And $a = 2 + 1 = 3$ $\Big\}$

$$\therefore \frac{5}{1.2}.x + \frac{7}{2.3}.x^2 + \ldots\ldots \frac{2n+3}{n.(n+1)}x^n = (3x-1)\times$$

$$S - \frac{3x^{n+1}}{n+1} + 1.$$

Let $x = \dfrac{1}{3}$

$$\text{Then } \frac{5}{1.2}\times\frac{1}{3}+\frac{7}{2.3}\times\frac{1}{3^2}+\ldots\frac{2n+3}{n.(n+1)}\times\frac{1}{3^n} = 1 - \frac{1}{3^n\times(n+1)}$$

To sum $\dfrac{1}{1^3} - \dfrac{1}{3^3} + \dfrac{1}{5^3} - \ldots\ldots\ \infty$. See 694.

713. To sum $1 + 5 + 9 + 13 + \ldots\ldots 4n - 3$.

$$\Delta\ S = 4n + 1$$

$$\therefore S = \frac{(4n+3).(4n+1)}{8} + C = \frac{(4n-3)(4n+1)+3}{8} = n.(2n-1)$$

714. To sum $\dfrac{1}{1.2.3.4} + \dfrac{3}{2.3.4.5} + \dfrac{6}{3.4.5.6} + \ldots\ldots$

$$\frac{\;\dfrac{n.(n+1)}{2}\;}{n.(n+1).(n+2).(n+3)}.$$

$$\Delta\ S = \frac{1}{2}.\frac{(n+1).(n+2)}{(n+1).(n+2).(n+3).(n+4)}$$

$$= \frac{1}{2}.\frac{1}{(n+3).(n+4)}$$

$$\therefore S = C - \frac{1}{2}.\ \frac{1}{n+3} = \frac{1}{6} - \frac{1}{2}.\frac{1}{(n+3)} = \frac{n}{6.(n+3)}$$

715. To sum $\dfrac{1}{1.2.3} + \dfrac{2^2}{2.3.4} + \ldots\ \dfrac{n^2}{n.(n+1).(n+2)}$.

$$\Delta\ S = \frac{(n+1)^2}{(n+1).(n+2).(n+3)} = \frac{n+2-1}{(n+2).(n+3)}$$

$$= \frac{1}{(n+3)} - \frac{1}{(n+2).(n+3)}$$

$$\therefore\ S = \Sigma.\frac{1}{n+3} + \frac{1}{n+2} + C$$

Now $\Sigma\frac{1}{v} = \frac{l.v}{\Delta v} - \frac{1}{2v} - \frac{\Delta v}{12v^2} + \frac{(\Delta v)^3}{120v^4} - \frac{(\Delta v)^5}{252v^6} + \&c.$ (See 652.)

$$\therefore\ \Sigma\frac{1}{(n+3)} = l.(n+3) - \frac{1}{2.(n+3)} - \frac{1}{12.(n+3)^2}$$

$$+ \frac{1}{120.(n+3)^4} - \&c.$$

$$\therefore\ S = l.(n+3) + \frac{1}{n+2} - \frac{1}{2.(n+3)} - \frac{1}{12.(n+3)^2}$$

$$+ \frac{1}{120.(n+3)^4} - \frac{1}{252.(n+3)^6} + \frac{1}{240.(n+3)^8} - \&c. + C.$$

Now, in order to find C, let $n = 2$.

$$S = \frac{1}{6} + \frac{1}{6} = \frac{1}{3} = l.\,5 + \frac{1}{4} - \frac{1}{10} - \frac{1}{300} + \frac{1}{75000}$$

$$- \&c. + C$$

$$\therefore\ C = \frac{1}{12} + \frac{1}{10} + \frac{1}{300} - \frac{1}{75000} - l.\,5 \text{ nearly}$$

$$= \begin{Bmatrix} .08333333 \\ .1 \\ .00333333 \end{Bmatrix} - \begin{Bmatrix} .0000333 \\ 1.6094379 \end{Bmatrix} = -\ 1.4228045 \text{ nearly.}$$

$$\text{Hence } S = l.(n+3) + \frac{n+4}{2.(n+2)(n+3)} - \frac{1}{12.(n+3)^2}$$

$$+ \frac{1}{120.(n+3)^4} - \&c. - 1.4228045 \text{ nearly.}$$

To sum $1 + 2.5 + 3.5^2 + 4.5^3 + \ldots\ldots n.5^{n-1}$.

Here $\Delta\ S = (n+1).\ 5^n$

$$\therefore\ S = \Sigma\,(n+1)\,5^n = (n+1)\,\Sigma\,5^n - \Sigma\,\{\Delta\,(n+1)\,\Sigma.5^{n+1}\}$$

But $\Sigma.5^n = \frac{5^n}{5-1} = \frac{5^n}{4}$

$$\Sigma.5^{n+1} = \frac{5^{n+1}}{4}$$

And $\Delta\ (n+1) = 1$

$$\therefore S = (n+1).\frac{5^n}{4} - \Sigma.\frac{5^{n+1}}{4}$$

$$= (n+1).\frac{5^n}{4} - \frac{5^{n+1}}{16} + C$$

$$= \frac{5^n \times (4n-1)}{16} + \frac{1}{16}.$$

716. To sum $\frac{1}{2} - \frac{1}{2.3} + \frac{1}{2.3^2} - \ldots\ldots \infty.$

The common ratio of this geometric series being $\left(-\frac{1}{3}\right)$, we have

$$S = \frac{a}{1-r} = \frac{\frac{1}{2}}{1+\frac{1}{3}} = \frac{3}{8}.$$

To sum $\frac{1}{2.4.6} + \frac{1}{4.6.8} + \ldots\ldots \frac{1}{8.n.(n+1)(n+2)}.$

Here $\Delta\ S = \frac{1}{8.(n+1).(n+2).(n+3)}$.

$$\therefore S = C - \frac{1}{16.(n+1).(n+2)} = \frac{1}{32} - \frac{1}{16.(n+1).(n+2)}.$$

717. To sum $\frac{1}{2} - \frac{1}{4} + \frac{1}{8} - \ldots\ldots \infty.$

Here the common ratio is $\left(-\frac{1}{2}\right)$.

$$\therefore S = \frac{a}{1-r} = \frac{\frac{1}{2}}{1+\frac{1}{2}} = \frac{1}{3}.$$

Again, $S = 1 + 2 + 3 + \dots n$
$\therefore S = n + (n-1) + (n-2) + \dots 1$ } and by addition we have
$2S = (n+1) + (n+1) \dots n \text{ terms} = n.(n+1)$
$$\therefore S = \frac{n.(n+1)}{2}.$$

718. To sum $1.4.5 + 2.5.6 + \dots n.(n+3)(n+4)$.

Here $\Delta S = (n+1).(n+4).(n+5) = (n+3-2).(n+4).(n+5)$
$= (n+3).(n+4).(n+5) - 2(n+4).(n+5)$

$\therefore S = \dfrac{(n+2).(n+3).(n+4).(n+5)}{4} - \dfrac{2}{3}.(n+3) \times$
$(n+4).(n+5) + C$
$$= \frac{(n+3).(n+4).(n+5).(3n-2)}{12} + 10.$$

To sum $\dfrac{10}{1.2.4} + \dfrac{14}{2.3.5} + \dots \dfrac{6+4n}{n.(n+1).(n+3)}$.

Here $\Delta S = \dfrac{10+4n}{(n+1).(n+2).(n+4)} = \dfrac{4(n+4)-6}{(n+1).(n+2).(n+4)}$

$= \dfrac{4}{(n+1).(n+2)} - \dfrac{6.(n+3)}{(n+1).(n+2).(n+3).(n+4)} = \dfrac{4}{(n+1).(n+2)}$

$- \dfrac{6}{(n+1).(n+2).(n+3)} + \dfrac{6}{(n+1)\dots(n+4)}$

$\therefore S = C - \dfrac{4}{n+1} + \dfrac{3}{(n+1).(n+2)} - \dfrac{2}{(n+1).(n+2).(n+3)}$

$= C - \dfrac{4n^2 + 17n + 17}{(n+1).(n+2).(n+3)} = \dfrac{17}{6} - \dfrac{4n^2 + 17n + 17}{(n+1).(n+2).(n+3)}.$

To sum $\dfrac{1}{1.4} - \dfrac{1}{3.6} + \dfrac{1}{5.8} - \dots \infty$.

By actual division we have

$$dx - x^2dx + x^4dx - \dots \infty = \frac{dx}{1+x^2}$$

$$\therefore x - \frac{x^3}{3} + \frac{x^5}{5} - \dots \infty = \int \frac{dx}{1+x^2}.$$

Multiplying by x^3dx and again integrating, we have

$$\frac{x^4}{1.4} - \frac{x^6}{3.6} + \frac{x^8}{5.8} - \ldots. \infty = \int x^3 dx \int \frac{dx}{1+x^2}$$

$$= \frac{x^3}{3}\int\frac{dx}{1+x^2} - \int\frac{x^3dx}{3.(1+x^2)} = \frac{x^3}{3}.\int\frac{dx}{1+x^2} - \frac{1}{3}.\int xdx$$

$$+ \frac{1}{3}\int\frac{xdx}{1+x^2} = \frac{x^3}{3}.\tan.^{-1}x + \frac{1}{6}\,l.(1+x^2) - \frac{x^2}{6} + \mathrm{C}$$

Let $x = 0$ and 1 successively,

Then the difference of the resulting values of the integral $= \frac{\pi}{12} + \frac{l.2}{6} - \frac{1}{6} = \frac{1}{1.4} - \frac{1}{3.6} + \frac{1}{5.8} - \ldots.$ the value of the series required.

719. To prove $1 + \frac{1}{2} + \frac{1}{3} + \ldots. \infty = \infty$.

We have $dx + xdx + \ldots. \infty = \frac{dx}{1-x}$

$$\therefore x + \frac{x^2}{2} + \frac{x^3}{3} + \ldots.. \infty = \int\frac{dx}{1-x} = -\,l.(1-x)$$

Now, when $x = 0$, $l.(1-x) = l.\,1 = 0$

And when $x = 1$, $l.(1-x) = l.(0) = -\infty$

But $1 + \frac{1}{2} + \frac{1}{3} + \ldots.. \infty = -\,l.(1-x)$ taken between $x = 0$ and 1.

$$\therefore 1 + \frac{1}{2} + \ldots. \infty = -(-\infty) = \infty$$

720. To sum $1 + 3 + 5 \ldots. 1 + 49 \times 2$.

An arithmetic series $=$ sum of first and last terms $\times$ half the number of terms,

$$\therefore \mathrm{S} = (1 + 1 + 98)\frac{50}{2} = 100 \times 25 = 2500.$$

To sum $\frac{1}{1.2} + \frac{1}{2.3} + \ldots. \infty$. See 685.

721. To sum $1 - \frac{1}{3 \,.\, 3} + \frac{1}{5 \,.\, 3^2} - \ldots\ldots \infty$.

By division, we have

$$\frac{dx}{x^{\frac{1}{2}}} - x^{\frac{1}{2}}dx + x^{\frac{3}{2}}dx - \ldots\ldots \propto = \frac{dx}{\sqrt{x}.(1+x)}$$

Then $x^{\frac{1}{2}} - \frac{x^{\frac{3}{2}}}{3} + \frac{x^{\frac{5}{2}}}{5} - \ldots\ldots \infty = \frac{1}{2} . \int \frac{dx}{\sqrt{x}.(1+x)}$

Let $\sqrt{x} = u$,

Then $\int \frac{dx}{\sqrt{x}.(1+x)} = \int \frac{2du}{1+u^2} = 2 \tan.^{-1} \sqrt{x}$

$\therefore\ x^{\frac{1}{2}} - \frac{x^{\frac{3}{2}}}{3} + \ldots\ldots \infty = \tan.^{-1} \sqrt{x} + C = \tan.^{-1} \sqrt{x} - p\pi$

Let $x = \frac{1}{3}$

Then $\frac{1}{\sqrt{3}} - \frac{1}{3 . 3\sqrt{3}} + \frac{1}{5 \,.\, 3^2 \sqrt{3}} - \ldots\ldots \infty =$

$\tan.^{-1} \left(\frac{1}{\sqrt{3}}\right) - p\pi = p\pi + 30° - p\pi = 30°.$

$\therefore\ 1 - \frac{1}{3 \,.\, 3} + \frac{1}{5 . 3^2} - \ldots\ldots \infty = \sqrt{3} \times 30° = \frac{\pi}{2\sqrt{3}}.$

To sum $\frac{1}{1 \,.\, 2 \,.\, 5} - \frac{1}{2 \,.\, 3 \,.\, 7} + \frac{1}{3 \,.\, 4 \,.\, 9} - \ldots\ldots \infty$.

We have $dx - xdx + x^2dx - \ldots\ldots \infty = \frac{dx}{1+x}$

$\therefore\ x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots\ldots \infty = \int \frac{dx}{1+x}$

Multiply by dx and integrate,

Then $\frac{x^2}{2} - \frac{x^3}{2 \,.\, 3} + \frac{x^4}{3 \,.\, 4} - \ldots\ldots \infty = \int dx \int \frac{dx}{1+x}$

Multiplying by $\frac{dx}{\sqrt{x}}$, and again integrating, &c., we have $\frac{x^{\frac{5}{2}}}{2 \,.\, 5}$ $-$

$\frac{x^{\frac{7}{2}}}{2.3.7} + \ldots\ldots \infty = \frac{1}{2}.\int \frac{dx}{\sqrt{x}} \int dx \int \frac{dx}{1+x} = \sqrt{x} \times$
$\int dx \int \frac{dx}{1+x} - \int \sqrt{x}\, dx \int \frac{dx}{1+x} = \sqrt{x} \times x \int \frac{dx}{1+x} - \sqrt{x} \times$
$\int \frac{xdx}{1+x} - \frac{2}{3}.x^{\frac{3}{2}} \int \frac{dx}{1+x} + \frac{2}{3} \int \frac{x^{\frac{3}{2}}dx}{1+x} = \left(\frac{1}{3}.x^{\frac{3}{2}} + \sqrt{x}\right)$
$\times \int \frac{dx}{1+x} - \frac{5}{9}\, x^{\frac{3}{2}} - \frac{2}{3} \int \frac{\sqrt{x}\, dx}{1+x} = \left(\frac{1}{3}.x^{\frac{3}{2}} + \sqrt{x}\right) \times$
$l.(1+x) - \frac{5}{9}\, x^{\frac{3}{2}} - \frac{4}{3}\sqrt{x} + \frac{4}{3}.\tan.^{-1}\sqrt{x}.$

Let the integral be taken between $x = 0$ and 1.

Then $\frac{1}{2.5} - \frac{1}{2.3.7} + \ldots\ldots \infty = \frac{4}{3}.l.2 - \frac{17}{9} + \frac{4}{3}.\frac{\pi}{4}$

$$= \frac{4}{3}.l.2 + \frac{\pi}{3} - \frac{17}{9}.$$

To sum $1 + 2^4 + 3^4 + \ldots\ldots n^4$, we have

$\Delta\, S = (n+1)^4 = (n-1).n.(n+1).(n+2) + 2.n \times (n+1).(n+2) + n.(n+1) + (n+1)$, by the method of indeterminate coefficients.

$$\therefore S = \frac{n.(n^2-1).(n^2-4)}{5} + \frac{n.(n^2-1).(n+2)}{2}$$
$$+ \frac{n.(n^2-1)}{3} + \frac{n.(n+1)}{2} + C = \frac{n^5}{5} + \frac{n^4}{2} + \frac{n^3}{3} - \frac{n}{30},$$

there being no correction.

722. To sum $\frac{1}{3} - \frac{1}{6} + \frac{1}{12} - \ldots\ldots \infty$.

The series being geometrical, and the common ratio $= -\frac{1}{2}$, we have $S = \frac{\frac{1}{3}}{1+\frac{1}{2}} = \frac{2}{9}$.

To sum $\frac{1}{1.2.3} + \frac{1}{2.3.4} + \ldots\ldots$ See 686.

723. To sum $1 . 2 . 5 + 2 . 3 . 6 + \ldots\ldots n . (n+1) . (n+4)$.

$\Delta\ S = (n+1) . (n+2) . (n+5) = (n+1) . (n+2) . (n+3) + 2 (n+1) . (n+2)$

$$\therefore S = \frac{n . (n+1) . (n+2) . (n+3)}{4} + \frac{2}{3} . n . (n+1) \times (n+2) + C$$

$$= \frac{n . (n+1) . (n+2) . (3n+17)}{12}.$$

To sum $\frac{2}{1 . 3 . 4} + \frac{3}{2 . 4 . 5} + \frac{4}{3 . 5 . 6} + \ldots \frac{n+1}{n . (n+2) . (n+3)}$

$$\Delta\ S = \frac{n+2}{(n+1) . (n+3) . (n+4)} = \frac{1}{(n+3) . (n+4)} + \frac{1}{(n+1) . (n+3) . (n+4)}$$

But $\frac{1}{(n+1) . (n+3) . (n+4)} = \frac{n+2}{(n+1) . (n+2) . (n+3) . (n+4)}$

$$= \frac{1}{(n+2) . (n+3) . (n+4)} + \frac{1}{(n+1) \ldots\ldots (n+4)}$$

Hence $\Delta\ S = \frac{1}{(n+3) . (n+4)} + \frac{1}{(n+2) . (n+3) . (n+4)} + \frac{1}{(n+1) \ldots . (n+4)}$

$$\therefore S = C - \frac{1}{(n+3)} - \frac{1}{2 . (n+2) . (n+3)} - \frac{1}{3 . (n+1) . (n+2) . (n+3)} = \frac{17}{36} - \frac{6n^2 + 21n + 17}{6 . (n+1) . (n+2) . (n+3)}.$$

To sum $\frac{1}{1 . 5} - \frac{1}{3 . 7} + \ldots\ldots \infty$.

Assume $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots\ldots \infty = S$

Then $\frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \ldots . = S - 1 + \frac{1}{3}$

And by subtraction $\frac{4}{1 \cdot 5} - \frac{4}{3 \cdot 7} + \frac{4}{5 \cdot 9} - \ldots\ldots \infty = 1 - \frac{1}{3} = \frac{2}{3}$

$$\therefore \frac{1}{1 \cdot 5} - \frac{1}{3 \cdot 7} + \ldots\ldots \infty = \frac{1}{6}.$$

Otherwise,

$$dx - x^2 dx + x^4 dx - \ldots\ldots \infty = \frac{dx}{1+x^2}$$

Integrate, multiply by $x^3 dx$ and again integrate; then

$$\frac{x^5}{1 \cdot 5} - \frac{x^7}{3 \cdot 7} + \ldots\ldots = \int x^3 dx \int \frac{dx}{1+x^2} = \frac{x^4}{4} \int \frac{dx}{1+x^2} - \frac{1}{4} \int \frac{x^4 dx}{1+x^2}$$

Now in taking this integral between $x = 0$ and 1, when x falls without the integral sign, its index being positive, we may put it $= 1$ without affecting the ultimate result.

$$\therefore \frac{x^5}{1 \cdot 5} - \frac{x^7}{3 \cdot 7} + \ldots\ldots = \frac{1}{4} \cdot \int \frac{1-x^4}{1+x^2} dx = \frac{1}{4} \cdot \int (1 - x^2) dx = \frac{x}{4} - \frac{x^3}{12}$$

Let $x = 1$.

Then $\frac{1}{1 \cdot 5} - \frac{1}{3 \cdot 7} + \ldots\ldots \infty = \frac{1}{4} - \frac{1}{12} = \frac{1}{6}$ as before.

724. To sum $\frac{a}{c} + \frac{a+b}{ce} + \frac{a+2b}{ce^2} + \ldots\ldots \infty$.

By division, we have

$$x^{\frac{a}{b}} + x^{\frac{a}{b}+1} + x^{\frac{a}{b}+2} + \ldots\ldots \infty = \frac{x^{\frac{a}{b}}}{1-x}$$

Differentiating and multiplying $\frac{b}{dx}$,

$$a\,x^{\frac{e}{i}-1} + (a+b)\,.\,x^{\frac{e}{i}} + (a+2b)\,x^{\frac{e}{i}+1} + \ldots = \frac{ax^{\frac{e}{i}-1}}{1-x}$$

$$+ \frac{bx^{\frac{e}{i}}}{(1-x)^2}$$

$$\therefore a + (a+b)\,x + (a+2b)\,x^2 + \ldots\ldots = \frac{a}{1-x} + \frac{bx}{(1-x)^2}$$

Let $x = \frac{1}{e}$, and divide by c.

Then $\frac{a}{c} + \frac{a+b}{ce} + \frac{a+2b}{ce^2} + \ldots = \frac{ae}{c\,.\,(e-1)} + \frac{be}{c\,.\,(e-1)^2}$

$= \frac{e}{c} \times \frac{a\,.\,\overline{e-1}+b}{(e-1)^2}$, which may be verified by actual division.

725. To find the scale of relation in the series

Sin. A + sin. (A + B) + sin. (A + 2B) + ...

We have from the form sin. P + sin. Q $= 2$ sin. $\frac{P+Q}{2} \times$ cos. $\frac{P-Q}{2}$,

Sin. A + sin. (A + 2B) = 2 sin. (A + B) . cos B
Sin. (A+B) + sin.(A+3B) = 2 sin. (A+2B) . cos B
&c. = &c.

$\therefore$ Sin. (A+2B) = 2 cos. B $\times$ sin. (A+B) − sin. A
Sin. (A+3B) = 2 cos. B $\times$ sin. (A+2B) − sin. (A+B)
&c. = &c.

Hence the series is recurring, and its scale of relation is

2 cos. B − 1.

The sum of the series will, therefore, be

$$S = \frac{\text{Sin. A} + \text{sin.}\,(A+B) - 2\,\text{cos. B. sin. A}}{2 - 2\,\text{cos. B}}$$

$$= \frac{\text{cos.}\,\left(A - \frac{B}{2}\right)}{2\,\text{sin.}\,\frac{B}{2}}.$$

726. To sum $1 + \frac{1}{3} + \frac{1}{9} + \ldots\ldots \infty$, which is a geometric series, whose common ratio is $\frac{1}{3}$, we have

$$S = \frac{a}{1-r} = \frac{1}{1-\frac{1}{3}} = \frac{3}{2}.$$

To sum $1 - 2 + 4 - 8 + \ldots\ldots\ n$ terms, which is a geometric series whose common ratio is -2, we have

$$S = \frac{ar^n - a}{r-1} = \frac{1 \times (-2)^n - 1}{-2-1} = \frac{1 - (-2)^n}{3} =$$

$\frac{1 \pm 2^n}{3}$ according as n is odd or even.

727. To sum $\frac{1}{1\,.\,4\,.\,7} + \frac{1}{4\,.\,7\,.\,10} + \frac{1}{7\,.\,10\,.\,13}$ $+ \ldots\ldots \infty$.

$$\text{Let } \frac{1}{1\,.\,4\,.\,7} + \ldots\ldots \frac{1}{(3n-2)\,.\,(3n+1)\,.\,(3n+4)} = S$$

$$\text{Then } \Delta\, S = \frac{1}{(3n+1)\,.\,(3n+4)\,.\,3n+7)}$$

$$\therefore S = C - \frac{1}{6 \times (3n+1)\,.\,(3n+4)} = \frac{1}{24} -$$

$\frac{1}{6\,.\,(3n+1)\,(3n+4)}$ which gives the sum of the series to n terms.

Let $n = \infty$.

$$\text{Then } \frac{1}{1\,.\,4\,.\,7} + \frac{1}{4.7.10} + \ldots\ldots \infty = \frac{1}{24}.$$

To sum $2^2 + 5^2 + \ldots\ldots (3n-1)^2$.

$$\Delta\, S = (3n+2)^2 = (3n+2)\,.\,(3n+5-3)$$
$$= (3n+2)\,.\,(3n+5) - 3 \times (3n+2)$$

$$\therefore S = \frac{(3n-1).(3n+2).(3n+5)}{3\times 3} - \frac{3\times(3n-1).(3n+2)}{3\times 2} + C$$

$$= \frac{(3n-1).(3n+2).(6n+1)}{18} + \frac{1}{9}.$$

To sum $\frac{1}{1\,.\,2} + \frac{1}{5\,.\,6} + \frac{1}{9.10} + \ldots\ldots \infty.$

By division $dx + x^4dx + x^8dx + \ldots\ldots \infty = \frac{dx}{1-x^4}$

$$\therefore x + \frac{x^5}{5} + \frac{x^9}{9} + \ldots\ldots \infty = \int\frac{dx}{1-x^4}$$

Multiply by dx and again integrate, then

$$\frac{x^2}{1\,.\,2} + \frac{x^6}{5\,.\,6} + \frac{x^{10}}{9.10} + \ldots\ldots \infty = \int dx \int\frac{dx}{1-x^4} =$$

$x\int\frac{dx}{1-x^4} - \int\frac{xdx}{1-x^4}$. But $\frac{1}{1-x^4} = \frac{1}{2}\times\left(\frac{1}{1-x^2}\right.$
$\left.+ \frac{1}{1+x^2}\right)$

$\therefore \int\frac{dx}{1-x^4} = \frac{1}{2}\cdot\int\frac{dx}{1-x^2} + \frac{1}{2}\int\frac{dx}{1+x^2} = \frac{1}{4}\times$
$l.\ \frac{1+x}{1-x} + \frac{1}{2}\,.\,\tan.^{-1}x$, and $\int\frac{xdx}{1-x^4} = \frac{1}{2}\cdot\int\frac{xdx}{1-x^2} +$
$\frac{1}{2}\int\frac{xdx}{1+x^2} = \frac{1}{4}\,.\,l.\ \frac{1+x^2}{1-x^2}.$

$\therefore \frac{x^2}{1\,.\,2} + \frac{x^6}{5\,.\,6} + \ldots\ldots \infty = \frac{x}{4}\,.\,l.\,\frac{1+x}{1-x} - \frac{1}{4}\,.\,l.\ \frac{1+x^2}{1-x^2}$
$+ \frac{1}{2}\,.\,\tan.^{-1}x.$

Let the integral be taken between $x = 0$ and 1.

Then $\frac{1}{1\,.\,2} + \frac{1}{5\,.\,6} + \ldots\ldots \infty = \frac{1}{4}\,.\,l.\ \frac{2}{0} - \frac{1}{4}\,.\,l.\ \frac{2}{0} +$
$\frac{1}{2}\,.\,45^\circ = \frac{\pi}{8}.$

To sum $\frac{2}{1.3.3}+\frac{3}{3.5.3^2}+\frac{4}{5.7.3^3}+\ldots\frac{n+1}{(2n-1).(2n+1).3^n}$.

In this series we will exemplify Demoivre's method.

Let $1+\frac{x}{3}+\frac{x^2}{5}+\ldots\frac{x^{n-1}}{2n-1}+\frac{x^n}{2n+1}=S$. Multiplying $ax-b$, and reducing to a common denominator, &c., we get $\frac{3a-b}{3}x+\frac{5a-3b}{3.5}.x^2+\ldots\frac{(2n+1)a-(2n-1)b}{(2n-1).(2n+1)}x^n=$ $(ax-b).S+b-\frac{ax^{n+1}}{2n+1}$

Put $3a-b=2$

And $2a-2b=1$, the common difference of the numerators.

Hence $a=\frac{3}{4}$ and $b=\frac{1}{4}$.

$\therefore\ \frac{2x}{1.3}+\frac{3x^2}{3.5}+\ldots\frac{n+1}{(2n-1).(2n+1)}x^n=\frac{3x-1}{4}.S$ $+\frac{1}{4}-\frac{3x^{n+1}}{4.(2n+1)}$

Let $n=\frac{1}{3}$. Then $\frac{3x-1}{4}=0$,

And $\frac{2}{1.3.3}+\frac{3}{3.5.3^2}+\ldots\frac{n+1}{(2n-1).(2n+1).3^n}=\frac{1}{4}-$ $\frac{1}{4.(2n+1).3^n}$.

728. To sum $3+\frac{1}{2}+\frac{1}{12}+\ldots\ldots\infty$, which is a geometric series, whose common difference is $\frac{1}{6}$, we have

$$S=\frac{a}{1-r}=\frac{3}{1-\frac{1}{6}}=\frac{18}{5}=3\,\frac{3}{5}.$$

To sum $1.3.5+3.5.7+\ldots\ldots(2n-1)(2n+1).(2n+3)$.

$\Delta\ S=(2n+1).(2n+3).(2n+5)$

$$\therefore S = \frac{(2n-1).(2n+1).(2n+3).(2n+5)}{8} + C$$

$$= \frac{(4n^2-1).(2n+3).(2n+5)}{8} + \frac{15}{8}.$$

729. To sum $\frac{1}{5} - \frac{2}{15} + \frac{4}{45} - \ldots$ to n terms, which is a geometric series, whose common ratio is $\left(-\frac{2}{3}\right)$, we have

$$S = \frac{ar^n - a}{r-1} = \frac{\frac{1}{5} \times \left(-\frac{2}{3}\right)^n - \frac{1}{5}}{-\frac{2}{3} - 1} = \frac{3^n - (-2)^n}{25 \times 3^{n-1}}$$

$= \frac{3^n \pm 2^n}{25 \times 3^{n-1}}$ according as n is odd or even.

730. To sum $\frac{1}{2.4.6} + \frac{1}{4.6.8} + \ldots \infty$.

Let $S = \frac{1}{2.4.6} + \frac{1}{4.6.8} + \ldots \frac{1}{2n \times 2.(n+1).2.(n+2)}$

Then $\Delta S = \frac{1}{8.(n+1).(n+2).(n+3)}$

$$\therefore S = C - \frac{1}{16.(n+1)(n+2)} = \frac{1}{32} - \frac{1}{16.(n+1)(n+2)}.$$

To sum $\frac{1}{1.3} - \frac{1}{2.4} + \frac{1}{3.5} - \ldots \infty$. See 702.

731. To sum $\frac{1}{m.(m+r)} + \frac{1}{(m+r).(m+2r)} + \ldots \infty$.

Let $\frac{1}{m} + \frac{1}{m+r} + \frac{1}{m+2r} + \ldots \infty = S$

Then $\frac{1}{m+r} + \frac{1}{m+2r} + \frac{1}{m+3r} + \ldots \infty = S - \frac{1}{m}$

$\therefore$ by subtraction,

$$\frac{r}{m.(m+r)} + \frac{r}{(m+r).(m+2r)} + \ldots\ldots \infty = \frac{1}{m}$$

$$\therefore \frac{1}{m.(m+r)} + \frac{1}{(m+r).(m+2r)} + \ldots\ldots \infty = \frac{1}{rm}.$$

Otherwise.

$$\text{Let } \frac{1}{m.(m+r)} + \frac{1}{(m+r).(m+2r)} + \ldots \frac{1}{(m+\overline{n-1}.r)\,(m+nr)} = S$$

$$\text{Then } \Delta S = \frac{1}{(m+nr).(m+\overline{n+1}r.)}.$$

$\therefore S = C - \dfrac{1}{r.(m+nr)} = \dfrac{1}{mr} - \dfrac{1}{r.(m+nr)}$ the sum to n terms.

Let $n = \infty$.

Then $S = \dfrac{1}{mr}$, as before.

To sum $\dfrac{5}{1.2.3} + \dfrac{7}{2.3.4} + \ldots\ldots \dfrac{2n+3}{n.(n+1).(n+2)}$.

$$\Delta S = \frac{2n+5}{(n+1).(n+2).(n+3)} = \frac{2}{(n+1).(n+2)}$$

$$- \frac{1}{(n+1).(n+2).(n+3)}.$$

$$C - \frac{2}{n+1} + \frac{1}{2.(n+1).(n+2)} = \frac{7}{4} - \frac{4n+7}{2(n+1).(n+2)}.$$

To sum $1 + 2.3 + 3.3^2 + 4.3^3 + \ldots\ldots n.3^{n-1} = S$.

$$\Delta S = (n+1).\,3^n$$

$\therefore S = \Sigma\,(n+1)3^n = \dfrac{(n+1)3^n}{3-1} - \dfrac{3^{n+1}}{(3-1)^2} + C$, from the form $\Sigma.ua^x = u\,\dfrac{a^x}{a-1} - \Delta u\,\dfrac{a^{x+1}}{(a-1)^2} + \Delta^2 u.\dfrac{a^{x+2}}{(a-1)^3} -$ &c. See Appendix to Translation of *Lacroix*.

$$\text{Hence } S = \frac{(n+1)3^n}{2} - \frac{3^{n+1}}{4} + \frac{1}{4}$$

$$= \frac{3^n.(2n-1)}{4} + \frac{1}{4}.$$

732. Given $1 + \frac{1}{2^2} + \frac{1}{3^2} + \dots \infty = \frac{\pi^2}{6}$ (see 684) to find $\frac{1}{1.\,2^2} + \frac{1}{2^2.\,3^2} + \dots \infty$.

Put $\frac{1}{1^2.\,2^2} + \frac{1}{2^2.\,3^2} + \dots \frac{1}{n^2.(n+1)^2} = S$

Now $\frac{1}{n^2.(n+1)^2} = \frac{2}{n+1} - \frac{2}{n} + \frac{1}{n^2} + \frac{1}{(n+1)^2}$ which splits the series into four others whose n^{th} terms are $\frac{2}{n+1}$, $\frac{2}{n}$, $\frac{1}{n^2}$ and $\frac{1}{(n+1)^2}$ respectively.

Hence then

$$\frac{1}{1.2^2} + \frac{1}{2^2 3^2} + \dots\infty = \begin{cases} \frac{2}{2} + \frac{2}{3} + \dots \infty - \frac{2}{1} - \frac{2}{2} - \frac{2}{3} - \dots\infty \\ \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots \infty \\ \frac{1}{2^2} + \frac{1}{3^2} + \dots \infty \end{cases}$$

$$= \left\{\begin{matrix} -2 \\ +\frac{\pi^2}{6} \\ +\frac{\pi^2}{6} - 1 \end{matrix}\right\} = \frac{\pi^2}{3} - 3$$ which will be found to accord with the result obtainable by the method of Definite Integrals.

733. By the common rule for arithmetic series we have

$$S = 1 + 3 + \dots 2n - 1 = (1 + 2n - 1)\frac{n}{2} = n^2$$

And $S' = 2 + 4 + \dots 2n - 2 = (2 + 2n - 2)\frac{n-1}{2} = n.(n-1)$

$$\therefore S : S' :: n^2 : n.(n-1) :: n : n-1.$$

734. To sum $\frac{3}{1.2^2} + \frac{5}{2^2.3^2} + \frac{7}{3^2.4^2} + \dots \frac{2n+1}{n^2.(n+1)^2}$

$$\Delta\, S = \frac{2n+3}{(n+1)^2.(n+2)^2} = \frac{1}{(n+1)^2} - \frac{1}{(n+2)^2} = -\Delta \,.\, \frac{1}{(n+1)^2}$$

$$\therefore S = C - \frac{1}{(n+1)^2} = 1 - \frac{1}{(n+1)^2}.$$

Let $n = \infty$

Then $S = 1 - \frac{1}{\infty} = 1.$

Otherwise.

$$\left.\begin{aligned} &\text{Let } 1 + \frac{1}{2^2} + \frac{1}{3^2} + \dots \frac{1}{n^2} = S \\ &\text{Then } \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \dots \frac{1}{(n+1)^2} = S - 1 + \frac{1}{(n+1)^2} \end{aligned}\right\} \text{Subtracting}$$

the second from the first we get

$$\frac{3}{1.2^2} + \frac{5}{2^2.3^2} + \dots\dots \frac{2n+1}{n^2.(n+1)^2} = 1 - \frac{1}{(n+1)^2}, \text{ as before.}$$

To sum $\frac{1}{1.2.3.4.5} + \frac{2^2}{4.5.6.7.8} + \frac{3^2}{7.8.9.10.11}$

$$+ \dots\dots \frac{n^2}{(3n-2).(3n-1)3n.(3n+1).(3n+2)}$$

$$\Delta\, S = \frac{(n+1)^2}{(3n+1).(3n+2).(3n+3).(3n+4).(3n+5)} =$$

$$\frac{1}{3} \cdot \frac{n+1}{(3n+1).(3n+2) \times (3n+4).(3n+5)}$$

Now since $\frac{1}{(3n+4).(3n+5)}$ is the next successive value of $\frac{1}{(3n+1).(3n+2)}$ it is probable that Δ . S is of the form $\Delta . \frac{N}{(3n+1).(3n+2)}$.

To ascertain this, let $\frac{1}{(3n+1).(3n+2).(3n+4).(3n+5)}$ $= \frac{An+B}{(3n+1).(3n+2)} + \frac{an+b}{(3n+4).(3n+5)}$ reduce the fractions to a common denominator, and equate coefficients of like powers of n in the numerators. Thence we find

$$\Delta\, S = \frac{n+1}{3} \times \frac{1}{4} \times \left\{\frac{n+2}{(3n+4).(3n+5)} - \frac{n}{(3n+1).(3n+2)}\right\}$$

$$= \frac{1}{12} \times \left\{\frac{(n+1).(n+2)}{(3n+4).(3n+5)} - \frac{n.(n+1)}{(3n+1).(3n+2)}\right\}$$

$$= \frac{1}{12} \times \Delta\ \frac{n.(n+1)}{(3n+1).(3n+2)}.$$

$$\therefore S = C + \frac{1}{12}.\frac{n.(n+1)}{(3n+1).(3n+2)} = \frac{n.(n+1)}{12.(3n+1).(3n+2)}.$$

Let $n = \infty$

Then $S = \frac{\infty \times \infty}{108 \times \infty \times \infty} = \frac{1}{108}$.

Otherwise.

The series may be reduced to this form

$$\frac{1}{1.2 \times 4.5} + \frac{2}{4.5 \times 7.8} + \ldots \frac{n}{(3n-2).(3n-1) \times (3n+1).(3n+2)} = 3.S.$$

Assume $\frac{1}{1.2} + \frac{1}{4.5} + \ldots\ldots \frac{1}{(3n-2).(3n-1)} = s$

Then $\frac{1}{4.5} + \frac{1}{7.8} + \ldots \frac{1}{(3n+1).(3n+2)} = s$

$+ \frac{1}{(3n+1).(3n+2)} - \frac{1}{2}$

$\therefore$ by subtraction we get

$$\frac{18}{1.2\times4.5}+\frac{2\times18}{4.5\times7.8}+\ldots\frac{n\times18}{(3n-2).(3n-1)\times(3n+1).(3n+2)}$$

$$=\frac{1}{2}-\frac{1}{(3n+1).(3n+2)}=\frac{9n^2+9n}{2.(3n+1).(3n+2)}.$$ Hence

$$3S=\frac{n^2+n}{4.(3n+1).(3n+2)}$$

$$\therefore S=\frac{n.(n+1)}{12.(3n+1).(3n+2)}$$ as before.

735. To sum $\frac{5}{1.4}-\frac{7}{2.5}+\frac{9}{3.6}-\ldots\pm\frac{2n+3}{n.(n+3)}$

Let $\frac{1}{1}-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\ldots\pm\frac{1}{n}=s$

Then $-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\frac{1}{7}-\frac{1}{8}+\ldots\mp\frac{1}{n+3}=s$

$$-1+\frac{1}{2}-\frac{1}{3}\mp\frac{1}{n+1}\pm\frac{1}{n+2}\mp\frac{1}{n+3}=s-\frac{5}{6}$$

$$\mp\frac{n^2+4n+5}{(n+1).(n+2).(n+3)}.$$ Hence by subtraction, we have

$$S=\frac{5}{1.4}-\frac{7}{2.5}+\ldots\ldots\pm\frac{2n+3}{n.(n+3)}=\frac{5}{6}$$

$$\pm\frac{n^2+4n+5}{(n+1).(n+2).(n+3)}$$ according as n is odd or even.

Let $n=\infty$

Then $S=\frac{5}{6}\pm\frac{\infty^2}{\infty^3}=\frac{5}{6}.$

To sum $\frac{1}{1.3.4}+\frac{4}{2.4.5}+\ldots\frac{3n-2}{n.(n+2).(n+3)}.$

$$\Delta S=\frac{3n+1}{(n+1).(n+3).(n+4)}=\frac{(3n+1).(n+2)}{(n+1).(n+2).(n+3).(n+4)}$$

$$=\frac{3(n+1).(n+2)-2(n+1)-2}{(n+1)\ldots.(n+4)}$$

$$= \frac{3}{(n+3).(n+4)} - \frac{2}{(n+2).(n+3).(n+4)} - \frac{2}{(n+1)..(n+4)}.$$

$$\therefore S = C - \frac{3}{n+3} + \frac{1}{(n+2).(n+3)} + \frac{2}{3.(n+1).(n+2).(n+3)}$$

$$= C - \frac{9n^2+24n+13}{3(n+1).(n+2).(n+3)} = \frac{13}{18} - \frac{9n^2+24n+13}{3.(n+1).(n+2).(n+3)}.$$

Let $n = \infty$

Then $S = \frac{13}{18} - \frac{9\infty^2}{3\infty^3} = \frac{13}{18}$.

To sum $1 - \frac{1}{3} + \frac{1.2}{3.4} - \frac{1.2.3}{3.4.5} + \ldots. \infty$

Let $s = \frac{x^3}{3} + \frac{2}{3.4}x^4 + \frac{2.3}{3.4.5}x^5 + \ldots. \infty$

Then $\frac{ds}{x^2} = dx + \frac{2}{3}xdx + \frac{2.3}{3.4}x^2dx + \ldots. \infty$

$$\therefore \int\frac{ds}{x^2} = x + \frac{x^2}{3} + \frac{2}{3.4}x^3 + \ldots.$$

$$= x + \frac{1}{x}.s$$

$$\therefore \frac{ds}{x^2} = dx + \frac{ds}{x} - \frac{sdx}{x^2}, \text{ or}$$

$$ds.(1-x) = x^2dx - sdx$$

$\therefore ds + \frac{dx}{1-x}s = \frac{x^2dx}{1-x}$, which being a Linear Equation of the first order and degree, assume

$s = Pz$ (P being a function of x at present undetermined)

Then $zdP + Pdz + \frac{dx}{1-x}.\ Pz = \frac{x^2dx}{1-x}$

Put $Pdz + \frac{dx}{1-x}.\ Pz = 0$. Then

$\frac{dz}{z} = -\frac{dx}{1-x}$, and $dP = \frac{x^2dx}{(1-x).z}$;

$\therefore lz = l.(1-x)$, or $z = 1 - x$

$$\text{Hence } dP = \frac{x^2dx}{(1-x)^2} = dx + \frac{2xdx - 2dx}{x^2-2x+1} + \frac{dx}{(1-x)^2}$$

$$\therefore\ P = x + l.\,(x^2 - 2x + 1) - \frac{1}{1-x}$$

Hence $s = Pz = x \times (1-x) + (1-x).\,l.\,(1-x)^2 - 1$, which being taken between $x = 0$ and (-1), gives

$$-\frac{1}{3} + \frac{2}{3.4} - \frac{2.3}{3.4.5} + \ldots.. \infty = 4\,l.2 - 2$$

$$\therefore 1 - \frac{1}{3} + \frac{1.2}{3.4} - \frac{1.2.3}{3.4.5} + \ldots. \infty = 4\,l.2 - 1.$$

736. To sum $2 + 2\frac{1}{3} + 2\frac{2}{3} + 3 + 3\frac{1}{3} + \ldots$ 13 terms which is an arithmetic series, whose common difference is $\frac{1}{3}$, we have

$$S = (2a + \overline{n-1}\,.\,b)\,\frac{n}{2} = \left(4 + 12 \times \frac{1}{3}\right).\frac{13}{2} = 8 \times \frac{13}{2} = 52.$$

To sum $\frac{1}{1.3} + \frac{1}{3.5} + \frac{1}{5.7} + \ldots. \frac{1}{(2n-1).(2n+1)}$.

$$\left.\begin{aligned} &\text{Assume } \frac{1}{1} + \frac{1}{3} + \ldots. \frac{1}{2n-1} = s \\ &\text{Then } \frac{1}{3} + \frac{1}{5} + \ldots. \frac{1}{2n+1} = s - 1 + \frac{1}{2n+1} \end{aligned}\right\}\ \therefore \text{ by sub-}$$

traction,

$$\frac{2}{1.3} + \frac{2}{3.5} + \ldots. \frac{2}{(2n-1).(2n+1)} = 1 - \frac{1}{2n+1} = \frac{2n}{2n+1}$$

$$\therefore \frac{1}{1.3} + \frac{1}{3.5} + \ldots. \frac{1}{(2n-1).(2n+1)} = \frac{n}{2n+1}.$$

Otherwise.

$$\Delta\ s = \frac{1}{(2n+1).\,(2n+3)}$$

$$\therefore s = C - \frac{1}{2.(2n+1)} = \frac{n}{2n+1}, \text{ as before.}$$

737. Figurate numbers of the first order arise from taking the successive sums of the natural numbers; of the second order, from the successive sums of the first order, and so on. Thus,

Order.	Figurates.	General Terms.
Natural Series.	1, 2, 3, 4, 5	n
1st Order	1, 3, 6, 10, 15	$n \cdot \frac{n+1}{2}$
2d Order	1, 4, 10, 20, 35	$\frac{n.(n+1).(n+2)}{2.3}$
&c.	&c.	&c.

The general or n^{th} term of one series is evidently obtainable by integrating the $(n+1)^{th}$ term of the preceding series.

Now the n^{th} terms of the p^{th}, $(p+1)^{th}$, $(p+2)^{th}$ orders being

$$\frac{n.(n+1)\ldots.(n+p)}{1.2\ldots.p+1}, \frac{n.(n+1)\ldots.(n+p+1)}{1.2\ldots.p+2}, \frac{n.(n+1)\ldots.(n+p+2)}{1.2\ldots.p+3}$$

respectively, we shall have the general or n^{th} terms of the series formed by dividing the figurates of the p^{th} order by the corresponding ones of the $(p+1)^{th}$ and $(p+2)^{th}$ orders respectively; viz.,

$\frac{p+2}{n+p+1}$ and $\frac{(p+2).(p+3)}{(n+p+1).(n+p+2)}$: and it remains for us to

shew that the series $(p+2) \times \left\{\frac{1}{p+2} + \frac{1}{p+3} + \ldots. \infty\right\}$ is infinite,

and that $(p+2).(p+3).\left\{\frac{1}{(p+2).(p+3)} + \frac{1}{(p+3).(p+4)} + \ldots. \infty\right\}$ is finite.

By division,

$$x^{p+1}dx + x^{p+2}dx + \ldots. \infty = \frac{x^{p+1}dx}{1-x}$$

$$\therefore \frac{x^{p+2}}{p+2} + \frac{x^{p+3}}{p+3} + \ldots. \infty = \int \frac{x^{p+1}dx}{1-x}$$

$$= \int \left\{x^p dx + x^{p-1}dx + x^{p-2}dx + \ldots. dx + \frac{dx}{1-x}\right\}$$

$= \frac{x^{p+1}}{p+1} + \frac{x^p}{p} + \frac{x^{p-1}}{p-1} + \ldots\ldots x - l.(1-x)$, which being taken between the limits of $x = 0$ and 1, we have

$$\frac{1}{p+2} + \frac{1}{p+3} + \ldots\ldots \infty = \frac{1}{p+1} + \frac{1}{p} + \frac{1}{p-1} + \ldots\ldots 1 + \infty$$

$$= \infty$$

$$\therefore (p+2).\left\{\frac{1}{p+2} + \frac{1}{p+3} + \ldots\ldots \infty\right\} = \infty.$$

$$\left.\begin{aligned} \text{Again, let } \frac{1}{p+2} + \frac{1}{p+3} + \ldots\ldots \infty &= s \\ \text{Then } \frac{1}{p+3} + \frac{1}{p+4} + \ldots\ldots \infty &= s - \frac{1}{p+2} \end{aligned}\right\}$$

$\therefore$ by subtraction

$$\frac{1}{(p+2).(p+3)} + \frac{1}{(p+3).(p+4)} + \ldots\ldots \infty = \frac{1}{p+2}$$

$$\therefore (p+2).(p+3) \times \left\{\frac{1}{(p+2).(p+3)} + \frac{1}{(p+3).(p+4)} + \ldots\ldots \infty\right\}$$

$= p + 3$, a finite quantity, when p is finite.

738. To sum $\frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \ldots\ldots \infty$.

The factors of sin. x being x, $x \pm \pi$, $x \pm 2\pi$, &c. (because sin. $x = 0$ is satisfied by substituting for x the values 0, $\pm \pi$, $\pm 2\pi$, &c.) we have

$$\text{sin. } x = x.(x^2 - \pi)(x^2 - 2^2\pi^2) \ldots\ldots \infty$$

$$= x - \frac{x^3}{2.3} + \frac{x^5}{2.3.4.5}. \quad \text{Hence}$$

$1 - \frac{x^2}{2.3} + \frac{x^4}{2.3.4.5} - \ldots \infty = 0$, is an equation whose roots are $\pm \pi$, $\pm 2\pi$, &c.

Put $y = \frac{1}{x^2}$ and multiply by y^∞.

Then $y^{\infty} - \frac{y^{\infty-1}}{2.3} + \frac{y^{\infty-2}}{2.3.4.5} - \ldots = 0$ is an equation whose roots are $\frac{1}{\pi^2}, \frac{1}{2^2\pi^2}, \frac{1}{3^2\pi^2}$, &c.

Now, if p, q, r, &c. be the coefficients of 2^d, 3^d, &c. terms of an equation, and a, b, c, &c. its roots,

$$p = a + b + c + \ldots$$

$$p^2 - 2q = a^2 + b^2 + c^2 + \ldots$$

$$p^3 - 3qp + 3r = a^3 + b^3 + 3^3 + \ldots$$ See *Waring's* Theorem.

$$\&c. = \&c.$$

$$\therefore \frac{1}{\pi^2} + \frac{1}{2^2\pi^2} + \frac{1}{3^2\pi^2} + \ldots = p = \frac{1}{2.3}$$

$$\therefore \frac{1}{1} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots = \frac{\pi^2}{6}$$

$$\text{Also } \frac{1}{1} + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \frac{1}{5^4} + \ldots \infty = (p^2 - 2q)\,\pi^4$$

$= \left(\frac{1}{36} - \frac{1}{60}\right)\pi^4 = \frac{\pi^4}{90}$, and similarly the sums of the other even powers of $\frac{1}{1}, \frac{1}{2}, \frac{1}{3}$, &c. may be found. We have found $\therefore$

$$\left.\begin{aligned} &1 + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \frac{1}{5^4} + \frac{1}{6^4} + \ldots = \frac{\pi^4}{90} \\ &\therefore \frac{1}{2^4} \qquad + \frac{1}{4^4} \qquad + \frac{1}{6^4} + \ldots = \frac{\pi^4}{16 \times 90} \end{aligned}\right\} \therefore \text{ by subtraction}$$

$$\frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \ldots \infty = \frac{\pi^4}{90} - \frac{\pi^4}{16 \times 90} = \frac{\pi^4}{96}.$$

739. Since we have,

$$S_1 = a + (a + d) + (a + 2d) + \ldots l$$

$$S_2 = a^2 + (a + d)^2 + (a + 2d)^2 + \ldots l^2$$

$$\&c. = \&c.$$

$$S_{m-1} = a^{m-1} + (a + d)^{m-1} + (a + 2d)^{m-1} + \ldots l^{m-1}$$

$$\Delta.S_1 = l+d,\ \Delta.S_2 = (l+d)^2,\ \Delta.S_3 = (l+d)^3, \&c. = \&c.$$

And $\Delta . S_{m-2} = (l+d)^{m-2}$, $\Delta . S_{m-1} = (l+d)^{m-1}$.

Now $\Delta\ (l+d)^m = (l+d+d)^m - (l+d)^m = md\ (l+d)^{m-1}$

$+ m.\dfrac{m-1}{2} d^2\ (l+d)^{m-2} + m\ .\ \dfrac{m-1}{2}.\dfrac{m-2}{3} d^3.(l+d)_{m-3} + \ldots\ldots$

to m terms $= md \times \Delta . S_{m-1} + m.\dfrac{m-1}{2} d^2 \times \Delta . S_{m-2} + m.\dfrac{m-1}{2} \times$

$\dfrac{m-2}{3} d^3 \times \Delta . S_{m-3} + \ldots .$, and integrating, we have,

$(l+d)^m = md \times S_{m-1} + m.\dfrac{m-1}{2} d^2 \times S_{m-2} + m.\dfrac{m-1}{2} d^3 \times S_{m-3}$
$+$ &c. $+ C$

In order to determine C, we will take one term of the series only.

Then $l+d = a+d$, $S_{m-1} = a^{m-1}$, $S_{m-2} = a^{m-2}$, &c.

$\therefore (a+d)^m = mda^{m-1} + m.\dfrac{m-1}{2} d^2 a^{m-2} +$ &c. to m terms $+ C$

$$= (a+d)^m - a^m + C.$$

$$\therefore C = a^m$$

Hence we finally obtain

$(l+d)^m = a^m + md \times S_{m-1} + \dfrac{m.\overline{m-1}}{2} d^2 \times S_{m-2} +$

$m.\dfrac{m-1}{2}.\dfrac{m-2}{3} d^3 \times S_{m-3} +$ &c. $(m+1)$ terms.

This Theorem will enable us to sum all series of the form

$a^p + (a+d)^p + (a+2d)^p + \ldots .\ (a + \overline{n-1}.d)^p$

Ex. $a + (a+d) + \ldots a + (n-1)d = \dfrac{(l+d)^2 - a^2 - nd^2}{2d}$

$= (2a + \overline{n-1}\,.\,d)\,\dfrac{n}{2}$.

It will also very readily give us the relation between the successive sums of the powers of the roots of an equation, when the roots are in arithmetic progression.

740. To sum cos. A + cos. 2 A + cos. 3 A + ... cos. nA = S.

Multiply both sides by $2 \sin. \frac{A}{2}$. Then by the form

$2 \cos. \frac{P+Q}{2} . \sin. \frac{P-Q}{2} = \sin. (P+Q) - \sin. (P-Q)$, we have

$$2 \sin. \frac{A}{2} . \cos. A = \sin. \frac{3A}{2} - \sin. \frac{A}{2}$$

$$2 \sin. \frac{A}{2} . \cos. 2A = \sin. \frac{5A}{2} - \sin. \frac{3A}{2}$$

$$2 \sin. \frac{A}{2} . \cos. 3A = \sin. \frac{7A}{2} - \sin. \frac{5A}{2}$$

$$\&c. = \&c.$$

$$2 \sin. \frac{A}{2} . \cos. n A = \sin. \frac{2n+1}{2} . A - \sin. \frac{2n-1}{2} . A$$

And by addition

$$2 \sin. \frac{A}{2} . S = \sin. \frac{2n+1}{2} A - \sin. \frac{A}{2} = 2 \sin. \frac{n}{2} A . \sin. \frac{n+1}{2} A$$

$$\therefore S = \sin. \frac{(n+1)A}{2} \times \frac{\sin. \frac{nA}{2}}{\sin. \frac{A}{2}}$$

Let $nA = 2\pi$

Then $\sin. \frac{(n+1)A}{2} = \sin. \frac{2\pi + \frac{2\pi}{n}}{2} = \sin. \left(\pi + \frac{\pi}{n}\right) = -\sin. \frac{\pi}{n}$

Also $\sin. \frac{nA}{2} = \sin. \pi = 0$

$\therefore$ in this case, $S = 0$,

Or $\cos. \frac{2\pi}{n} + \cos. \frac{4\pi}{n} + \cos. \frac{6\pi}{n} + \ldots \cos. 2\pi = 0$

To sum $\frac{1}{2} - \frac{2}{3} + \frac{3}{4} - \frac{4}{5} + \ldots \infty$.

By division, $x - x^2 + x^3 - \dots\dots \infty = \frac{x}{1+x}$. Differentiate, multiply by x, and integrate; the result will be

$$\frac{x^2}{2} - \frac{2x^3}{3} + \frac{3x^4}{4} - \dots\dots \infty = \int\frac{xdx}{(1+x)^2} = \int\frac{dx}{1+x}$$

$$- \int\frac{dx}{(1+x)^2} = l.(1+x) + \frac{1}{1+x} + C = l.(1+x) + \frac{1}{1+x} - 1$$

Let $x = 1$

Then $\frac{1}{2} - \frac{2}{3} + \frac{3}{4} - \dots\dots \infty = l.2 - \frac{1}{2}$.

741. To sum $2^2 + 4^2 + 6^2 + \dots\dots (2n)^2$.

$$\Delta\, S = 2^2.(n+1)^2 = 4n.(n+1) + 4(n+1)$$

$$\therefore S = \frac{4.(n-1).n(n+1)}{3} + 2.n.(n+1) + C$$

$$= \frac{2}{3}.\, n.\,(n+1)\,(2n+1).$$

Again, let $1^3 + 2^3 + \dots\dots n^3 = s$.

Then $\Delta\, s = (n+1)^3 = n.(n+1).(n+2) + n + 1$

$$\therefore s = \frac{(n-1)n\,.(n+1)\,.(n+2)}{4} + \frac{n.(n+1)}{2} + C$$

$$= \frac{n^2.(n+1)^2}{4} = \left(n.\frac{n+1}{2}\right)^2$$

But $1 + 2 + 3 + \dots\dots n = (1+n).\frac{n}{2}$

$$\therefore 1^3 + 2^3 + \dots\dots n^3 = \left(n.\frac{n+1}{2}\right)^2 = (1+2+3+\dots\dots n)^2.$$

742. The n^{th} terms of the progressions are

$1 + \overline{n-1}$, $1 + \overline{n-1}.\,2$, $1 + \overline{n-1}.\,3$, ... $1 + \overline{n-1}.\,p$ respectively.

$$\therefore S = (1 + \overline{n-1}) + (1 + \overline{n-1.2}) + \ldots (1 + \overline{n-1.p})$$

$$= p + (n-1).(1+2+3+\ldots p) = p + (n-1).\frac{p.(p+1)}{2}$$

$$= \frac{(n-1).p^2 + (n+1)p}{2}.$$

743. To sum $\frac{5}{1.2.1.3} + \frac{9}{2.3.3.5} + \frac{13}{3.4.5.7} + \ldots\ldots$

$\frac{4n+1}{n.(n+1)(2n-1)(2n+1)}$.

Assume $\frac{1}{1.1} + \frac{1}{2.3} + \frac{1}{3.5} + \ldots \frac{1}{n.(2n-1)} = s$

Then $\frac{1}{2.3} + \frac{1}{3.5} + \frac{1}{4.7} + \ldots \frac{1}{(n+1).(2n+1)} = s - 1$

$+ \frac{1}{(n+1).(2n+1)}$

By subtraction and reducing to a common denominator, we have

$$\frac{5}{1.2.1.3} + \frac{9}{2.3.3.5} + \frac{13}{3.4.5.7} + \ldots\ldots \frac{4n+1}{n.(n+1).(2n-1).(2n+1)}$$

$$= 1 - \frac{1}{(n+1).(2n+1)}.$$

Let $n = \infty$

Then $\frac{5}{1.2.1.3} + \frac{9}{2.3.3.5} + \ldots\ldots \infty = 1.$

The more scientific and infallible method of summing series of the above form is that of Definite Integrals so frequently exemplified in the progress of this subject. The applicability of *Bernoulli's* method of subtraction in this instance as well as in those of No. 734, depending upon the difference of any two successive denominators in the assumed series being some multiple of the corresponding numerator in the given series, must be very limited.

To sum $\frac{1}{1 . 3^2} + \frac{2}{3 . 5^2} + \frac{3}{5 . 7^2} + \ldots \infty. = S.$

The general term $\frac{n}{(2n-1) . (2n+1)^2}$ may be decomposed by the usual methods into the Partial Fractions,

$$\frac{1}{2.(2n-1).(2n+1)} + \frac{1}{4.(2n+1)^2} + \frac{1}{8.(2n+1)} - \frac{1}{8.(2n-1)}$$

$\therefore\ S = \frac{1}{2} . \left\{\frac{1}{1 . 3} + \frac{1}{3 . 5} + \ldots \infty\right\} + \frac{1}{4} . \left\{\frac{1}{3^2} + \frac{1}{5^2}\right.$
$\left. + \ldots \infty\right\} + \frac{1}{8} \times \left\{\frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \ldots \infty - \frac{1}{1} - \frac{1}{3} - \frac{1}{5}\right.$
$\left. - \ldots \infty\right\} = \frac{1}{2} . \left\{\frac{1}{1 . 3} + \frac{1}{3 . 5} + \ldots \infty\right\} + \frac{1}{4} . \left\{\frac{1}{3^2} + \frac{1}{5^2}\right.$
$\left. + \ldots \infty\right\} - \frac{1}{8}.$

$$\left.\begin{array}{l} \text{Let } 1 + \frac{1}{3} + \frac{1}{5} + \ldots \infty = s \\ \text{Then } \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \ldots \infty = s - 1 \end{array}\right\} \therefore \text{ by subtraction,}$$

&c., we get

$$\frac{1}{1 . 3} + \frac{1}{3 . 5} + \ldots \infty = \frac{1}{2}$$

Also $\frac{1}{3^2} + \frac{1}{5^2} + \ldots \infty = \frac{\pi^2}{8} - 1$ (See 689 or 738)

Hence $S = \frac{1}{4} + \frac{\pi^2}{32} - \frac{1}{4} - \frac{1}{8} = \frac{\pi^2}{32} - \frac{1}{8}.$

To sum $1 + \frac{3}{1 . 3} + \frac{3 . 4}{2.3^2} + \frac{4 . 5}{2.3^3} + \ldots \frac{n.(n+1)}{2.3^{n-1}}.$

$$\Delta . S = \frac{(n+1).(n+2)}{2 \times 3^n}$$

$$\therefore S = \frac{1}{2}\Sigma.\left\{(n+1).(n+2) \times \frac{1}{3^n}\right\}$$

$$= \frac{1}{2} \times \left\{(n+1).(n+2) \times \Sigma.\frac{1}{3^n} - \Delta.(n+1).\right.$$

$$\left.(n+2) \times \Sigma^2.\frac{1}{3^{n+1}} + \Delta^2.(n+1)(n+2)\,\Sigma^3.\frac{1}{3^{n+2}}\right\} \text{ See 654.}$$

Now, since $\Delta.\frac{1}{a^n} = \frac{1}{a^{n+1}} - \frac{1}{a^n} = \frac{1}{a^n} \times \left(\frac{1}{a} - 1\right)$

$$\therefore \Sigma.\frac{1}{a^n} = \frac{a}{1-a} \times \frac{1}{a^n}$$

Hence $\Sigma.\frac{1}{3^n} = -\frac{1}{2 \times 3^{n-1}}$

$$\Sigma^2.\frac{1}{3^{n+1}} = \frac{1}{2^2 \times 3^{n-1}}$$

$$\Sigma^3.\frac{1}{3^{n+2}} = \frac{1}{-2^3.3^{n-1}}$$

Also $\Delta.(n+1).(n+2) = 4n + 2$, and $\Delta^2(n+1)(n+2) = 2$.
$\therefore$ by substitution,

$$S = C - \frac{1}{2} \times \left\{\frac{(n+1)(n+2)}{2.3^{n-1}} + \frac{n+2}{2.3^{n-1}} + \frac{1}{2^2 \times 3^{n-1}}\right\}$$

$$= C - \frac{2n^2+8n+9}{8 \times 3^{n-1}} = \frac{27}{8} - \frac{2n^2+8n+9}{8.3^{n-1}}.$$

744. In any arithmetic series, whose first term is a, and common difference b, the n^{th} term is

$$a + \overline{n-1}.b.$$

Here $a = 13$, $b = -\frac{1}{3}$, and $n = 28$.

$\therefore$ the 28^{th} term of 13, $12\frac{2}{3}$ &c. is

$$18 + 27 \times \left(-\frac{1}{3}\right) = 18 - 9 = 4.$$

To sum $2 - \frac{1}{3} + \frac{1}{18} - \frac{1}{108} + \ldots\ldots \infty$, which is a geometric series, whose first term is 2, and common ratio $-\frac{1}{6}$, we have, (see *Wood.*)

$$S = \frac{a}{1-r} = \frac{2}{1+\frac{1}{6}} = \frac{12}{7}.$$

745. To sum $1^2 + 3^2 + 5^2 + \ldots\ldots 12$ terms $= S$.

$$\Delta S = (2n+1)^2 = (2n+1)(2n+3) - 2.(2n+1)$$

$$\therefore S = \frac{(2n-1).(2n+1).(2n+3)}{6} - \frac{(2n-1).(2n+1)}{2} + C$$

$$= \frac{n}{3} \times (4n^2 - 1).$$

Let $n = 12$

Then $S = 4 \times 575 = 2300$.

To sum $x + 2x^2 + 3x^3 + \ldots\ldots nx^n$, we have the geometric series $x + x^2 + x^3 + \ldots\ldots x^n = \frac{x \times x^n - x}{x-1} = \frac{x^{n+1} - x}{x-1}$, and differentiating and dividing by $\frac{dx}{x}$ we get

$$x + 2x^2 + 3x^3 + \ldots. nx^n = \frac{x}{(x-1)^2} \times (nx^n \times \overline{x-1} - \overline{x^n - 1}).$$

To sum $\frac{1}{1^2} + \frac{1}{2^2} + \ldots. \infty$, see 689.

746. To prove that

$$-n \times 1^m + n.\frac{n-1}{2} 2^m - n.\frac{n-1}{2}.\frac{n-2}{3}. 3^m + \ldots\ldots \pm n^m = 0$$

(n and m being integers and $m < n$) we have

$1 - nx + n.\frac{n-1}{2}x^2 - \ldots \pm x^n = (1-x)^n$, $\therefore$ differentiating

$$-n + n.\frac{n-1}{2}. \times 2x - \ldots \pm nx^{n-1} = n\,(1-x)^{n-1}$$

Multiply by x, differentiate and divide by dx,

Then $-n + n.\frac{n-1}{2}.\,2^2x - \ldots\ldots \pm n^2x^{n-1} = n.\,(1-x)^{n-1} - n.\,(n-1)\,x.\,(1-x)^{n-2}$.

Multiply by x, again differentiate, &c. and we get

$-n + n.\frac{n-1}{2}.\,2^3x - \ldots \pm n^3\,x^{n-1} = n.(1-x)^{n-1} - 3n.(n-1) \times x.\,(1-x)^{n-2} + n.(n-1).(n-2)\,x^2.(1-x)^{n-3}$

&c. $=$ &c.

Hence it is manifest, that by repeating the operation a sufficient number of times, we shall at length arrive at the form

$$-n.\,1^m + n.\frac{n-1}{2}.\,2^mx - \ldots \pm n^m\,x^{n-1} = Q \times (1-x)^{n-m},$$

(Q containing $(1-x)^{m-1}$, $(1-x)^{m-2}$, ... $(1-x)^2$, $(1-x)$ with their coefficients)

Let $x = 1$.

Then $-n.\,1^m + n.\frac{n-1}{2}.\,2^m - n\frac{n-1}{2}.\frac{n-2}{3}.\,3^m + \ldots\ldots \pm n^m$

$= Q' \times (1-1)^{n-m}$ which $= 0$, or $\frac{1}{0}$ (to be determined in each case)] according as $n - m$ is positive or negative, or as m is $<$ or $>$ n. When $n = m$, since the coefficient of $(1-x)^{n-n}$ is evidently $\pm\, n.(n-1)\,(n-2) \ldots$ to n terms $= \pm\, n \times (n-1) \times (n-2) \ldots 3.2.1$, and the other terms vanish when $x = 1$, we have

$-n.\,1^n + n.\frac{n-1}{2}.\,2^n - n.\frac{n-1}{2}.\frac{n-2}{3}.\,3^n + \ldots \pm n^n = \pm n.(n-1) \times \ldots 3.2.1$

or $n^n - n.(n-1)^n + n.\frac{n-1}{2}.(n-2)^n \ldots \mp n.1^n = n.(n-1)\ldots$ $3.2.1$. See also 750.

747. To sum $\frac{1}{1^2} + \frac{1}{2^2} + \ldots \infty$. See 684.

To sum $\frac{10.18}{2.4.9.12} + \frac{12.21}{4.6.12.15} + \ldots \frac{(2n+8).(3n+15)}{4.n.(n+1)\times(3n+6).(3n+9)} = S.$

$$\Delta S = \frac{(2n+10).(3n+18)}{4.(n+1).(n+2).(3n+9).(3n+12)}$$

$$= \frac{(n+5).(n+6)}{6(n+1).(n+2).(n+3).(n+4)}$$

$$= \frac{1}{6.(n+1).(n+2)} + \frac{2}{3.(n+1).(n+2).(n+3)}$$

$$+ \frac{1}{3.(n+1).(n+2).(n+3).(n+4)}.$$

$$\therefore S = C - \frac{1}{6.(n+1)} - \frac{1}{3.(n+1).(n+2)} - \frac{1}{9.(n+1).(n+2).(n+3)}$$

$$= C - \frac{3n^2+21n+38}{18.(n+1)(n+2)(n+3)} = \frac{19}{54} - \frac{3n^2+21n+38}{18.(n+1).(n+2).(n+3)}.$$

748. To sum $-9-7-5-\ldots 20$ terms, which is an arithmetic series, whose common difference is 2, we have

$$S = (2a + \overline{n-1}.b).\frac{n}{2} = (-18+19.2)\,10 = 200.$$

To sum $1 + \frac{2}{3} + \frac{4}{9} + \ldots 10$ terms, which is a geometric series, whose common ratio is $\frac{2}{3}$, we have

$$S = \frac{ar^n - a}{r - 1} = \frac{(\frac{2}{3})^{10} - 1}{\frac{2}{3} - 1} = \frac{2^{10} - 3^{10}}{(2-3)3^9}$$

$$= \frac{1024 - 59049}{-3^9} = \frac{58025}{3^9} = \frac{58025}{19683}.$$

To sum $1.2 + 2.3 + \ldots\ldots n$ terms, see 685.

749. To sum $\frac{1}{1.3} - \frac{1}{2.4} + \ldots\ldots n$ terms. See 706.

To sum $\frac{1}{2.3} + \frac{1}{4.5} + \frac{1}{6.7} + \ldots\ldots \infty = S.$

By division, $xdx + x^3dx + \ldots\ldots \infty = \frac{xdx}{1-x^2}$

$$\therefore \frac{x^2}{2} + \frac{x^4}{4} + \ldots\ldots \infty = \int \frac{xdx}{1-x^2}$$

Multiply by dx, and again integrate. Then

$$\frac{x^3}{2.3} + \frac{x^5}{4.5} + \ldots\ldots \infty = \int dx \int \frac{xdx}{1-x^2} = x \int \frac{xdx}{1-x^2}$$

$$- \int \frac{x^2dx}{1-x^2} = x - \frac{x-1}{2}.\, l.\,(1-x) - \frac{1}{2}.\, l.\,(1+x).$$

$$\therefore \frac{1}{2.3} + \frac{1}{4.5} + \ldots \infty = 1 - \frac{1}{2}.\, l.2.$$

750. To prove that

$$n^n - n\,(n-2)^n + n.\frac{n-1}{2}.\,(n-4)^n - \ldots\ldots \text{ to } \frac{n}{2} \text{ or } \frac{n+1}{2}$$

terms $= 1.2.3\ldots\ldots n \times 2^{n-1}$, we have by the theorem,

$$\Delta^n u_x = u_{x+n} - \frac{n}{1}.\, u_{x+n-1} + n.\frac{n-1}{2}\, u_{x+n-2} - \ldots\ldots (n+1). \text{ terms,}$$

$\Delta^n . x^m = (x + n)^m - \frac{n}{1} . (x+n-1)^m + n . \frac{n-1}{2} (x+n-2)^m$

$- \ldots . (n + 1)$ terms.

Let $m = n$, and $x = -\frac{n}{2}$

Then $\Delta^n . \left(-\frac{n}{2}\right)^n = \left(\frac{n}{2}\right)^n - \frac{n}{1} . \left(\frac{n-2}{2}\right)^n + n . \frac{n-1}{2} \times$

$\left(\frac{n-4}{2}\right)^n - \ldots (n + 1)$ terms.

But generally (as it may be easily proved)

$\Delta^n . x^n = 1.2.3 \ldots . n$, whatever be the value of x.

$\therefore 1.2 \ldots . n \times 2^n = n^n - n . (n - 2)^n + n . \frac{n-1}{2} (n - 4)^n - \ldots .$

$\pm n . \frac{n-1}{2} . (n - 2 . \overline{n-2})^n \mp n . (n - 2 \overline{n-1})^n \pm (n - 2n)^n,$

according as n is even or odd.

Now, when n is even

$+ (n - 2 . \overline{n-2})^n = + (n - 4)^n$

And when n is odd

$- (n - 2 . \overline{n-2})^n = - 1 \times - (n - 4)^n = (n - 4)^n$

And the same may be proved of the other terms.

$\therefore$ collecting extremes, and dividing by 2, we get

$1.2 \ldots n \times 2^{n-1} = n^n - n . (n - 2)^n + n . \frac{n-1}{2} (n - 4)^n - \ldots$ to $\frac{n}{2}$

or $\frac{n+1}{2}$ terms, according as n is even or odd, which was to be proved.

If we put $x = 0$, in the equation

$\Delta^n . x^m = (x + n)^m - n . (x + n - 1)^m + n . \frac{n-1}{2} . (x + n - 2)^m$

$- \ldots (n + 1)$ terms we have

$\Delta^n . 0 = n^m - n . (n - 1)^m + n . \frac{n-1}{2} . (n - 2)^m - \ldots$

And since $\Delta^n x^n = 1.2 \ldots. n$

$$\Delta^{n+1} x^n = 0$$

$$\&c. = \&c.$$

$\Delta^{n+p} x^n = 0$, whatever be the value of x.

$$\therefore n^n - n.(n-1)^n + n.\frac{n-1}{2}.(n-2)^n - \ldots. = 1.2.3.\ldots. n$$

And $(n+p)^n - (n+p).(n+p-1)^n + (n+p).\frac{n+p-1}{2} \times$

$(n+p-2)^n - \ldots. = 0$

whatever be the positive integer values of p and n. See 746.

By giving x different values in the equation

$\Delta^n.x^m = (x+n)^m - n.(x+n-1)^m + \&c.$ many other singular results may be obtained.

751. To sum $\frac{1}{1^2} - \frac{1}{2^2} + \ldots. \infty$. See 684.

To sum $1^3 + 3^3 + 5^3 + \ldots. (2n-1)^3 = S$, we have

$\Delta S = (2n+1)^3 = A.(2n+1).(2n+3).(2n+5) + B.(2n+1).(2n+3) + C.(2n+1) + D$ by supposition; whence by equating coefficients of the same powers of n, we get $A = 1$, $B = -6$, $C = 4$ and $D = 0$.

$\therefore \Delta S = (2n+1)(2n+3).(2n+5) - 6.(2n+1).(2n+3) + 4.(2n+1)$.

$$\therefore S = \frac{(2n-1).(2n+1).(2n+3).(2n+5)}{8} - (2n-1).(2n+1) \times$$

$$(2n+3) + (2n-1).(2n+1) + C = \frac{(4n^2-1)^2}{8} - \frac{1}{8}.$$

752. To sum $1^2 + 3^2 + 7^2 + 15^2 + \ldots (2^n - 1)^2 = S$.

$$\Delta S = (2^{n+1} - 1)^2 = 2^{2n+2} - 2.2^{n+1} + 1$$

$$= 4^{n+1} - 2^{n+2} + 1$$

$$\therefore S = \Sigma . 4^{n+1} - \Sigma . 2^{n+2} + \Sigma . 1$$

$$= \frac{4^{n+1}}{4-1} - \frac{2^{n+2}}{2-1} + n + C$$

$$= \frac{4^{n+1}}{3} - 2^{n+2} + n + \frac{8}{3}.$$

To sum $\frac{1}{2.3} \cdot \frac{1}{2} + \frac{1}{3.4} \cdot \frac{1}{2^2} + \ldots\ldots \infty$, we have by division

$$xdx + x^2dx + \ldots\ldots \infty = \frac{xdx}{1-x}$$

$$\therefore \frac{x^2}{2} + \frac{x^3}{3} + \ldots\ldots \infty = \int \frac{xdx}{1-x}$$

Multiply by dx and again integrate,

Then $\frac{x^3}{2.3} + \frac{x^4}{3.4} + \ldots\ldots \infty = \int dx \int \frac{xdx}{1-x} = x \int \frac{xdx}{1-x}$

$-\int \frac{x^2dx}{1-x} = x \times \{-x - l.(1-x)\} + \frac{x^2}{2} + x +$

$l.(1-x) + C = l.(1-x) \times (1-x) + x - \frac{x^2}{2}.$

Let $x = \frac{1}{2}$, and divide by $\frac{1}{4}$; then

$$\frac{1}{2.3} \cdot \frac{1}{2} + \frac{1}{3.4} \cdot \frac{1}{2^2} + \ldots\ldots = 2 . l. \left(\frac{1}{2}\right) + 2 - \frac{1}{2}$$

$$= \frac{3}{2} - 2\, l.\, 2.$$

753. To sum $2 + 3 + \frac{9}{2} + \ldots\ldots$ 20 terms, which is a geometric series, whose common ratio is $\frac{3}{2}$, we have

$$S = \frac{ar^n - a}{r-1} = \frac{2 \cdot (\frac{3}{2})^{20} - 2}{\frac{3}{2} - 1} = \left(\frac{3}{2}\right)^{20} - 1$$

$\therefore$ Log. $(1 + S) = \log. \left(\frac{3}{2}\right)^{20} = 20 \cdot l. \frac{3}{2} = 20 \cdot (\log. 3 -$ log. 2), whence, and by aid of the tables, $1 + S$,

And $\therefore$ S may be found.

To sum $\frac{1}{1 \cdot 3} + \frac{1}{2 \cdot 4} + \frac{1}{3 \cdot 5} + \ldots$ n terms. See 683.

To sum $\frac{1}{1 \cdot 2 \cdot 3} + \frac{1}{3 \cdot 4 \cdot 5} + \ldots \frac{1}{(2n+1) \cdot 2n \cdot (2n+1)} +$ &c. to ∞.

The n^{th} term $= \frac{1}{(2n-1) \cdot 2n \cdot (2n+1)} = \frac{1}{(2n-1) \cdot (2n+1)} + \frac{1}{2n+1} - \frac{1}{2n}$

It appears then that the series may be transformed to $\frac{1}{1 \cdot 3} + \frac{1}{3 \cdot 5} + \ldots \infty - \left(\frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} + \ldots \infty\right)$.

$$\text{Let } \frac{1}{1 \cdot 3} + \frac{1}{3 \cdot 5} + \ldots \frac{1}{(2n-1) \cdot (2n+1)} = s$$

$$\text{Then } \Delta s = \frac{1}{(2n+1) \cdot (2n+3)}$$

$$\therefore s = C - \frac{1}{2 \cdot (2n+1)} = \frac{1}{2} - \frac{1}{2 \cdot (2n+1)}$$

$$= \frac{n}{2n+1} \cdot = \frac{1}{2} \text{ when } n = \infty.$$

Also, since $l. (1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots \infty$

$$\therefore l. 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots \infty$$

$$\therefore - \left(\frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \ldots \infty\right) = l. 2 - 1.$$

Substituting these values we $\therefore$ get

$$\frac{1}{1\,.\,2\,.\,3}+\frac{1}{3\,.\,4\,.\,5}+\,.....\,\infty=\frac{1}{2}+l.\,2-1=l.\,2-\frac{1}{2}.$$

754. To sum $1\,.\,2\,.\,4+2\,.\,3\,.\,5+....\,n$ terms. See 704.

To sum $\frac{1}{3}+\frac{1}{2}+\frac{3}{4}+\frac{9}{8}+....\,n$ terms, which is a geometric series of the common ratio $\frac{3}{2}$, we have

$$S=\therefore\frac{ar^n-a}{r-1}=\frac{\frac{1}{3}\,.\,(\frac{3}{2})^n-\frac{1}{3}}{\frac{3}{2}-1}=\frac{(\frac{3}{2})^n-1}{6}$$

$\therefore$ Log. $(6\,S+1)=n\,.\,(\log.\,3-\log.\,2)$ which will give, by tabular reference, $6\,S+1$, and $\therefore$ S the sum required.

To sum $1\,.\,2+2\,.\,3x+3\,.\,4x^2+.....\,\infty$.

By division we have

$$x+x^2+x^3+.....\,\infty=\frac{x}{1-x}$$

Differentiate and divide by dx

$$\text{Then } 1+2x+3x^2+.....\,\infty=\frac{1}{1-x}+\frac{x}{(1-x)^2}=\frac{1}{(1-x)^2}$$

$$\therefore x^2+2x^3+3x^4+.....\,\infty=\frac{x^2}{(1-x)^2}$$

Differentiate and divide again by dx, &c.

$$\text{Then } 1\,.\,2\,x+2\,.\,3x^2+3\,.\,4\,.\,x^3\,.....=\frac{2x}{(1-x)^3}$$

$$\text{And } 1\,.\,2\,x+2\,.\,3\,x+3\,.\,4\,.\,x^2+\,....\,=\frac{2}{(1-x)^3}.$$

755. To sum $\frac{5}{1\,.\,2\,.\,3}+\frac{7}{2\,.\,3\,.\,4}+\,....\,\frac{2n+3}{n\,.\,(n+1)\,.\,(n+2)}$

$=$ S, we have

$$\Delta \,.\, S = \frac{2n+5}{(n+1)\,(n+2)\,(n+3)} = \frac{2\,(n+3)-1}{(n+1)\,(n+2)\,(n+3)} =$$

$$\frac{2}{(n+1)\,.\,(n+2)} - \frac{1}{(n+1)\,(n+2)\,(n+3)}$$

$$\therefore S = C\,\frac{2}{n+1} + \frac{1}{2\,.\,(n+1)\,.\,(n+2)} = C -$$

$$\frac{4n+7}{2\,.\,(n+1)\,(n+2)}$$

$$= \frac{5}{6} - \frac{4n+7}{2\,.\,(n+1)\,(n+2)}.$$

To sum $\frac{2}{3\,.\,5} - \frac{3}{5\,.\,7} + \frac{4}{7\,.\,9} - \ldots\ldots \infty$.

Let $\frac{1}{3} - \frac{x}{5} + \frac{x^2}{7} - \ldots\ldots \infty = s$

Then multiplying by $ax - b$ and arranging the terms, we get

$$\frac{5a+3b}{3\,.\,5}\,.\,x - \frac{7a+5b}{5\,.\,7}\,x^2 + \ldots\ldots \infty = s \times (ax-b) + \frac{b}{3}$$

Put $5a+3b = 2$, and the common difference of the numerators

$2a+2b = 3 - 2 = 1$, the common difference in the given series.

Hence $b = \frac{1}{4}$ and $a = -\frac{1}{4}$, and substituting

$$\frac{2x}{3\,.\,5} - \frac{3x^2}{5\,.\,7} + \ldots\ldots = \frac{s}{4}\,.\,\times(x-1) + \frac{1}{12}.$$

Let $x = 1$,

Then $\frac{2}{3\,.\,5} - \frac{3}{5\,.\,7} + \ldots\ldots = \frac{1}{12}.$

Given $\frac{1}{1} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots\ldots = \frac{\pi^2}{6}$ (see 684) to find $\frac{1}{1^2.2.3}$ $+ \frac{1}{2^2.3.4} + \ldots\ldots \frac{1}{n^2.(n+1)\,.\,(n+2)} +$ &c. to $\infty = S$.

The general term $\frac{1}{n^2.(n+1)\,.\,(n+2)}$ is decomposable, by the ordinary methods, into

$$\frac{1}{2n^2} - \frac{3}{4n} + \frac{3}{4(n+1)} + \frac{1}{4\,.\,(n+1)\,.\,(n+2)},$$

We have ∴

$$S = \frac{1}{2}\cdot\left(\frac{1}{1}+\frac{1}{2^2}+\frac{1}{3^2}+\ldots\infty\right) - \frac{3}{4}\cdot\left(1+\frac{1}{2}+\frac{1}{3}+\ldots\infty-\frac{1}{2}-\frac{1}{3}-\frac{1}{4}-\ldots\infty\right)+\frac{1}{4}\times\left(\frac{1}{2\,.\,3}+\frac{1}{3\,.\,4}+\ldots\infty\right) = \frac{1}{2}\cdot\frac{\pi^2}{6}-\frac{3}{4}+\frac{1}{4}\cdot\left(\frac{1}{2\,.\,3}+\frac{1}{3\,.\,4}+\ldots\right).$$

$$\text{Let } \frac{1}{2}+\frac{1}{3}+\ldots\infty = s$$

$$\text{Then } \frac{1}{3}+\frac{1}{4}+\ldots\infty = s-\frac{1}{2}$$

$$\therefore \frac{1}{2\,.\,3}+\frac{1}{3\,.\,4}+\ldots\infty = \frac{1}{2} \text{ by subtraction.}$$

Hence then, we finally obtain,

$$S = \frac{\pi^2}{12}-\frac{3}{4}+\frac{1}{8} = \frac{\pi^2}{12}-\frac{5}{8}.$$

756. To sum cosec. A + cosec. 2A + ... cosec. $2(n-1)$A $=$ S, we have generally,

$$\text{Cosec. }\theta = \frac{1}{\sin.\theta}$$

$$= \frac{\cos.^2\frac{\theta}{2}+\sin.^2\frac{\theta}{2}}{2\sin.\frac{\theta}{2}.\cos.\frac{\theta}{2}} = \frac{2\cos.^2\frac{\theta}{2} - \cos.\theta}{2\sin.\frac{\theta}{2}.\cos.\frac{\theta}{2}}$$

$$= \frac{\cos.\frac{\theta}{2}}{\sin.\frac{\theta}{2}} - \frac{\cos.\theta}{\sin.\theta} = \cot.\frac{\theta}{2} - \cot.\theta.$$

∴ by substitution we have

$$\text{Cosec. } A = \cot. \frac{A}{2} - \cot. A$$

$$\text{Cosec. } 2A = \cot. A - \cot. 2A$$

$$\text{Cosec. } 4A = \cot. 2A - \cot. 2^2A$$

$$\&c. = \&c.$$

$$\text{Cosec. } 2^{n-2} . A = \cot. 2^{n-3} . A - \cot\ 2^{n-2} . A$$

$$\text{Cosec. } 2^{n-1} . A = \cot. 2^{n-2} . A - \cot. 2^{n-1} . A,$$ which

added cross-wise give

$$S = \cot. \frac{A}{2} - \cot. 2^{n-1} A.$$

757. Let r be the common ratio, and S the sum of the series. Then since $b = ar$, we have $r = \frac{b}{a}$,

$$\text{and } S = \frac{a}{1-r} \text{ (See Wood) } = \frac{a}{1-\frac{b}{a}} = \frac{a^2}{a-b}.$$

758. To sum $\frac{1}{x+a} + \frac{a}{(x+a).(x+b)} + \frac{ab}{(x+a).(x+b).(x+c)}$ + ... to n terms.

Assume $1 + \frac{a}{x+a} + \frac{ab}{(x+a).(x+b)} + \ldots \frac{ab \ldots}{(x+a).(x+b)\ldots} = s$

Then $\frac{a}{x+a} + \frac{ab}{(x+a).(x+b)} + \frac{abc}{(x+a).(x+b).(x+c)} + \ldots$ $\frac{ab \ldots}{(x+a)\ldots} = s - 1 + \frac{abc \ldots n}{(x+a)(x+b)\ldots n}$. Whence, by subtraction, we get $\frac{x}{x+a} + \frac{ax}{(x+a).(x+b)} + \frac{abx}{(x+a).(x+b).(x+c)}$ + ... n terms $= 1 - \frac{abc \ldots n \text{ terms}}{(x+a).(x+b).(x+c) \ldots n \text{ terms}}$.

$$\therefore S = \frac{1}{x} - \frac{abc \ldots n \text{ terms}}{x(x+a).(x+b) \ldots n \text{ terms}}$$

To sum $\frac{4}{1\,.\,3} - \frac{12}{5\,.\,7} + \frac{20}{9\,.\,11} - \dots\, \infty = S.$

$$S = \left\{\begin{array}{l} \frac{1}{1} - \frac{1}{5} + \frac{1}{9} - \frac{1}{13} + \dots\, \infty \\ +\frac{1}{3} - \frac{1}{7} + \frac{1}{11} - \frac{1}{15} + \dots\, \infty \end{array}\right\}$$

Now $dx - x^4 dx + x^8 dx - \dots\, \infty = \frac{dx}{1+x^4}$

$\therefore x - \frac{x^5}{5} + \frac{x^9}{9} - \dots\dots\, \infty = \int \frac{dx}{1+x^4} = \frac{1}{4\sqrt{2}}$ $\times\, l.\, \frac{1 + x\sqrt{2} + x^2}{1 - x\sqrt{2} + x^2} + \frac{1}{2\sqrt{2}} \tan.^{-1}.\, \frac{x\sqrt{2}}{1-x^2}$ by the common method.

Let this be taken between $x = 0$ and 1,

Then $1 - \frac{1}{5} + \frac{1}{9} - \dots = \frac{1}{4\sqrt{2}}\,.\, l.\, \frac{2+\sqrt{2}}{2-\sqrt{2}} +$ $\frac{1}{2\sqrt{2}} \times \frac{\pi}{2} = \frac{1}{2\sqrt{2}} \times l.\, (\sqrt{2}+1) + \frac{1}{2\sqrt{2}} \times \frac{\pi}{2}.$

Again $x^2 dx - x^6 dx + \dots\, \infty = \frac{x^2 dx}{1+x^4}$

$\therefore \frac{x^3}{3} - \frac{x^7}{7} + \dots = \int \frac{x^2 dx}{1+x^4} = \frac{1}{4\sqrt{2}}\,.\, l.\, \frac{1 - 2x\sqrt{2} + x^2}{1 + 2x\sqrt{2} + x^2}$ $+ \frac{1}{2\sqrt{2}} \tan.^{-1}.\, \frac{\sqrt{1 - 2x\sqrt{2} + 2x^2} + \sqrt{1 + 2x\sqrt{2} + 2x^2}}{2}$

which, being taken between the limits of $x = 0$ and 1, gives

$\frac{1}{3} - \frac{1}{7} + \frac{1}{11} - \dots\, \infty = \frac{1}{4\sqrt{2}}\,.\, l.\, \frac{\sqrt{2}-1}{\sqrt{2}+1} +$ $\frac{1}{2\,.\sqrt{2}} \times \frac{\pi}{2}.$

Hence $S = \frac{1}{4\sqrt{2}} \cdot \left\{ l.\frac{\sqrt{2}+1}{\sqrt{2}-1} - l.\frac{\sqrt{2}+1}{\sqrt{2}-1} \right\} + \frac{1}{2\sqrt{2}} \times \left(\frac{\pi}{2} + \frac{\pi}{2}\right) = \frac{\pi}{2\sqrt{2}}.$

Another mode of summing series of this description may be seen in *Euler's Analysis Infinitorum*, vol. 1, p. 140, who has shewn generally, that

$$\frac{\pi}{n \cdot \sin. \frac{m\pi}{n}} = \frac{1}{m} + \frac{1}{n-m} - \frac{1}{n+m} - \frac{1}{2n-m} + \frac{1}{2n+m} + \frac{1}{3n-m} - \&c.$$

$$\text{Or} = \frac{n}{m.(n-m)} - \frac{3n}{(n+m).(2n-m)} + \frac{5n}{(2n+m)(3n-m)} - \&c.$$

which coincides with the above series, when $m = 1$ and $n = 4$. We take this opportunity of earnestly recommending to the Student the perusal of the above work. It is replete with every kind of analytical artifice, and, indeed, excellently adapted, in all respects, to form the Mathematician.

759. To sum $na^{n-1}b \times x + n.\frac{n-1}{2}.a^{n-2}b^2 \times 2x + n.\frac{n-1}{2}.\frac{n-2}{3} \times a^{n-3}b^3 \times 3x + \&c. = S.$

We have $(1+z)^n = 1 + nz + n.\frac{n-1}{2}.z^2 + n.\frac{n-1}{2}.\frac{n-2}{3}.z^3 + \&c.$

$\therefore$ differentiating and dividing by dz, we get

$$n.(1+z)^{n-1} = n + n.\frac{n-1}{2}.z \times 2 + n.\frac{n-1}{2}.\frac{n-2}{3}.z^2 \times 3 + \&c.$$

$$\therefore nz\,(1+z)^{n-1} = nz + n\,.\,\frac{n-1}{2}\,.\,z^2 \times 2 + n.\frac{n-1}{2}\,.\,\frac{n-2}{3}\,.\,z^3 \times 3 + \&c.$$

Let $z = \frac{b}{a}$, substitute, and multiply by ax^n; then

$$n\,a^{n-1}\,bx\left(1+\frac{b}{a}\right)^{n-1} = n\,a^{n-1}\,b\,.\,x + n\,.\frac{n-1}{2}\,.\,a^{n-2}\,b^2 \times 2x + \&c.$$

Or $S = na^{n-1}\,bx\left(\frac{a+b}{a}\right)^{n-1} = n\,bx\,.\,(a+b)^{n-1}$.

760. To sum $\cos. A + \frac{1}{2}\cos. 2A + \frac{1}{3}\cos. 3A + \ldots = S$, we have

$$\frac{dS}{-dA} = \sin. A + \sin. 2A + \sin. 3A + \ldots\ldots$$

Let $\sin. A + \sin. 2A + \ldots = s$

Then, by the form $2\sin. m\,.\,\sin. n = \cos. (m-n) - \cos. (m+n)$, we have

$$2\sin\,.\,\frac{A}{2}\,.\,\sin. A = \cos. \frac{A}{2} - \cos. \frac{3A}{2}$$

$$2\sin\,\frac{A}{2}\,.\,\sin. 2A = \cos. \frac{3A}{2} - \cos. \frac{5A}{2}$$

$$2\sin.\,\frac{A}{2}\,.\,\sin. 3A = \cos. \frac{5A}{2} - \cos. \frac{7A}{2}$$

$$\&c. = \&c. \text{ to } \infty.$$

Hence, adding crosswise, we get

$$2\sin. \frac{A}{2}\,.\,s = \cos. \frac{A}{2}.$$

$$\therefore s = \frac{1}{2}\,.\,\cot. \frac{A}{2}$$

$$\therefore dS = -\frac{dA}{2}\,.\,\cot. \frac{A}{2} = \frac{-d\,.\,\frac{A}{2}\cos.\frac{A}{2}}{\sin. \frac{A}{2}} = \frac{-d\,.\,\sin. \frac{A}{2}}{\sin. \frac{A}{2}}$$

$$\therefore S = -l.\left(\sin. \frac{A}{2}\right) + C$$

To find C we will put $A = \pi$.

Then $C = \cos. \pi + \frac{\cos. 2\pi}{2} + \frac{\cos. 3\pi}{3} + \ldots. \infty + l. (1)$

$= -(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots.) = -l. 2.$

$\therefore S = -l. \sin. \frac{A}{2} - l. 2 l. \frac{1}{2 \sin. \frac{A}{2}}.$

Again, since $f - g$ is the scale of relation of the series

$$1 + 5x + 19x^2 + 65x^3 + 211x^4 + \ldots. = S,$$

we have

$$\left.\begin{aligned} 19 &= 5f - g \\ \text{and } 65 &= 19f - 5g \end{aligned}\right\} \text{ whence } f = 5, \text{ and } g = 6.$$

$$\begin{aligned} \text{Hence } 1 &= 1 \\ 5x &= 5x \\ 19x^2 &= 5 \times 5x^2 - 6 \times 1x^2 \\ 65x^3 &= 5 \times 19x^3 - 6 \times 5x^3 \\ \&c. &= \&c. \end{aligned}$$

$\therefore S = 1 + 5x + 5 \times x (S - 1) - 6x^2 . S$

$\therefore S = \frac{1}{1 - 5x + 6x^2} = \frac{1}{(1-2x).(1-3x)}$

$= \frac{3}{1 - 3x} - \frac{2}{1 - 2x}$ (by indeterminate coefficients)

$\left.\begin{aligned} &= 3 + 9x + 27x^2 + 3^4 x^3 + \&c. \\ &- (2 + 4x + 8x^2 + 2^4 x^3 + \&c.) \end{aligned}\right\}$ by common division.

Let $x = 1$, and we get the series

$\left.\begin{aligned} 1 + 5 + 19 + 65 + \&c. = 3 + 9 + 27 + 81 + \ldots. \\ - (2 + 4 + 8 + 16 + \ldots.) \end{aligned}\right\}$ which are geometric, the common ratios being 3 and 2 respectively. See 671.

761. To sum $6 + 2 - 2 - 6 - \ldots.$ 19 terms $= S$, which is an Arithmetic Series, whose common difference is (-4), we have

$$S = (2a + \overline{n-1} . b) \frac{n}{2} = \{2 \times 6 + \overline{19-1} . (-4)\} . \frac{19}{2}$$

$= -570.$

To sum 8 + 20 + 50 + ... to 15 terms, which is a Geometric Series, whose common ratio is $\frac{5}{2}$, we have

$$S = \frac{ar^n - a}{r-1} = \frac{8 \cdot (\frac{5}{2})^{15} - 8}{\frac{5}{2} - 1} = \frac{4}{3} \times \left\{\left(\frac{5}{2}\right)^{15} - 1\right\}$$

$$\therefore 1 + \frac{3}{4} \cdot S = \left(\frac{5}{2}\right)^{15}$$

$$\therefore \log. (1 + \frac{3}{4} S) = 15 \cdot \log. \frac{5}{2} = 15 (\log. 5 - \log. 2)$$

and the tables will give us $1 + \frac{3}{4} S$; and therefore S.

To sum $\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \ldots\ldots n$ terms. See 685.

762. To sum $\frac{1}{1 \cdot 5} + \frac{1}{3 \cdot 7} + \ldots n$ terms and ∞. See 706.

To sum $1 \cdot 4 + 2 \cdot 5 + 4 \cdot 6 + \ldots\ldots 2^{n-1} \times (n+3) = S$, we have

$$\Delta\ S = 2^n (n+4)$$

$$\therefore S = \Sigma \cdot (n+4)\, 2^n = (n+4)\, \Sigma \cdot 2^n - \Delta\ (n+4)\, \Sigma^2 \cdot 2^{n+1}$$

$$= (n+4) \cdot \frac{2^n}{2-1} - 1 \times \frac{2^{n+1}}{(2-1)^2} + C = (n+4)\, 2^n -$$

$$2^{n+1} - 2 = 2^n \cdot (n+2) - 2.$$

To sum $\frac{1}{1 \cdot 2} - \frac{2}{2 \cdot 3} + \frac{3}{3 \cdot 4} - \ldots\ldots \infty = S$, we have, by striking out the numerators,

$$S = \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \ldots\ldots$$

But, since $dx - xdx + x^2dx - \ldots\ldots \infty = \frac{dx}{1+x}$

$$\therefore x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots\ldots \infty = \int \frac{dx}{1+x} = l.\ (1+x)$$

$$\therefore 1 - \frac{1}{2} + \frac{1}{3} - \ldots\ldots \infty = l.\ (1+1) = l.\ 2.$$

Hence $S = 1 - l.\ 2$.

763. To sum $\frac{2}{10} - \frac{2}{100} + \ldots\ldots \infty$ which is a geometric series whose common ratio $= \left(-\frac{1}{10}\right)$, we have

$$S = \frac{a}{1-r} = \frac{\frac{2}{10}}{1+\frac{1}{10}} = \frac{2}{11}.$$

To sum $5 + 7 + 9 + \ldots.$ 50 terms, which is an arithmetic series, whose common difference $= 2$, we have

$$S = (2a + \overline{n-1}\,.\,b)\frac{n}{2} = (10 + 49 \times 2)\,25$$

$$= 2600.$$

764. To sum $1 + \frac{2}{2\,.\,3} + \frac{2^2}{3\,.\,3^2} + \ldots.\ \infty$, we have

$$dx + xdx + x^2dx + \ldots.\ \infty = \frac{dx}{1-x}$$

$$\therefore x + \frac{x^2}{2} + \frac{x^3}{3} + \ldots.\ \infty = \int\frac{dx}{1-x} = -\,l.\,(1-x) + C$$

$$= -\,l.\,(1-x)$$

$$\therefore 1 + \frac{x}{2} + \frac{x^2}{3} + \ldots\ldots \infty = -\frac{1}{x}\,l.\,(1-x)$$

Let $x = \frac{2}{3}$. Then

$$1 + \frac{2}{2\,.\,3} + \frac{2^2}{3\,.\,3^2} + \frac{2^3}{4\,.\,3^3} + \ldots\ldots = -\frac{3}{2} \times l.\,\frac{1}{3}$$

$$= \frac{3}{2}\,.\,l.\,3.$$

To sum $\frac{5}{1\,.\,2\,.\,3\,.\,4} + \frac{9}{3.4.5.6} + \ldots \frac{4n+1}{(2n-1)\,.\,2n\,.\,(2n+1).(2n+2)} = S.$

$$\Delta\ S = \frac{4n+5}{(2n+1)\,.\,(2n+2)\,.\,(2n+3)\,.\,(2n+4)}$$

$$= \frac{4n+5}{4\,.\,(n+1)\,.\,(n+2)\,.\,(2n+1)\,.\,(2n+3)}$$

$$= -\frac{1}{4} \times \left\{\frac{1}{(n+2).(2n+3)} - \frac{1}{(n+1).(2n+1)}\right\}$$

$$= -\frac{1}{4} \times \Delta \cdot \frac{1}{(n+1).(2n+1)}$$

$$\therefore S = C - \frac{1}{4.(n+1).(2n+1)} = \frac{1}{4} - \frac{1}{4.(n+1).(2n+1)}.$$

Let $n = \infty$.

$$\text{Then } S = \frac{1}{4} - \frac{1}{4 . \infty^2} = \frac{1}{4}.$$

See also 768.

765. Let $(a+b)^m = a^m + ma^{m-1} b + m . \frac{m-1}{2} a^{m-2} b^2 + m . \frac{m-1}{2} . \frac{m-2}{3} a^{m-3} b^3 + m . \frac{m-1}{2} . \frac{m-2}{3} \times \frac{m-3}{4} . a^{m-4} b^4 +$ &c. be the expansion of a binomial. Then also we have $(a-b)^m = a^m - m . a^{m-1} b + m . \frac{m-1}{2} . a^{m-2} b^2 - m . \frac{m-1}{2} . \frac{m-2}{3} . a^{m-3} b^3 + m . \frac{m-1}{2} . \frac{m-2}{3} \times \frac{m-3}{4} . a^{m-4} b^4 -$ &c. Hence taking the difference of these series and dividing the results by 2, we get

$$m . a^{m-1} . b + m . \frac{m-1}{2} . \frac{m-2}{3} . a^{m-3} b^3 + m . \frac{m-1}{2} \times \frac{m-2}{3} . \frac{m-3}{4} . \frac{m-4}{5} . a^{m-5} b^5 . + \&c. = \frac{(a+b)^m - (a-b)^m}{2},$$

the sum required.

We also obtain $a^m + m . \frac{m-1}{2} a^{m-2} b^2 + \ldots\ldots = \frac{(a+b)^m + (a-b)^m}{2}$.

766. Let a be the first term, and r the common ratio of the series.

Then $s = a + ar + ar^2 + \ldots\ldots \infty$

$$= \frac{a}{1-r}. \quad \text{(See Wood.)}$$

Also $S = a^2 + a^2r^2 + a^2r^4 + \ldots\ldots \infty = \dfrac{a^2}{1-r^2}$

$$\text{Hence } \frac{s}{S} = \frac{a}{1-r} \times \frac{1-r^2}{a^2} = \frac{1+r}{a}.$$

$$\left.\begin{aligned} \therefore\ 1+r &= \frac{as}{S} \\ \text{and } 1-r &= \frac{a}{s} \end{aligned}\right\} \therefore\ a \times \left(\frac{s}{S} \times \frac{1}{s}\right) = 2$$

$$\text{and } a = \frac{2s\,S}{s^2 + S}$$

$$\therefore \text{ also } r = 1 - \frac{a}{s} = 1 - \frac{2S}{s^2+S} = \frac{s^2 - S}{s^2 + S}.$$

The series $\therefore$ is

$$\frac{2sS}{s^2+S} + 2s\,S\,.\frac{s^2 - S}{(s^2+S)^2} + 2s\,S\,.\frac{(s^2-S)^2}{(s^2+S)^3} + \&c.$$

767. Let $\dfrac{1}{2^n} + \dfrac{1}{4^n} + \dfrac{1}{6^n} + \ldots\ldots \&c. = S.$

$$\text{Then } 1 + \frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{4^n} + \&c. = 2^n\,S$$

$$\text{and } \frac{1}{2^n} \qquad + \frac{1}{4^n} + \&c. = S$$

$$\therefore\ 1 \qquad + \frac{1}{3^n} + \frac{1}{5^n} + \&c. = S\,.(2^n - 1)$$

$$\text{Hence } \frac{\dfrac{1}{2^n} + \dfrac{1}{4^n} + \dfrac{1}{6^n} + \ldots\ldots \infty}{\dfrac{1}{1^n} + \dfrac{1}{3^n} + \dfrac{1}{5^n} + \ldots\ldots \infty} = \frac{2^n - 1}{1},$$ the relation required.

768. To sum $\frac{2a+b}{a^2.(a+b)^2} + \frac{2a+3b}{(a+b)^2.(a+2b)^2} + \ldots$

$\frac{2a+\overline{2n-1}.b}{(a+\overline{n-1}.b)^2.(a+nb)^2} = S,$

we have $\Delta S = \frac{2a+(2n+1)b}{(a+nb)^2.(a+\overline{n+1}.b)^2} =$

$\frac{\frac{1}{b}.\Delta.(a+nb)^2}{(a+nb)^2.(a+\overline{n+1}.b)^2}$

Now, generally $\frac{\Delta.u}{u.u_1} = \frac{u_1-u}{u.u_1} = \frac{1}{u} - \frac{1}{u_1} = -\Delta.\frac{1}{u}.$

$$\therefore \Delta.S = -\frac{1}{b}.\Delta.\frac{1}{(a+nb)^2}$$

$$\therefore S = C - \frac{1}{b.(a+nb)^2} = \frac{1}{ba^2} - \frac{1}{b.(a+nb)^2}$$

$$= \frac{n.(2a+nb)}{a^2.(a+nb)^2}.$$ See 764.

Let $n = \infty$

Then $S = \frac{1}{ba^2}$.

To sum $\frac{1}{1.4.7} + \frac{1}{4.7.10} + \ldots$ n terms. See 727.

To sum $\frac{1}{4.7.12} + \frac{1}{6.10.16} + \frac{1}{8.13.20} + \ldots \infty = S,$

we have

$$8.S = \frac{1}{2.3.7} + \frac{1}{3.4.10} + \frac{1}{4.5.13} + \ldots \infty$$

Now $x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \ldots = \int \frac{dx}{1-x}$

$$\therefore \frac{x^2}{2} + \frac{x^3}{2.3} + \frac{x^4}{3.4} + \ldots = \int dx \int \frac{dx}{1-x}$$

And multiplying by $x^{-\frac{5}{3}}dx$, and again integrating,

$$\frac{x^{\frac{4}{3}}}{1.2.4}+\frac{x^{\frac{7}{3}}}{2.3.7}+\frac{x^{\frac{10}{3}}}{3.4.10}+\ldots=\frac{1}{3}\int\frac{dx}{x^{\frac{5}{3}}}\int dx\int\frac{dx}{1-x}$$

$$=-\frac{1}{2x^{\frac{2}{3}}}\cdot\int dx\int\frac{dx}{1-x}+\frac{1}{2}\cdot\int\frac{dx}{x^{\frac{2}{3}}}\int\frac{dx}{1-x}$$

$$=-\frac{x}{2x^{\frac{2}{3}}}\int\frac{dx}{1-x}+\frac{1}{2x.^{\frac{2}{3}}}\int\frac{xdx}{1-x}+\frac{3}{2}x^{\frac{1}{3}}\int\frac{dx}{1-x}-$$

$$\frac{3}{2}\int\frac{x^{\frac{1}{3}}dx}{1-x}.$$

Now in taking the integral between the limits $x=0$ and 1, the ultimate result will not be affected by putting x, whenever it appears in the numerator without the integral sign, $=1$, the latter limit. The integral will $\therefore$ reduce to

$$\frac{-1}{2x^{\frac{2}{3}}}\times\int dx\,.\,\frac{1-x}{1-x}+\frac{3}{2}\cdot\int dx\,.\,\frac{1-x^{\frac{1}{3}}}{1-x}$$

$$=-\frac{x^{\frac{1}{3}}}{2}+\frac{3}{2}\cdot\int\frac{dx}{1+x^{\frac{1}{3}}+x^{\frac{2}{3}}}=4\,x^{\frac{1}{3}}-\frac{9}{4}l.(1+x^{\frac{1}{3}}+x^{\frac{2}{3}})$$

$-\dfrac{3\sqrt{3}}{2}\,.\,\tan.^{-1}\dfrac{2x^{\frac{1}{3}}+1}{\sqrt{3}}$, as we easily learn by putting $x^{\frac{1}{3}}=u$, substituting, &c.

$\therefore$ taking the integral between $x=1$ and 0, we have

$$8\,S+\frac{1}{1.2.4}=4-\frac{9}{4}.\,l.\,3-\frac{3\sqrt{3}}{2}\,.\,\tan.^{-1}\sqrt{3}+$$

$$\frac{3\sqrt{3}}{2}.\tan.^{-1}\frac{1}{\sqrt{3}}.$$

$$\text{But, }\tan.\frac{\pi}{6}=\frac{\sin.\frac{\pi}{6}}{\cos.\frac{\pi}{6}}=\frac{\frac{1}{2}}{\frac{\sqrt{3}}{2}}=\frac{1}{\sqrt{3}}$$

$$\text{and }\tan.\frac{\pi}{3}=\sqrt{3}.$$

$$8 \cdot S + \frac{1}{8} = 4 - \frac{9}{4} l. 3 - \frac{3\sqrt{3}}{2} \cdot \left(\frac{\pi}{3} - \frac{\pi}{6}\right)$$

$$\therefore S = \frac{31}{64} - \frac{9}{32} \cdot l. 3 - \frac{\sqrt{3}}{32} \cdot \pi.$$

769. To sum $(S+a) + (S+a+ar) (S+a+ar+ar^2) +$ &c. to n terms, $= s$, we have

$$s = n S + na + (n-1) ar + (n-2) ar^2 + (n-3) ar^3 + \dots 2 ar^{n-2} + ar^{n-1}$$

$$= n S + na \times (1 + r + r^2 + \dots r^{n-1})$$

$$- ar \times (1 + 2r + 3r^2 + \dots \overline{n-1} \cdot r^{n-2})$$

$$= n S + na \cdot \frac{r^n - 1}{r - 1} - ar \cdot (1 + 2r + 3r^2 + \dots \overline{n-1} \cdot r^{n-2})$$

Now $r + r^2 + r^3 + \dots r^{n-1} = \dfrac{r^n - r}{r - 1}$; $\therefore$ differentiating and dividing by dr, we have

$$1 + 2r + 3r^2 + \dots (n-r) r^{n-2} = \frac{nr^{n-1} - 1}{r - 1} - \frac{r^n - r}{(r-1)^2}$$

Hence $s = n S + na \cdot \dfrac{r^n - 1}{r - 1} - ar \cdot \dfrac{nr^{n-1} - 1}{r - 1} + ar \cdot \dfrac{r^n - r}{(r-1)^2}$

$$= n S + a \cdot \frac{r - n}{r - 1} + ar^2 \cdot \frac{r^{n-1} - 1}{(r-1)^2}$$

But $S = \dfrac{ar^n - a}{r - 1}$

$$\therefore s = \frac{\overline{n+1} \cdot ar^{n+1} - nar^n - \overline{2n+1} \cdot ar + 2an}{r - 1}.$$

Hence we learn that this form is an integer, whatever may be the positive integer values of n, a, r.

770. To sum $\dfrac{1}{1 \cdot 3} - \dfrac{1}{3 \cdot 5} + \dots \infty$. See 687.

To sum $1 + 3 + 7 + 15 + \dots 2^n - 1$, we have

$$\Delta S = 2^{n+1} - 1$$

$$\therefore S = \Sigma .. 2^{n+1} - \Sigma . 1$$
$$= 2^{n+1} - n + C = 2^{n+1} - n - 2.$$

To sum $\frac{1}{1.5.9} + \frac{1}{5.9.13} + \ldots \frac{1}{(4n-3).(4n+1).(4n+5)} = S.$

$$\Delta . S = \frac{1}{(4n+1).(4n+5).(4n+9)}$$

$$\therefore S = C - \frac{1}{8.(4n+1).(4n+5)}$$

$$= \frac{1}{40} - \frac{1}{8.(4n+1).(4n+5)}$$

$$= \frac{2n^2 + 3n}{5.(4n+1).(4n+5)}$$

771. To sum $\frac{11}{1.2.3.4} + \frac{17}{4.5.6.7} + \ldots\ldots$

$\frac{6n+5}{(3n-2).(3n-1).3n.(3n+1)} = S,$

we have, by decomposing the general or n^{th} term into its partial fractions,

$$\frac{6n+5}{(3n-2).(3n-1).3n.(3n+1)} = \frac{3}{2} \cdot \frac{1}{3n-2} -$$

$\frac{7}{2} \cdot \frac{1}{3n-1} + \frac{5}{2} \cdot \frac{1}{3n} - \frac{1}{2} \cdot \frac{1}{3n+1}$; we have $\therefore$

$$S = \begin{cases} \frac{3}{2} \times \left(\frac{1}{1} + \frac{1}{4} + \ldots \frac{1}{3n-2}\right) \\ -\frac{7}{2} \times \left(\frac{1}{2} + \frac{1}{5} + \ldots \frac{1}{3n-1}\right) \\ +\frac{5}{2} \times \left(\frac{1}{3} + \frac{1}{6} + \ldots \frac{1}{3n}\right) \\ -\frac{1}{2} \times \left(\frac{1}{4} + \frac{1}{7} + \ldots \frac{1}{3n+1}\right) \end{cases}$$

Now $x^{3n}dx + x^{3n+3}dx + x^{3n+6}\,dx + \ldots \infty = \frac{x^{3n}dx}{1-x^3}$

$$\therefore \frac{x^{3n+1}}{3n+1} + \frac{x^{3n+4}}{3n+4} + \&c. \ldots \infty = \int \frac{x^{3n} dx}{1 - x^3}$$

Let $n = 0$

$$\text{Then } \frac{x}{1} + \frac{x^4}{4} + \ldots \infty = \int \frac{dx}{1 - x^3}$$

$$\text{Thence } \frac{x}{1} + \frac{x^4}{4} + \ldots \frac{x^{3n-2}}{3n-2} = \int \frac{dx}{1 - x^3} - \int \frac{x^{3n} dx}{1 - x^3}$$

$$= \int \frac{1 - x^{3n}}{1 - x^3} dx$$

$$\text{Similarly } \frac{x^2}{2} + \frac{x^5}{5} + \ldots \frac{x^{3n-1}}{3n-1} = \int \frac{1 - x^{3n}}{1 - x^3} . x dx$$

$$\frac{x^3}{3} + \frac{x^6}{6} + \ldots \frac{x^{3n}}{3n} = \int \frac{1 - x^{3n}}{1 - x^3} x^2 dx$$

$$\frac{x^4}{4} + \frac{x^7}{7} + \ldots \frac{x^{3n+1}}{3n+1} = \int \frac{1 - x^{3n}}{1 - x^3} x^3 dx$$

We $\therefore$ have

$$S = \frac{1}{2} \times \left\{ 3 \int \frac{1 - x^{3n}}{1 - x^3} dx - 7 \int \frac{1 - x^{3n}}{1 - x^3} x dx + 5 \int \frac{1 - x^{3n}}{1 - x^3} x^2 dx - \int \frac{1 - x^{3n}}{1 - x^3} . x^3 dx \right\},$$

the integrals being taken between the limits of $x = 1$ and 0; which, however, does not appear expressible in less than n terms, except when $n = \infty$. In this case we have

$$S = \frac{1}{2} \times \left\{ 3 \int \frac{dx}{1 - x^3} - 7 \int \frac{x dx}{1 - x^3} + 5 \int \frac{x^2 dx}{1 - x^3} - \int \frac{x^3 dx}{1 - x^3} \right\} = \frac{1}{2} \int \frac{(3 - 4x + x^2)\, dx}{1 + x + x^2}$$

taken between $x = 1$ and 0.

$$\text{But } \int \frac{(3 - 4x + x^2)\, dx}{1 + x + x^2} = \int dx - \int \frac{\frac{5}{2} dx + 5 x dx}{1 + x + x^2} + \frac{9}{2} . \int \frac{dx}{1 + x + x^2} = x - \frac{5}{2} . l. (1 + x + x^2) + 3 \sqrt{3} \times \tan.^{-1} \frac{2x + 1}{\sqrt{3}}.$$

Hence $S = \frac{1}{2} - 0 - \frac{5}{4}.(l.\,3 - l.\,1) + \frac{3\sqrt{3}}{2}.\ \left(\tan.^{-1}\sqrt{3} - \tan.\frac{1}{\sqrt{3}}\right)$

But $\sqrt{3} = \frac{(\sqrt{\frac{3}{4}})}{(\frac{1}{2})} = \frac{\sin.\,60^{\circ}}{\cos.\,60^{\circ}} = \tan.\,60^{\circ} = \tan.\,\frac{\pi}{3}$

And $\frac{1}{\sqrt{3}} = \frac{1}{\tan.\,60^{\circ}} = \cot.\,60^{\circ} = \tan.\,30^{\circ} = \tan.\,\frac{\pi}{6}$

And $l.\,1 = 0$

$$\therefore S = \frac{1}{2} - \frac{5}{4}l.\,3 + \frac{3\sqrt{3}}{2} \times \left(\frac{\pi}{3} - \frac{\pi}{6}\right)$$

$$= \frac{1}{2} - \frac{5}{4}l.\,3 + \sqrt{3}.\,\frac{\pi}{4}.$$

To sum the several series,

$$1 + 2.\left(\frac{e}{2}\right)^2 + \frac{4.3}{2}.\left(\frac{e}{2}\right)^4 + \frac{6.5.4}{2.3}.\left(\frac{e}{2}\right)^6 + \dots\ \infty \quad (1)$$

$$1 + 3.\left(\frac{e}{2}\right)^2 + \frac{5.4}{2}.\left(\frac{e}{2}\right)^4 + \frac{7.6.5}{2.3}.\left(\frac{e}{2}\right)^6 + \dots\ \infty \quad (2)$$

$$1 + 4.\left(\frac{e}{2}\right)^2 + \frac{6.5}{2}.\left(\frac{e}{2}\right)^4 + \frac{8.7.6}{2.3}.\left(\frac{e}{2}\right)^6 + \dots\ \infty \quad (3)$$

&c. &c. &c.

which are coefficients in the expression for the radius vector of an orbit in terms of the cosines of the multiples of the angle between the radius vector and axis-major, reckoned from the perihelion. See 679, page 497.

Let $x + 2x^3 + \frac{3.4}{2}x^5 + \dots\ \infty = s$

Then $xdx + 2.3\,x^3dx + \frac{3.4.5}{2}.\,x^5\,dx + \dots\ \infty = xds$

And $2x^2 + \frac{3.4}{2}x^4 + \frac{4.5.6}{2.3}x^6 + \dots\ \infty = 4\int xds = \text{also}\,\frac{s}{x} - 1$

$$\therefore 4xds = \frac{ds}{x} - \frac{sdx}{x^2}$$

Hence $\frac{ds}{s} = \frac{dx}{(1-4x^2)x} = \frac{dx}{x} - \frac{dx}{1+2x} + \frac{dx}{1-2x}$

$$\therefore ls = lx - \frac{1}{2}l.(1+2x) - \frac{1}{2}l.(1-2x)$$

$$= l.\frac{x}{\sqrt{1-4x^2}}$$

$$\therefore s = \frac{x}{\sqrt{1-4x^2}} \quad \therefore \frac{s}{x} = \frac{1}{\sqrt{1-4x^2}}$$

Let $x = \frac{e}{2}$

Then $1 + 2.\left(\frac{e}{2}\right)^2 + \frac{4.3}{2}.\left(\frac{e}{2}\right)^4 + \ldots\ldots \infty = \frac{1}{\sqrt{1-e^2}} \ldots (a)$

Again, since $1 + 2x^2 + \frac{3.4}{2}x^4 + \ldots\ldots \infty = \frac{1}{\sqrt{1-4x^2}}$,

multiplying by $2xdx$ and integrating, we have

$$x^2 + x^4 + \frac{4}{2}x^6 + \ldots = \int \frac{2xdx}{\sqrt{1-4x^2}} = -\frac{1}{2}\sqrt{1-4x^2} + C$$

$$= \frac{1}{2} - \frac{1}{2}\sqrt{1-4x^2};$$ which, being

divided by x, differentiated and divided by dx, gives

$$1 + 3x^2 + \frac{4.5}{2}x^4 + \frac{5.6.7}{2.3}x^6 + \ldots = \frac{1-\sqrt{1-4x^2}}{2x^2\sqrt{1-4x^2}} \ldots (A)$$

Let $x = \frac{e}{2}$

Then $1 + 3\left(\frac{e}{2}\right)^2 + \frac{4.5}{2}.\left(\frac{e}{2}\right)^4 + \frac{5.6.7}{2.3}.\left(\frac{e}{2}\right)^6$

$+ \ldots\ldots \infty = 2 \times \frac{1-\sqrt{1-e^2}}{e^2\sqrt{1-e^2}} \ldots (b)$

Again, multiplying (A) by $2x^3dx$, and integrating, we have

$$\frac{x^4}{2} + x^6 + \frac{5}{2}x^8 + \frac{6.7}{2.3}x^{10} + \ldots \infty = \frac{1}{4} - \frac{1}{4}\sqrt{1-4x^2} - \frac{x^2}{2},$$

which being divided by x^2 differentiated &c. gives

$$1 + 4x^2 + \frac{5.6}{2}x^4 + \ldots = \frac{1-2x^2-\sqrt{1-4x^2}}{2x^3.\sqrt{1-4x^2}} \ldots (B)$$

Let $x = \frac{e}{2}$.

Then $1 + 4\left(\frac{e}{2}\right)^2 + \frac{5.6}{2}\cdot\left(\frac{e}{2}\right)^4 + \frac{6.7.8}{2.3}\cdot\left(\frac{e}{2}\right)^6$

$$+\ldots\infty = 2^2 \times \frac{2 - e^2 - 2\sqrt{1-e^2}}{e^4\sqrt{1-e^2}} = 2^2 \times \frac{(1-\sqrt{1-e^2})^2}{e^4\sqrt{1-e^2}}$$

By repeating the above processes it will be found that

$$1 + 5x^2 + \frac{6.7}{2}.x^4 + \ldots\infty = \frac{1}{2^3x^6\sqrt{1-4x^2}} \times (1-\sqrt{1-4x^2})^3$$

$$1 + 6x^2 + \frac{7.8}{2}.x^4 + \ldots\infty = \frac{1}{2^4x^8\sqrt{1-4x^2}} \times (1-\sqrt{1-4x^2})^4$$

&c. = &c.

And we $\therefore$ have also

$$1+5\left(\frac{e}{2}\right)^2 + \frac{6.7}{2}\cdot\left(\frac{e}{2}\right)^4 + \ldots\infty = \frac{2^3}{e^6\sqrt{1-e^2}} \times \{4-3e^2$$

$$+(e^2-4)\sqrt{1-e^2}\} = \frac{2^3}{e^6\sqrt{1-e^3}} \times (1-\sqrt{1-e^2})^3$$

$$1+6\left(\frac{e}{2}\right)^2 + \frac{7.8}{2}\cdot\left(\frac{e}{2}\right)^4 + \ldots\infty = \frac{2^4}{e^8\sqrt{1-e^2}} \times$$

$$\{8.(1-e^2)+e^4+4(e^2-2)\sqrt{1-e^2}\} = \frac{2^4}{e^8\sqrt{1-e^2}} \times (-\sqrt{1-e^2})^4$$

&c. = &c.

Reverting, therefore, to Prob. 679, we have

$$r = a.\sqrt{1-e^2}\left\{1 - \frac{2}{e}.(1-\sqrt{1-e^2})\cos.\theta + \frac{2}{e^2}\times\right.$$

$$(1-\sqrt{1-e^2})^2\cos.2\theta - \frac{2}{e^3}(1-\sqrt{1-e^2})^3\cos.3\theta$$

$$+\frac{2}{e^4}.(1-\sqrt{1-e^2})^4 \times \cos.4\theta - \text{\&c. \&c.}$$

To sum $\frac{5}{1.2.3}\cdot\frac{1}{2^2} + \frac{6}{2.3.4}\cdot\frac{1}{2^3} + \ldots \frac{n+4}{n.(n+1)(n+2)}\cdot\frac{1}{2^{n+1}} = S.$

$$\Delta S = \frac{n+5}{(n+1).(n+2).(n+3)}\cdot\frac{1}{2^{n+2}}$$

$$= \frac{1}{(n+1).(n+2)2^{n+2}} - \frac{1}{(n+2).(n+3).2^{n+3}}$$

$$= -2\Delta . \frac{1}{(n+1).(n+2)2^{n+2}}$$

$$\therefore S = C - \frac{1}{(n+1)(n+2).2^{n+1}} = \frac{1}{4} - \frac{1}{(n+1)(n+2).2^{n+1}}$$

To sum $\frac{1}{1.2} - \frac{1}{4.5} + \frac{1}{7.8} - \ldots. \infty$

we have

$$dx - x^3 dx + x^6 dx - \ldots. \infty = \frac{dx}{1+x^3}$$

$$\therefore x - \frac{x^4}{4} + \frac{x^7}{7} - \ldots.. \infty = \int \frac{dx}{1+x^3}$$

And $\frac{x^2}{1.2} - \frac{x^5}{4.5} + \frac{x^8}{7.8} - \ldots. \infty = \int dx \int \frac{dx}{1+x^3}$

$= x \int \frac{dx}{1+x^3} - \int \frac{x dx}{1+x^3} = \frac{x}{3} l.(1+x) - \frac{x}{6} l.(1-x+x^2) + \frac{x}{\sqrt{3}} \times$
$\sin.^{-1} \frac{x\sqrt{3}}{2\sqrt{1-x+x^2}} - \left(\frac{1}{6} l.(1-x+x^2) - \frac{1}{3} . l.(1+x) \right.$
$\left. + \frac{1}{\sqrt{3}} \sin.^{-1} \frac{x\sqrt{3}}{2\sqrt{1-x+x^2}} \right) = \frac{x+1}{3} . l.(1+x) - \frac{x+1}{6} \times$
$l.(1-x+x^2) + \frac{x-1}{\sqrt{3}} . \sin.^{-1} \frac{x\sqrt{3}}{2\sqrt{1-x+x^2}}$, which, taken between $x = 0$ and 1, gives

$$\frac{1}{1.2} - \frac{1}{4.5} + \frac{1}{7.8} - \ldots \infty = \frac{2}{3} . l. 2.$$

This may be verified as follows,

$$l. 2 = l.(1+1) = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \frac{1}{9} - \&c.$$

$$= \frac{1}{1.2} + \frac{1}{3} - \frac{1}{4.5} - \frac{1}{6} + \frac{1}{7.8} + \frac{1}{9} - \&c.$$

$$= \frac{1}{1.2} - \frac{1}{4.5} + \frac{1}{7.8} - \&c. + \frac{1}{3} . (1 - \frac{1}{2} + \frac{1}{3} - \&c.$$

$$= \frac{1}{1.2} - \frac{1}{4.5} + \&c. + \frac{1}{3} \times l. 2.$$

$$\therefore \frac{2}{3}.l.2 = \frac{1}{1.2} - \frac{1}{4.5} + \dots \infty.$$

772. To sum $\frac{3}{1.2.2} + \frac{4}{2.3.2^2} + \dots \frac{n+2}{n.(n+1)2^n} = S.$

$$\Delta S = \frac{n+3}{(n+1)(n+2).2^{n+1}} = -2\left(\frac{1}{(n+2).2^{n+2}} - \frac{1}{(n+1)2^{n+1}}\right)$$

$$= - \Delta . \frac{1}{(n+1).2^n}.$$

$$\therefore S = C - \frac{1}{(n+1)2^n} = 1 - \frac{1}{(n+1).2^n}.$$

To sum $\frac{1}{1.5} + \frac{1}{3.7} + \frac{1}{5.9} + \dots \frac{1}{(2n-1)(2n+3)} = S.$

$$\Delta S = \frac{1}{(2n+1).(2n+5)} = \frac{2n+3}{(2n-1)(2n+3).(2n+5)}$$

$$= \frac{1}{(2n+1)(2n+3)} - \frac{2}{(2n+1).(2n+3).(2n+5)}$$

$$\therefore S = C - \frac{1}{2.(2n+1)} + \frac{1}{2.(2n+1).(2n+3)}$$

$$= C - \frac{n+1}{(2n+1).(2n+3)} = \frac{1}{3} - \frac{n+1}{(2n+1).(2n+3)}.$$

773. To sum $\frac{1}{1.3.3} - \frac{2}{3.5.3^2} + \frac{3}{5.7.3^3} - \dots \infty.$

Assume $1 - \frac{x}{3} + \frac{x^2}{5} - \dots\dots \infty = s$; multiply by $ax - b$

and arrange the terms in due order; then

$$\frac{3a-b}{1.3}x - \frac{5a-3b}{3.5}x^2 + \dots \infty = s.(ax-b) + b.$$

Let $3a - b = 1$ and $2a - 2b$ (the common difference of the numerators) $= 1$. Hence $a = \frac{1}{4}$ and $b = -\frac{1}{4}$.

Now $s = \frac{1}{x^{\frac{1}{2}}} \times (x^{\frac{1}{2}} - \frac{x^{\frac{3}{2}}}{3} + \frac{x^{\frac{5}{2}}}{5} - \ldots . \infty) = \frac{1}{x^{\frac{1}{2}}} \times \tan.^{-1} x^{\frac{1}{2}}$

(See *Lacroix*, or the new edition of *Simpson's* Fluxions.)

Let $x = \frac{1}{3}$. Then $s = \sqrt{3} \times \tan.^{-1} \frac{1}{\sqrt{3}}$

$= \sqrt{3}. \tan.^{-1} \frac{(\frac{1}{2})}{\sqrt{\frac{3}{4}}} = \sqrt{3} \tan.^{-1} (\tan. \frac{\pi}{6}) = \sqrt{3} \frac{\pi}{6} = \frac{\pi}{2\sqrt{3}}.$

Hence by substitution, we have

$$\frac{1}{1.3.3} - \frac{2}{3.5.3^2} + \ldots . \infty = \frac{\pi}{2.\sqrt{3}} \times \left(\frac{1}{4 \times 3} + \frac{1}{4}\right) - \frac{1}{4}$$
$$= \frac{\pi}{6\sqrt{3}} - \frac{1}{4}$$

To sum $\frac{2}{1.3.3} + \frac{3}{3.5.3^2} + \ldots \frac{n+1}{(2n-1).(2n+1).3^n} = \mathrm{S}.$

$$\Delta . \mathrm{S} = \frac{n+2}{(2n+1).(2n+3)3^{n+1}} = -\frac{3}{4} . \Delta \frac{1}{(2n+1).3^{n+1}}$$

$$\therefore \mathrm{S} = \mathrm{C} - \frac{1}{4} . \frac{1}{(2n+1).3^n} = \frac{1}{4} - \frac{1}{4} . \frac{1}{(2n+1).3^n}.$$

774. To sum $r - \frac{r^{\frac{3}{2}}}{2} + \frac{r^2}{4} - \ldots . n$ terms, which is a geometric series of the common ratio $(-\frac{r^{\frac{1}{2}}}{2})$, we have

$$\mathrm{S} = \frac{a\varrho^n - a}{\varrho - 1} \text{ (See Wood's Alg.) } = \frac{r \times \left(-\frac{r^{\frac{1}{2}}}{2}\right)^n - r}{-\frac{r^{\frac{1}{2}}}{2} - 1}$$

Hence $\pm (1 - \frac{r^{\frac{1}{2}} + 2}{2r} . \mathrm{S}) = \left(\frac{r^{\frac{1}{2}}}{2}\right)^n$ according as n is even or odd.

$\therefore \log. \{ \pm (1 - \frac{r^{\frac{1}{2}} + 2}{2r} . \mathrm{S})\} = \frac{n}{2} . \log. r - l. 2$, which, with the aid of the Tables, will give the value of S required.

To sum $1.2 + 2.5 + 3.8 + \ldots\, n.(3n-1) = S$.

$$\Delta\, S = (n+1).(3n+2) = 3n.(n+1) + 2.(n+1)$$

$$\therefore\; S = (n-1).\,n.(n+1) + n.(n+1) + C \;= n^2.(n+1) + C$$
$$= n^2.(n+1).$$

775. The series $1 + 2n + \dfrac{2n.(2n-1)}{1.2} + \dfrac{2n.(2n-1).(2n-2)}{1.2.3}$

$+ \ldots\; \dfrac{2n.(2n-1)\ldots.2n-\overline{p-2}}{1.2.3\ldots.p-1} + \ldots\; (2n+1)$ terms $= (1+1)^{2n}$

$= 2^{2n} = S$.

Now the middle term (T) being the $(n+1)^{\text{th}}$, we have

$$T = \frac{2n.(2n-1)\ldots..2n-\overline{n-1}}{1.2\ldots.n} = \frac{2n.(2n-1)\ldots.n+1}{1.2\ldots.n}$$

$$= \frac{2^n \times 1.3.5.7\ldots.\,2n-1 \times 2n.(2n-1)\ldots.(n+1)}{2.4.6\ldots.\,2n \times 1.3\ldots.\,2n-1}$$

$$= \frac{2^n \times 1.3.5.7\ldots.\,(2n-1) \times 2n.(2n-1)\ldots.(n+1)}{1.2.3\ldots.\,n.(n+1).(n+2)\ldots.2n}$$

$$= \frac{2^{2n} \times 1.3.5.7\ldots.\,(2n-1)}{2^n \times 1.2.3.4\ldots.n}$$

$$= \frac{S \times 1.3.5.7\ldots.2n-1}{2.4.6.8\ldots.\,2n}$$

$$\therefore\; S : T :: 2.4.6\ldots.\,2n : 1.3.5.7\ldots.\,2n-1.$$

776. To sum $\dfrac{1}{1\,.\,1} + \dfrac{1}{3\,.\,5} + \dfrac{1}{5\,.\,9}\ldots.\,\infty$.

$$dx + x^2dx + \ldots. = \frac{dx}{1-x^2}$$

$$\therefore\; x + \frac{x^3}{3} + \frac{x^5}{5} + \ldots. = \int\frac{dx}{1-x^2}$$

$$\therefore\; \frac{dx}{x^{\frac{1}{2}}} + \frac{x^{\frac{3}{2}}dx}{3} + \frac{x^{\frac{7}{2}}dx}{5} + \ldots. = \frac{dx}{x^{\frac{3}{2}}}.\int\frac{dx}{1-x^2}$$

$$\therefore\; \frac{x^{\frac{1}{2}}}{1} + \frac{x^{\frac{5}{2}}}{3.5} + \frac{x^{\frac{9}{2}}}{5.9} + \ldots = \frac{1}{2}\int\frac{dx}{x^{\frac{3}{2}}}.\int\frac{dx}{1-x^2}$$

$$= -\frac{1}{x^{\frac{1}{2}}}\int\frac{dx}{1-x^2}+\int\frac{x^{-\frac{1}{2}}dx}{1-x^2} = -\frac{1}{2\sqrt{x}}.l.\frac{1+x}{1-x}+\frac{1}{2}\times$$

$l.\dfrac{1+x}{1-x}+\tan.^{-1}x+C$ as we learn by putting $\sqrt{x}=u$, substituting, &c. &c.

$$\therefore \frac{x}{1.1}+\frac{x^3}{3.5}+\frac{x^5}{5.9}+\ldots\infty = \frac{\sqrt{x}-1}{2}.l.\frac{1+x}{1-x}$$

$+\sqrt{x}\tan.^{-1}x + C\sqrt{x}$, which being taken between $x=0$ and 1 gives

$$\frac{1}{1.1}+\frac{1}{3.5}+\frac{1}{5.9}+\ldots\infty = \tan.^{-1}1 = \frac{\pi}{4}.$$

This result may be verified by the well-known expression

$$\frac{\pi}{4} = 1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-\ldots\infty$$

$$= 1+\frac{2}{5}+\frac{2}{9}+\frac{2}{13}+\ldots-\frac{1}{3}-\frac{1}{5}-\frac{1}{7}-\ldots$$

$$= 1+\frac{2}{5}-\frac{1}{3}+\frac{2}{9}-\frac{1}{5}+\frac{2}{13}-\frac{1}{7}+\&c.$$

$$= 1+\frac{1}{3.5}+\frac{1}{5.9}+\ldots\infty$$ (by reducing to common denominators.)

777. To sum $\cos. a + \cos.(a+b) + \ldots \cos.(a+nb) = S$.

$$\Delta.S = \cos.(a+\overline{n+1}.b)$$

$$\therefore S = \Sigma.\cos.(a+\overline{n+1}.b) = \frac{\sin.(a+\overline{n+\frac{1}{2}}.b)}{2\sin.\frac{b}{2}}+C.$$ (See the *Appendix to Lacroix*, by J. F. W. Herschel, A.M., &c.)

Let $n=0$

$$\text{Then } C = \cos. a - \frac{\sin.\left(a+\frac{b}{2}\right)}{2\sin.\frac{b}{2}} = \frac{-\sin.\left(a-\frac{b}{2}\right)}{2\sin.\frac{b}{2}}$$

$$\therefore S = \frac{\sin.(a + \overline{n+\frac{1}{2}}.b) - \sin.\left(a - \frac{b}{2}\right)}{2\sin.\frac{b}{2}}$$

$$= \frac{\cos.\left(a + \frac{nb}{2}\right)\sin.\frac{(n+1)}{2}b}{\sin.\frac{b}{2}}.$$

Otherwise.

$$2\cos.a.\sin.\frac{b}{2} = \sin.\left(a + \frac{b}{2}\right) - \sin.\left(a - \frac{b}{2}\right)$$

$$2\cos.(a+b).\sin.\frac{b}{2} = \sin.\left(a + \frac{3b}{2}\right) - \sin.\left(a + \frac{b}{2}\right)$$

$$2\cos.(a+2b).\sin.\frac{b}{2} = \sin.\left(a + \frac{5b}{2}\right) - \sin.\left(a + \frac{3b}{2}\right)$$

$$\&c. = \&c.$$

$$2\cos.(a+\overline{n-1}.b.)\sin.\frac{b}{2} = \sin.(a+\overline{n-\frac{1}{2}}.b) - \sin.(a+\overline{n-\frac{3}{2}}.b)$$

$$2\cos.(a+nb).\sin.\frac{b}{2} = \sin.\ a+\overline{n+\frac{1}{2}}.b) - \sin.(a+\overline{n-\frac{1}{2}}.b)$$

$\therefore$ by adding crosswise, we have

$$2\sin.\frac{b}{2}\times S = \sin.(a + \overline{n+\frac{1}{2}}.\,b) - \sin.\left(a - \frac{b}{2}\right)$$

and $S = \dfrac{\sin.(a + \overline{n+\frac{1}{2}}.b) - \sin.\left(a - \frac{b}{2}\right)}{2\sin.\frac{b}{2}}$ which may be reduced as before.

778. To sum the $(p+1)^{\text{th}}$, $(p+n+1)^{\text{th}}$, $(p+2n+1)^{\text{th}}$, $(p+3n+1)^{\text{th}}$, &c. terms of the binomial $(a+x)^m$.

Let $A_1 = ma^{m-1}$, $A_2 = m.\frac{m-1}{2}.a^{m-2}$, $A_3 = m.\frac{m-1}{2}.\frac{m-2}{3}$ a^{m-3}, &c. = &c.

Then the sum may be expressed by

$$A_p x^p + A_{p+n} x^{p+n} + A_{p+2n} x^{p+2n} + A_{p+3n} x^{p+3n} + \&c. = \Sigma$$

which we are to express by a limited number of terms.

For this purpose, let

$r_1, r_2, r_3, \ldots\ldots r_n$ be the roots of $y^n - 1 = 0$

$$\begin{aligned} \text{And } S_1 &= r_1 + r_2 + r_3 + \ldots\ldots r^n \\ S_2 &= r_1^2 + r_2^2 + r_3^2 + \ldots\ldots r_n^2 \\ S_3 &= r_1^3 + r_2^3 + r_3^3 + \ldots\ldots r_n^3 \\ \&c. &= \&c. \end{aligned}$$

Then, since

$$(a+r_1x)^m = a^m + A_1 r_1 x + \ldots\ldots A_p r_1^p x^p + \ldots\ldots A_{p+n} r_1^{p+n} x^{p+n} + \ldots\ldots$$

$$\therefore (a+r_1x)^m . r_1^{n-p} = a^m r_1^{n-p} + \ldots\ldots A_p r_1^n \; x^p + \ldots\ldots A_{p+n} r_1^{2n} x^{p+n} + \ldots\ldots$$

In the same manner we get

$$(a+r_2x)^m r_2^{n-p} = a^m r_2^{n-p} + \ldots\ldots A_p r_2^n \; x^p + \ldots\ldots A_{p+n} r_2^{2n} x^{p+n} + \ldots\ldots$$

$$(a+r_3x)^m r_3^{n-p} = a^m r_3^{n-p} + \ldots\ldots A_p r_3^n \; x^p + \ldots\ldots A_{p+n} r_3^{2n} x^{p+n} + \ldots\ldots$$

$$\&c. = \&c.$$

$$(a+r_nx)^m r_n^{n-p} = a^m r_n^{n-p} + \ldots\ldots A_p r_n^n x^p + \ldots\ldots A_{p+n} r_n^{2n} x^{p+n} + \ldots\ldots$$

Whence

$$(a+r_1x)^m r_1^{n-p} + (a + r_2 x)^m r_2^{n-p} + \ldots\ldots (a+r_nx)^m r_n^{n-p} = a^m \times S_{n-p} + A_1 S_{n-p+1} x + A_2 S_{n-p+2} x^2 + \ldots A_p S_n x^p + A_{p+1} S_{n+1} x^{p+1} + A_{p+2} \times S_{n+2} x^{p+2} + \ldots\ldots + A_{p+n} S_{2n} x^{p+n} + A_{p+n+1} S_{2n+1} \; x^{p+n+1} + \ldots\ldots A_{p+2n} S^{3n} x^{p+2n} + \ldots\ldots$$

But S_{n-p}, $S_{n-p+1} \ldots\ldots S_{n-1}$, S_{n+1}, $S_{n+2} \ldots\ldots S_{2n-1}$, S_{2n+1}, $S_{2n+2} \ldots\ldots S_{3n-1}$, &c. &c. each $= 0$, and S_n, S_{2n}, S_{3n}, &c. each $= n$. (See 357.)

$$\therefore (a+r_1x)^m r_1^{n-p} + (a+r_2x)^m r_2^{n-p} + \ldots\ldots (a+r_nx)^m r_n^{n-p} = nA_p \times x^p + nA_{p+n} x^{p+n} + nA_{p+2n} x^{p+2n} + \&c. = n.\Sigma.$$

Hence then we have

$$\Sigma = \frac{1}{n} \times \{(a + r_1x)^m r_1^{n-p} + (a+ r_2x)^m r_2^{n-p} + \ldots\ldots (a+ r_nx)^m x^{n-p}\}$$

Take as an example; *required the sum of the even terms of* $(a+x)^m$, *beginning with the second.*

Here $p = 1$, $n = 2$, and $y^2 - 1 = 0$ gives $r_1 = 1$, $r_2 = -1$.

$\therefore \Sigma = \frac{1}{2} \times \{(a + x)^m - (a-x)^m\}$ which is verified by actually expanding by the Binomial Theorem.

779. To sum $\frac{8}{1.3.5.7} + \frac{16}{5.7.9.11} + \ldots\ldots$

$$\frac{8n}{(4n-3).(4n-1).(4n+1).(4n+3)}.$$

$$\Delta\,.\,S = \frac{8n+8}{(4n+1).(4n+3).(4n+5).(4n+7)} = \frac{1}{4} \times$$

$$\left(\frac{1}{(4n+1).(4n+3)} - \frac{1}{(4n+5).(4n+7)}\right) = -\frac{1}{4}.\ \Delta.\frac{1}{(4n+1).(4n+3)}.$$

$$\therefore\ S = C - \frac{1}{4.(4n+1).(4n+3)} = \frac{1}{12} - \frac{1}{4.(4n+1).(4n+3)}$$

$$= \frac{4}{3}.\frac{n.(n+1)}{(4n+1).(4n+3)}.$$

To sum $ax^{\alpha} + (a+b).\,x^{\alpha+\beta} + (a+2b)\,x^{\alpha+2\beta} + \ldots (a+\overline{n-1}.b) \times x^{\alpha+\overline{n-1}\beta}$.

$$\Delta\,.S = (a+nb)\,x^{\alpha+n\beta} = x^{\alpha} \times (a+nb)x^{n\beta}$$

$$\therefore\ S = x^{\alpha} \times \Sigma\,(a+nb)\,x^{n\beta}\ \text{(n being the only variable)}$$

$$= x^{\alpha} \times \left\{(a+nb)\,\Sigma.x^{n\beta} - \Delta.(a+nb).\ \Sigma^2.\,x^{\overline{n+1}.\beta}\right\}.\ \text{(See 654.)}$$

$$= x^{\alpha} \times \left\{(a+nb)\frac{x^{n\beta}}{x^{\beta}-1} - b.\,\frac{x^{\overline{n+1}.\beta}}{(x^{\beta}-1)^2}\right\} + C$$

Let $n = 1$

$$\text{Then } C = ax^{\alpha} - \frac{x^{\alpha+\beta}}{(x^{\beta}-1)^2}.\,(ax^{\beta} - \overline{a+b})$$

$$\therefore\ S = \frac{x^{\alpha+n\beta}}{(x^{\beta}-1)^2} \times \{(a+\overline{n-1}.b)\,x^{\beta} - \overline{a+nb}\} + ax^{\alpha}$$

$$- \frac{x^{\alpha+\beta}}{(x^{\beta}-1)^2} \times \{ax^{\beta} - \overline{a+b}\} = ax^{\alpha} + \frac{x^{\alpha+\beta}}{(x^{\beta}-1)^2} \times \{(a+$$

$$\overline{n-1}.b)\,x^{n\beta} - (a+nb)\,x^{(n-1)\beta} - ax^{\beta} + a + b\}.$$

780. To sum $\frac{1}{\sqrt{2}\,(1+\sqrt{2})} + \frac{1}{(1+\sqrt{2}).(2+\sqrt{2})} + \ldots\ldots\infty$.

$$\left.\begin{array}{l}\text{Let } \dfrac{1}{\sqrt{2}} + \dfrac{1}{1+\sqrt{2}} + \dfrac{1}{2+\sqrt{2}} + \ldots . \infty = S \\ \text{Then } \dfrac{1}{1+\sqrt{2}} + \dfrac{1}{2+\sqrt{2}} + \dfrac{1}{3+\sqrt{2}} + \ldots . \infty = S - \dfrac{1}{\sqrt{2}}\end{array}\right\}$$

$\therefore$ by subtraction

$$\frac{1}{\sqrt{2}.(1+\sqrt{2})} + \frac{1}{(1+\sqrt{2}).(2+\sqrt{2})} + \ldots . \infty = \frac{1}{\sqrt{2}}$$

Otherwise.

$$\text{Let } \frac{1}{\sqrt{2}.(1+\sqrt{2})} + \frac{1}{(1+\sqrt{2})(2+\sqrt{2})} + \ldots .$$

$$\frac{1}{(n-1+\sqrt{2}).(2+\sqrt{2})} = S.$$

$$\text{Then } \Delta\ S = \frac{1}{(n+\sqrt{2}).(n+1+\sqrt{2})} \text{ and integrating,}$$

$$S = C - \frac{1}{(n+\sqrt{2})} = \frac{1}{\sqrt{2}} - \frac{1}{n+\sqrt{2}} = \frac{n}{\sqrt{2}(n+\sqrt{2})}.$$

Let $n = \infty$

and S becomes $\dfrac{1}{\sqrt{2}}$, as before.

781. To sum. $\tan. A + \dfrac{1}{2}\tan.\dfrac{A}{2} + \dfrac{1}{4}\tan.\dfrac{A}{4} + \ldots . \infty.$

$$\text{Let } \tan. A + \frac{1}{2}\tan.\frac{A}{2} + \ldots . \frac{1}{2^{n-1}}\tan.\frac{A}{2^{n-1}} = S$$

$$\text{Then } \because \cot. 2\theta = \frac{1}{\tan. 2\theta} = \frac{1-\tan.^2\theta}{2\tan.\theta} = \frac{1}{2}.\cot.\theta - \frac{1}{2}.\tan.\theta$$

And $\therefore \tan.\theta = \cot.\theta - 2\cot. 2\theta$, we have

$$\tan. A = \cot. A - 2\cot. 2A$$

$$\frac{1}{2}.\tan.\frac{A}{2} = \frac{1}{2}\cot.\frac{A}{2} - \cot. A$$

$$\frac{1}{4}.\tan.\frac{A}{4} = \frac{1}{4}\cot.\frac{A}{4} - \frac{1}{2}\cot.\frac{A}{2}$$

&c. = &c.

$$\frac{1}{2^{n-2}}.\ \tan.\ \frac{A}{2^{n-2}} = \frac{1}{2^{n-2}}.\ \cot.\ \frac{A}{2^{n-2}} - \frac{1}{2^{n-3}}.\ \cot.\ \frac{A}{2^{n-3}}$$

$$\frac{1}{2^{n-1}}.\ \tan.\ \frac{A}{2^{n-1}} = \frac{1}{2^{n-1}}.\ \cot.\ \frac{A}{2^{n-1}} - \frac{1}{2^{n-2}}.\ \cot.\ \frac{A}{2^{n-2}}$$

and, adding crosswise, we get

$$S = \frac{1}{2^{n-1}}.\ \cot.\ \frac{A}{2^{n-1}} - 2\ \cot.\ 2A.$$

Let $n = \infty$

Then $\frac{1}{2^{n-1}}.\cot.\frac{A}{2^{n-1}} = \frac{1}{\infty} \times \cot.\ 0 = \frac{1}{\infty}\ \infty = \frac{\frac{1}{\infty}}{\frac{1}{\infty}} = \frac{0}{0}$ which

we must investigate by the usual methods, as follows.

By the method of indeterminate coefficients it will be found that

$$\cot.\ \theta = \frac{1}{\theta} - \frac{\theta}{3} - \frac{\theta^3}{45} - \frac{2\theta^5}{945} - \ldots.$$

Hence $\cot.\ \frac{A}{2^{n-1}} = \frac{2^{n-1}}{A} - \frac{A}{3.2^{n-1}} - \frac{A^3}{45 \times 2^{3n-3}} - \&c.$

$$\therefore\ \frac{\cot.\ \frac{A}{2^{n-1}}}{2^{n-1}} = \frac{1}{A} - \frac{A}{3.2^{n-2}} - \frac{A^3}{45 \times 2^{4n-4}} - \&c.$$

Let $n = \infty$. Then $\frac{\cot.\ \frac{A}{2^{n-1}}}{2^{n-1}} = \frac{1}{A}$, and we have

$$S = \frac{1}{A} - 2\ \cot.\ 2A.$$

782. To prove that $\tan.^{-1}\frac{x}{y} = \tan.^{-1}\frac{ex-y}{ey+x} + \tan.^{-1}\frac{e_1-e}{ee_1+1} +$ $\tan.^{-1}\frac{e_2-e_1}{e_1e_2+1} + \ldots.\tan.^{-1}\frac{e_n-e_{n-1}}{e_{n-1}e_n+1} + \tan.^{-1}\frac{1}{e^n}$, we have by the form $\tan.\ (\theta-\phi) = \frac{\tan.\ \theta - \tan.\ \phi}{1+\tan.\ \theta.\ \tan.\ \phi}$'

$$\tan.^{-1}\frac{ex-y}{ey+x}=\tan.^{-1}\frac{\frac{x}{y}-\frac{1}{e}}{1+\frac{x}{ey}}=\tan.^{-1}\frac{x}{y}-\tan.^{-1}\frac{1}{e}$$

$$\tan.^{-1}\frac{e_1-e}{ee_1+1}=\tan.^{-1}\frac{\frac{1}{e}-\frac{1}{e_1}}{1+\frac{1}{ee_1}}=\tan.^{-1}\frac{1}{e}-\tan.^{-1}\frac{1}{e_1}$$

$$\tan.^{-1}\frac{e_2-e_1}{e_1e_2+1}=\tan.^{-1}\frac{\frac{1}{e_1}-\frac{1}{e_2}}{1+\frac{1}{e_1e_2}}=\tan.^{-1}\frac{1}{e_1}-\tan.^{-1}\frac{1}{e_2}$$

&c. = &c.

$$\tan.^{-1}\frac{e_n-e_{n-1}}{e_n.e_{n-1}+1}=\tan.^{-1}\frac{\frac{1}{e_{n-1}}-\frac{1}{e_n}}{1+\frac{1}{e_{n-1}e_n}}=\tan.^{-1}\frac{1}{e_{n-1}}-\tan.^{-1}\frac{1}{e^n}$$

Hence, adding crosswise, we get

$$\tan.^{-1}\frac{x}{y}=\tan.^{-1}\frac{ex-y}{ey+x}+\tan.^{-1}\frac{e_1-e}{ee_1+1}+\tan.^{-1}\frac{e_2-e_1}{e_1e_2+1}$$

$$+\ldots\ \tan._{-1}\frac{e_{n-1}-e_n}{e_{n-1}e_n+1}.$$

From this Theorem many curious results may be derived, as

$$\frac{\pi}{4}=\tan.^{-1}\frac{1}{2}+\tan.^{-1}\frac{1}{2.2^2}+\tan.^{-1}\frac{1}{2.3^2}+\ldots\ \infty$$

$$\frac{\pi}{4}=2\tan.^{-1}\frac{1}{3}+\tan.^{-1}\frac{1}{7}.$$

$$\frac{\pi}{4}=\tan.^{-1}\frac{1}{2}+\tan.^{-1}\frac{1}{3},\ \text{\&c. \&c.}$$

783. To prove that

$$lc=l.a-\frac{b}{a}.\cos.C-\frac{b^2}{2a^2}.\cos.2C-\frac{b^3}{3a^3}\cos.3C-\ldots\ \infty$$

a, *b*, *c* being the sides of a plane triangle, and C the angle subtended by *c*.

By trigonometry we have

$$\cos. C = \frac{a^2 + b^2 - c^2}{2ab} = \text{also } \frac{e^{c\sqrt{-1}} + e^{-c\sqrt{-1}}}{2}$$

Whence $\frac{c^2}{a^2} = 1 + \frac{b^2}{a^2} - \frac{b}{a}. e^{c\sqrt{-1}} - \frac{b}{a} e^{-c\sqrt{-1}}$

$$= 1 - \frac{b}{a} e^{c.\sqrt{-1}} + \frac{b}{a} \times \frac{\left(\frac{b}{a} e^{c\sqrt{-1}} - 1\right)}{e^{c\sqrt{-1}}}$$

$$= \left(1 - \frac{b}{a} e^{c\sqrt{-1}}\right) \times \left(1 - \frac{b}{a} e^{-c\sqrt{-1}}\right)$$

and taking the hyperbolic logarithms, we have

$$2 l. \frac{c}{a} = l.\left(1 - \frac{b}{a} e^{c\sqrt{-1}}\right) + l. \left(1 - \frac{b}{a}. e^{-c\sqrt{-1}}\right)$$

$$= \left\{ \begin{array}{l} -\frac{b}{a} e^{c\sqrt{-1}} - \frac{b^2}{2a^2} e^{2c\sqrt{-1}} - \frac{b^3}{3a^3} e^{3c\sqrt{-1}} - \ldots . \infty \\ -\frac{b}{a}. e^{-c\sqrt{-1}} - \frac{b^2}{2a^2} e^{-2c\sqrt{-1}} - \frac{b^3}{3a^3} e^{-3c\sqrt{-1}} - \ldots . \infty \end{array} \right.$$

$$= -\frac{b}{a}. \left(e^{c\sqrt{-1}} + e^{-c\sqrt{-1}}\right) - \frac{b^2}{2a^2} \left(e^{2c\sqrt{-1}} + e^{-2c\sqrt{-1}}\right) + \&c.$$

$$= -2 \frac{b}{a} \cos. C - 2. \frac{b^2}{2a^2} \cos. 2 C - \&c.$$

by the known expressions

$$l. (1 - x) = - x - \frac{x^2}{2} - \frac{x^3}{3} - \ldots . \infty$$

$$\text{and } \cos. \theta = \frac{e^{\theta\sqrt{-1}} + e^{-\theta\sqrt{-1}}}{2}$$

Hence $l. \frac{c}{a} = - \frac{b}{a}. \cos. C - \frac{b^2}{2a^2}. \cos. 2C - \frac{b^3}{3a^3} \cos. 3C - \&c.$

which is *equivalent* to the formula required.

In a similar manner it may be shewn that

$B = \frac{b}{a} \sin. C + \frac{b^2}{2a^2} \sin. 2 C + \frac{b^3}{3a^3} \sin. 3C + \&c.$ B being the angle subtending b.

For it may easily be proved that

$$e^{2B\sqrt{-1}} = \frac{1 - \frac{b}{a}e^{-C\sqrt{-1}}}{1 + \frac{b}{a}e^{C\sqrt{-1}}}$$ which will easily, by taking the logarithms, &c., give the formula in question.

784. To sum $\frac{1}{\sqrt{x}} - \frac{\sqrt{2}}{x} + \frac{2}{x\sqrt{x}} - \ldots\ n$ terms, which is a geometric progression of the common ratio $\sqrt{\frac{2}{x}}$, we have

$$S = \frac{ar^n - a}{r - 1} = \frac{\frac{1}{\sqrt{x}} \times \left(\frac{2}{x}\right)^{\frac{n}{2}} - \frac{1}{\sqrt{x}}}{\sqrt{\frac{2}{x}} - 1} = \frac{\left(\frac{2}{x}\right)^{\frac{n}{2}} - 1}{\sqrt{2} - \sqrt{x}}$$

$$\therefore (\sqrt{2} - \sqrt{x}).\ S + 1 = \left(\frac{2}{x}\right)^{\frac{n}{2}}$$

and $l.\ (\overline{\sqrt{2} - \sqrt{x}.\ S} + 1) = \frac{n}{2}.\ (l.2 - lx)$ will give, by reference to the tables, the value of S.

To continue the Harmonic Progression .... 3, 4, 6 .... upwards and downwards, since the reciprocals of numbers in Harmonic Progression are in Arithmetic Progression, we must in the former case continually subtract the common difference of the denominators, and in the latter continually add that difference. To determine this difference, put

$$\frac{1}{a} = 3, \text{ and } \frac{1}{a-d} = 4$$

Then $a = \frac{1}{3}$, $a - d = \frac{1}{4}$ and $\therefore d = a - \frac{1}{4} = \frac{1}{12}$.

Hence the n^{th} term, either way from 3, will be represented by

$$T_n = \frac{1}{a \mp nd} = \frac{1}{\frac{1}{3} \mp \frac{n}{12}} = \frac{12}{4 \mp n}$$

Putting $\therefore n = \mp 1, \mp 2$, &c. we have

$\infty \ldots \frac{3}{2}, \frac{12}{7}, 2, \frac{12}{5}, 3, 4, 6, 12$, than which we cannot go farther so as to have *all the terms finite*, because the next term $= \frac{12}{4-4} = \frac{12}{0}$ an indefinite expression.

To shew the law and limits of continuation of an Harmonic Series, two of whose consecutive terms are

t, t', let

$$\frac{1}{a} = t \text{ and } \frac{1}{a-d} = t'$$

Hence $a = \frac{1}{t}$, and $d = a - \frac{1}{t'} = \frac{t'-t}{tt'}$.

$\therefore$ the general term is

$T_n = \frac{1}{a \mp nd} = \frac{tt'}{t' \mp n.(t'-t)}$ which becomes indefinite in the case of

$$t' - n.(t' - t) = 0, \text{ or when}$$

$$n = \frac{t'}{t'-t}$$

Hence it appears that the series may be continued both ways without limit when $\frac{t'}{t'-t}$ is not an integer, and that when it is limited it must be at the $\frac{t'}{t'-t}$th term upwards.

785. To sum $\frac{3}{1.2.2} + \frac{4}{2.3.2^2} + \ldots \frac{n+2}{n.(n+1).2^n}$ See 777.

To sum $\sin. x + \sin. 3x + \ldots \sin. (2n-1)x = S$.

By the form $2 \sin. P . \sin. Q = \cos. (P - Q) - \cos. (P + Q)$, we have

$$2 \sin. x . \sin. x = \cos. 0 - \cos. 2x$$
$$2 \sin. 3x . \sin. x = \cos. 2x - \cos. 4x$$
$$2 \sin. 5x . \sin. x = \cos. 4x - \cos. 6x$$
$$\&c. = \&c.$$

$$2\sin.(2n-3)x\sin.x = \cos.(2n-4)x - \cos.(2n-2)x$$
$$2\sin.(2n-1)x\sin.x = \cos.(2n-2)x - \cos.2nx$$

$\therefore$ adding crosswise, and dividing by $2\sin.x$, there results

$$S = \frac{\cos.0-\cos.2nx}{2\sin.x} = \frac{1-\cos.2nx}{2\sin.x} = \frac{\sin.^2 nx}{\sin.x}.$$

See 729.

786. To prove that if $S = 1 + \frac{1}{2^m} + \frac{1}{3^m} + \ldots\ldots\infty$, and $s = 1 - \frac{1}{2^m} + \frac{1}{3^m} \ldots\ldots\infty$, $S : s :: 2^{m-1} : 2^{m-1}-1$, we have

$$S = 1 + \frac{1}{2^m} + \frac{1}{3^m} + \frac{1}{4^m} + \frac{1}{5^m} + \frac{1}{6^m} + \frac{1}{7^m} \ldots\ldots$$

$$\text{and } \frac{2S}{2^m} = \frac{2}{2^m} + \frac{2}{4^m} + \frac{2}{6^m} + \ldots\ldots$$

$$\therefore S.\left(1-\frac{1}{2^{m-1}}\right) = 1 - \frac{1}{2^m} + \frac{1}{3^m} - \frac{1}{4^m} + \frac{1}{5^m} - \frac{1}{6^m} \ldots\ldots$$ by subtraction.

$$\therefore S.\left(1-\frac{1}{2^{m-1}}\right) = s$$

and $\frac{S}{s} = \frac{2^{m-1}}{2^{m-1}-1}$ the relation required.

787. To sum $\frac{1}{1.3} - \frac{1}{2.4} + \frac{1}{3.5} - \ldots\ldots\infty$. See 706.

To sum $\frac{2}{1.3.7} + \frac{4}{3.7.15} + \frac{8}{7.15.31} + \ldots\ldots$

$$\frac{2n}{(2^n-1)(2^{n+1}-1)(2^{n+2}-1)} = S.$$

$$\Delta.S = \frac{2^{n+1}}{(2^{n+1}-1)(2^{n+2}-1)(2^{n+3}-1)}$$

$$= -\frac{1}{3}.\left\{\frac{1}{(2^{n+2}-1).(2^{n+3}-1)} - \frac{1}{(2^{n+1}-1).(2^{n+2}-1)}\right\}$$

$$= -\frac{1}{3}\Delta : \frac{1}{(2^{n+1}-1).(2^{n+2}-1)}.$$

$$S = C - \frac{1}{3.(2^{n+1}-1)\ (2^{n+2}-1)} = \frac{1}{9} -$$

$$\frac{1}{3.(2^{n+1}-1)(2^{n+2}-1)}.$$ See 750.

To prove that

$$1^2 + n^{.2} + \left(n.\frac{n-1}{2}\right)^2 + \ldots.(n+1) \text{ terms} = \frac{1.2.3\ \ldots.\ 2n^2}{(1.2.3\ldots.n)}$$

we have (calling $1, n, n.\frac{n-1}{2}, n.\frac{n-1}{2}.\frac{n-2}{3}$, &c. $T_0, T_1, T_2 \ldots.$ T_n respectively)

$$(1+x)^n = 1 + T_1 x + T_2 x^2 + T_3 x^3 + \ldots.\ T_n x^n$$

which being multiplied into itself will give,

$$\begin{aligned}(1+x)^{2n} = T_0 + T_1 x + T_2 x^2 + T_3 x^3 + \ldots..\ T_0 T_n x^n \\ + T_1 x + T_1^2 x^2 + T_1 T_2 x^3 + \ldots.\ T_1 T_{n-1} x^n + \ldots. \\ + T_2 x^2 + T_2 T_1 x^3 + \ldots.\ T_2 T_{n-2} x^n + \ldots. \\ + T_3 x^3 + \ldots.\ T_3 T_{n-3} x^n + \ldots.. \\ \&c. \quad \&c. \\ T_n T_0 x^n\end{aligned}$$

Also by the Binomial Theorem, we have

$$(1 + x)^{2n} = 1 + 2nx + 2n.\frac{2n-1}{2}x^2 + \ldots\ldots$$

$$+ \frac{2n.(2n-1)(2n-2)\ldots.(n+1)}{1.2\ldots..n}x^n + \&c.$$

$\therefore$ equating coefficients of the same power (n) of x, we get

$$T_0 T_n + T_1 T_{n-1} + T_2 T_{n-2} + T_3 T_{n-3} + \ldots.T_{n-1} T_1 + T_n T_0 =$$

$$\frac{2n.(2n-1)\ldots.(n+1)}{1.2.3\ldots.n} = \frac{2n.(2n-1)\ldots(n+1).n.(n-1)\ldots3.2.1}{(1.2.3\ldots.n)^2}$$

$$= \frac{1.2.3\ldots.2n}{(1.2.3\ldots.n)^2}.$$

But the coefficients of the terms of a binomial expanded, equally distant from either extremity, being equal,

$$T_0 = T_n = 1, T_1 = T_{n-1} = n, T_2 = T_{n-2} = n \cdot \frac{n-1}{2}, \text{ \&c.}$$

$$\therefore 1 + n^2 + \left(n \cdot \frac{n-1}{2}\right)^2 + \dots n+1 \text{ terms} = \frac{1.2.3 \dots 2n}{(1.2.3 \dots n)^2}$$

$= \text{also } 2^n \times \dfrac{1 \cdot 3 \cdot 5 \dots 2n-1}{1 \cdot 2 \cdot 3 \dots 2n}$ (see 780) which is a more simple expression.

788. To sum $\dfrac{1}{1 \cdot 4} - \dfrac{1}{3 \cdot 6} + \dfrac{1}{5 \cdot 8} - \dots \infty$. See 731.

To sum $x \cdot \cos. A + x^2 \cos. 2A + x^3 \cos. 3A + \dots \infty = S$.

By the known form

$$\text{Cos.} (P + Q) = 2 \cos. P \cdot \cos. Q. - \cos. (P - Q)$$

we have $\cos. (m + 1) A = 2 \cos. mA \cdot \cos. A - \cos. (m - 1) A$.

Hence
$$\begin{aligned}
x \cos. A &= x \cos. A \\
x^2 \cos. 2A &= x^2 \cos. 2A \\
x^3 \cos. 3A &= 2x^3 \cos. A \cdot \cos. 2A - x^3 \cos. A \\
x^4 \cos. 4A &= 2x^4 \cos. A \cdot \cos. 3A - x^4 \cos. 2A \\
x^5 \cos. 5A &= 2x^5 \cos. A \cdot \cos. 4A - x^5 \cos. 3A \\
\text{\&c.} &= \text{\&c.}
\end{aligned}$$

$\therefore S = x \cdot \cos. A + x^2 \cos. 2A + 2x \cos. A \times (S - x \cdot \cos. A) - x^2 \times S$; whence

$$S = \frac{x \cdot \cos. A + x^2 \cdot (\cos. 2A - 2 \cos.^2 A)}{1 - 2x \cdot \cos. A + x^2}$$

$$= \frac{x \cdot \cos. A - x^2}{1 - 2x \cdot \cos. A + x^2}.$$

The *scale of relation* of this *recurring series* is $\therefore 2 \cos. A - 1$.

789. To sum $\dfrac{3}{1 \cdot 2 \cdot 2} + \dfrac{4}{2 \cdot 3 \cdot 2^2} + \dots \infty$.

$$\text{Assume } 1 + \frac{x}{2} + \frac{x^2}{3} + \dots \infty = s$$

Then, multiplying by $ax - b$, reducing to a common denominator, &c., we get

$$\frac{2a-b}{1.2}x + \frac{3a-2b}{2.3}x^2 + \ldots\ldots \infty = (ax-b)s + b$$

Let $2a - b = 3$

And $a - b = 1$ the common difference of the numerator.

Hence $a = 2$, and $b = 1$, and

$$\frac{3}{1.2}x + \frac{4}{2.3}.x^2 + \ldots\ldots \infty = (2x-1)s + 1$$

$$\text{Let } x = \frac{1}{2}$$

$$\text{Then } \frac{3}{1.2.2} + \frac{4}{2.3.2^2} + \ldots\ldots \infty = 0 + 1 = 1.$$

To sum $1.2 + 2.3 + \ldots n.(n+1)$, see 685.

790. To sum $1.2.4 + 2.3.5 + \ldots n.(n+1)\times(n+3) = S$.

$$\Delta S = (n+1).(n+2).(n+4)$$
$$= n.(n+1).(n+2) + 4(n+1).(n+2)$$
$$\therefore S = \frac{(n-1).(n+1).(n+2)}{4} + \frac{4n.(n+1).(n+2)}{3} + C$$
$$= \frac{3n+13}{12} \times n.(n+1).(n+2).$$

To sum $1 + \frac{1}{3.5} + \frac{1}{5.9} + \frac{1}{7.13} + \ldots\ldots \infty.$

Since $dx + x^2dx + x^4dx + \ldots\ldots \infty = \frac{dx}{1-x^2}$.

$$x + \frac{x^3}{3} + \frac{x^5}{5} + \ldots \infty = \int\frac{dx}{1-x^2} = \frac{1}{2}l.\frac{1+x}{1-x}.$$

Also $x + \frac{x^5}{5} + \frac{x^9}{9} + \ldots \infty = \int\frac{dx}{1-x^4} = \frac{1}{2}\int\frac{dx}{1-x^2}$

$$+ \frac{1}{2}.\int\frac{dx}{1+x^2} = \frac{1}{4}.l.\frac{1+x}{1-x} + \frac{1}{2}.\tan.^{-1}x.$$

Let $x = 1$, and multiply the latter series by 2; then we have

$$\left.\begin{aligned} 2 + \frac{2}{5} + \frac{2}{9} + \dots\infty &= \frac{1}{2}\,.\,l.\,\frac{2}{0} + \frac{\pi}{4} \\ \text{and } 1 + \frac{1}{3} + \frac{1}{5} + \dots\infty &= \frac{1}{2}\ l.\ \frac{2}{0} \end{aligned}\right\}$$

Whence, by subtraction,

$$1 + \frac{1}{3\,.\,5} + \frac{1}{5\,.\,9} + \dots\infty = \frac{\pi}{4}.$$

791. Given $1 + \frac{1}{2^2} + \frac{1}{3^2} + \dots\infty = \frac{\pi^2}{6}$ (see 684) to find the sum of $\frac{1}{1^2\,.\,2} + \frac{1}{2^2\,.\,3} + \dots\infty = S$.

The general term $\frac{1}{n^2\,.\,(n+1)}$ may be decomposed by the usual methods into

$$\frac{1}{n^2} - \frac{1}{n} + \frac{1}{n+1}$$

$$\therefore S = \left(1 + \frac{1}{2^2} + \frac{1}{3^2} + \dots\infty\right) - \left(\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \dots\right)$$

$$+ \left(\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots\right) = \frac{\pi^2}{6} - 1. \quad \text{(See 700).}$$

792. To sum $\frac{5}{1\,.\,2\,.\,3} + \frac{6}{2\,.\,3\,.\,4} + \dots \frac{n+4}{n.(n+1)(n+2)} = S.$

$$\Delta\, S = \frac{n+5}{(n+1).(n+2).(n+3)} = \frac{1}{(n+1)\,.\,(n+2)}$$

$$+ \frac{2}{(n+1).(n+2).(n+3)}.$$

$$\therefore S = C - \frac{1}{n+1} - \frac{1}{(n+1)\,.\,(n+2)}$$

$$= \frac{3}{2} - \frac{n+3}{(n+1)\,.\,(n+2)}.$$

To sum $\frac{6^2}{1.2.3.4} + \dots \frac{(n+5)^2}{n.(n+1).(n+2)(n+3)} = S.$

$$\Delta S = \frac{(n+6)^2}{(n+1).(n+2).(n+3).(n+4)}$$

$$= \frac{(n+3).(n+4) + 5.(n+4) + 4}{(n+1)(n+2)(n+3)(n+4)} = \frac{1}{(n+1).(n+2)}$$

$$+ \frac{5}{(n+1).(n+2)(n+3)} + \frac{4}{(n+1)\dots\dots(n+4)},$$

$$\therefore S = C - \left(\frac{1}{n+1} + \frac{5}{2.(n+1).(n+2)} + \frac{4}{3.(n+1).(n+2)(n+3)}\right)$$

$$= \frac{89}{36} - \frac{6n^2+45n+89}{6.(n+1).(n+2).(n+3)}.$$

PROBABILITIES.

793. The probability of failing both throws with the single die, is

$$P = \frac{5}{6} \times \frac{5}{6} = \frac{25}{36}$$

$\therefore$ the probability that once *at least* the ace will be thrown is

$$1 - \frac{25}{36} = \frac{11}{36}.$$

Again, omitting to consider the aces, the number of chances of failing to throw one ace or two is

$$\frac{10.9}{2} = 45$$

and the whole number of chances is

$$\frac{12.11}{2} = 66$$

$\therefore$ the probability of not throwing one ace at least is

$$\frac{45}{66} = \frac{15}{22}.$$

Consequently

$$P' = 1 - \frac{15}{22} = \frac{7}{22}$$

is the probability of throwing at least an ace in one trial with two dice, and we have

$$P : P' :: \frac{11}{36} : \frac{7}{22}$$

$$:: 11^2 : 7 \times 18$$

$$:: 121 : 126$$

794. If $\frac{a}{a+b}$ be the probability of an event's happening

in one trial; then by *Wood's Algeb.* p. 270, or any other elementary Treatise on the subject, it is shewn that the probability of its happening *at least* t times in n trials is,

$$\frac{a^n + na^{n-1}b + n\,.\frac{n-1}{2}\,a^{n-2}b^2 + \text{\&c. to } n-t+1 \text{ terms}}{(a+b)^n}.$$

Now by the problem, $a = 1$, $b = p - 1$

$$\therefore\ \mathrm{P} = \frac{1+n(p-1)+n.\frac{n-1}{2}\,(p-1)^2 + \text{\&c. to } n-1+1 \text{ terms}}{p^n}.$$

795. To generalize the problem somewhat, suppose, other circumstances remaining the same, that the number of bowls each plays with is n.

Then if the probability of A getting one ball or more at the first end is $\frac{1}{2}$; of his getting a second in is $\frac{n-1}{2n-1}$. Consequently that of his winning one *precisely* is

$$\frac{1}{2} - \frac{1}{2}\,\frac{n-1}{2n-1} = \frac{n}{2\,(2n-1)}.$$

Hence the probability of A's winning in the above manner is

$$\mathrm{P} = \frac{n}{4\,(2n-1)} \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ (1)$$

Again, since the probability of A's winning two or more the first end is

$$\frac{1}{2}\cdot\frac{n-1}{2n-1}$$

and that of his winning three or more is

$$\frac{1}{2}\cdot\frac{n-1}{2n-1}\cdot\frac{n-2}{2n-2} = \frac{n-2}{4\,(2n-1)}$$

$\therefore$ the probability of his winning two exactly is

$$\frac{1}{2}\cdot\frac{n-1}{2n-1} - \frac{n-2}{4\,(2n-1)} = \frac{n}{4\,(2n-1)}.$$

But since he then wants one of the game and B two, if B wins one at an end, they become equal; hence

B's chance of winning this way is

$$\frac{1}{4}\ \frac{n}{2n-1}$$

and his chance of winning by two or more is

$$\frac{1}{2}\cdot\frac{n-1}{2n-1}$$

∴ B's chance of winning is

$$\frac{1}{4}\ \frac{n}{2n-1}+\frac{1}{2}\ \frac{n-1}{2n-1}=\frac{3n-2}{8n-4}.$$

Hence A's chance of winning the game after winning two the first end is

$$1-\frac{3n-2}{8n-4}=\frac{5n-2}{8n-4}$$

and his chance of winning this second way is

$$P'=\frac{n}{4(2n-1)}\times\frac{5n-2}{8n-4}$$

$$=\frac{n(5n-2)}{16(2n-1)^2}\ \ldots\ldots\ldots\ldots\ (2)$$

Again, the probability of A's winning three or more at the first end is

$$\frac{1}{2}\cdot\frac{n-1}{2n-1}\cdot\frac{n-2}{2n-2}=\frac{1}{4}\cdot\frac{n-2}{2n-1},$$

A may win this way and we have

$$P''=\frac{1}{4}\cdot\frac{n-2}{2n-1}\ \ldots\ldots\ldots\ldots\ (3)$$

Again, suppose B to win one bowl *precisely* at the first end, then A wants three and B one of the game. Now to calculate A's chance on this supposition, we have

A's chance of winning precisely at first end

$$\frac{1}{2}\ \frac{n}{2n-1}$$

and his chance of winning the game afterwards

$$\frac{3n-2}{8n-4}.$$

$\therefore \dfrac{n.(3n-2)}{8(2n-1)^2}$ is part of A's chance of winning on this hypothesis. But A's chance of getting two precisely, the first end, being

$$\frac{1}{4}\cdot\frac{n}{2n-1}$$

which makes him even with B gives

$$\frac{1}{8}\cdot\frac{n}{2n-1}$$

for the second part of A's chance.

Also A's chance of winning three or more, *viz.*

$$\frac{1}{2}\cdot\frac{n-1}{2n-1}\cdot\frac{n-2}{2n-2}=\frac{1}{4}\cdot\frac{n-2}{2n-1}$$

gives the other part of his chance of winning the game. Hence the probability of A winning the game this third way is

$$P''' = \frac{n}{2\,(2n-1)}\left(\frac{n\,.\,(3n-2)}{8\,(2n-1)^2}+\frac{1}{8}\cdot\frac{n}{2n-1}+\frac{1}{4}\cdot\frac{n-2}{2n-1}\right)$$

$$= \frac{n\,(9n^2-13\,n+4}{16\,(2n-1)^3}\ldots\ldots\ldots\ldots\ldots\ldots(4)$$

Hence the total value of A's chance of winning is

$$P'+P'+P''+P''' = \frac{n}{4\,(2n-1)}+\frac{n\,.\,(5n-2)}{16\,(2n-1)^2}+\frac{1}{4}\cdot\frac{n-2}{2n-1}$$

$$+\frac{n\,.(9n^2-13n+4)}{16\,(2n-1)^3}=\frac{n-1}{2\,(2n-1)}+\frac{n\,(19n^2-22n+6)}{16\,(2n-1)^3}$$

$$=\frac{51n^3-86n^2+46n-8}{16\,(2n-1)^3}.$$

796. There are four ways in which

$$\left.\begin{array}{l}1,\ 2,\ 3,\ 4,\ 5,\ 6\\ 1,\ 2,\ 3,\ 4,\ 5,\ 6\end{array}\right\}$$

may be combined, taking one from each row, so as to make 5, *viz.*

$1 + 4 = 5, 2 + 3 = 5, 3 + 2 = 5, 4 + 1 = 5$; and in like manner it may be shewn that there are six ways in which seven may be formed from them.

Again, since each figure of the one line may be combined with each one of the other, there are on the whole

6×6, or 36 combinations.

Hence $36 - (6 + 4) = 26$, $36 - 6 = 30$, $36 - 4 = 32$, are the chances for throwing neither 5 nor 7, for throwing the 5 and for throwing the 7 the first throw, respectively.

Hence in three throws the varieties are as follows,

The total $= 36^3$

The number of chances of throwing neither 5 nor 7 $= 26^3$;

That of not throwing 7 once $= 30^3$;

That of not throwing 5 once is 32^3.

Hence

$36^3 - 30^3 =$ the number of chances for throwing 7 at least once, not excluding the chances for 5.

$32^3 - 26^3 =$ ditto for throwing 7 at least once without the possibility of throwing a 5.

$\therefore 36^3 - 30^3 - (32^3 - 26^3) = 4464$ is the number of chances of throwing at least once in three trials.

$$\therefore \frac{4464}{36^3} = \frac{4464}{46656} = \frac{1116}{11664} = \frac{279}{2916} = \frac{31}{324}$$

$\therefore$ the odds are 293 to 31.

797. That two of them, as A, B, *precisely*, will die in the time specified is

$$\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2}$$

and four things taken two and two, may be combined six ways.

$\therefore$ the probability that some two of them, and no more will die, is

$$\frac{6}{16} = \frac{3}{8}.$$

Again, that three will die precisely is found to be (in same manner)

$$4 \times \frac{1}{16} = \frac{1}{4}$$

and that all four will die is

$$\frac{1}{16}$$

$$\therefore \frac{6}{16} + \frac{4}{16} + \frac{1}{16} = \frac{11}{16}$$

gives the probability that two at least will die in the year.

798. The probabilities of A losing the first, the two first, &c., and the n first games are

$$\frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, \text{\&c.}, \frac{1}{2^n}$$

respectively; consequently the value of B's expectation is

$$\frac{S}{2} + \frac{2S}{2^2} + \frac{4S}{2^3} + \text{\&c.}\ \frac{2^{n-1}S}{2^n} = \frac{nS}{2}$$ which is

the equivalent A ought to receive.

799. The probabilities of their being alive are respectively

$$1 - \frac{5}{6} = \frac{1}{6} \text{ and } 1 - \frac{4}{5} = \frac{1}{5}$$

$$\therefore \frac{1}{6} \times \frac{1}{5} = \frac{1}{30}$$

is the probability of both being alive.

$$\therefore 1 - \frac{1}{30} = \frac{29}{30}$$

is the probability that both will not be alive.

Again, the probability that both will be dead is

$$\frac{5}{6} \times \frac{4}{5} = \frac{20}{30} = \frac{2}{3}$$

$$\therefore 1 - \frac{2}{3} = \frac{1}{3}$$

is the probability that both will not be dead in the given time.

800. The chance of throwing the ace three specified times and of missing the other two is

$$\frac{1}{6^3} \times \frac{5^2}{6^2}$$

and five things combined 3 and 3 give

$$\frac{5.4.3}{2.3} \text{ variations.}$$

$\therefore P = \dfrac{250}{6^5}$ = chance of throwing the ace three times precisely (without regard to order) in five trials.

Similarly

$$\frac{25}{6^5} \text{ and } \frac{1}{6^5}$$

are the chances of throwing the ace four times and five times precisely.

$$\therefore P' = \frac{25}{6^5} \text{ is, \&c., and } P : P' :: 250 : 256 :: 125 : 128.$$

801. Taking one figure from each of these rows, and adding them

1, 2, 3, 4, 5, 6
1, 2, 3, 4, 5, 6
1, 2, 3, 4, 5, 6

it will be found that there are 27 ways of making up the 10, and but 6 which give the number 5. Also 6^3 is the whole number of combinations.

$\therefore$ the probabilities of throwing 10 and 5 with three dice are as

$$\frac{27}{6^3} : \frac{6}{6^3} :: 9 : 2.$$

802. Let the skill of the parties A,B be as a, b; then

$$\frac{a}{a+b}, \frac{b}{a+b}$$

will represent the probabilities of either winning any one game. Now A may win in the following 10 ways, *viz.* by getting the

1 and 2, 1 and 3, 1 and 4, 1 and 5 games
2 and 3, 2 and 4, 2 and 5
3 and 4, 3 and 5
, 4 and 5

But

$$P_{1,2} = \left(\frac{a}{a+b}\right)^2$$

$$P_{1,3} = \left(\frac{a}{a+b}\right) \times \frac{b}{a+b} \times \frac{a}{a+b} = \frac{a^2 b}{(a+b)^3}$$

$$P_{1,4} = \frac{a^2}{(a+b)^2} \times \frac{b^2}{(a+b)^2} = \frac{a^2 b^2}{(a+b)^4}$$

$$P_{1,5} = \frac{a^2}{(a+b)^2} \times \frac{b^3}{(a+b)^3} = \frac{a^2 b^3}{(a+b)^5}$$

$$P_{2,3} = \frac{a^2}{(a+b)^2} \times \frac{b}{a+b}, P_{2,4} = \frac{a^2}{(a+b)^2} \times \frac{b^2}{(a+b)^2}, P_{2,5} = \frac{a^2 b^3}{(a+b)^5}$$

$$P_{3.4} = \frac{a^2}{(a+b)^2} \times \frac{b^2}{(a+b)^2}, P_{3,5} = \frac{a^2}{(a+b)^2} \cdot \frac{b^3}{(a+b)^3}$$

$$P_{4,5} = \frac{a^2 b^3}{(a+b)^5}$$

$$\therefore P = \left(\frac{a}{a+b}\right)^2 \times \left\{1 + 2\frac{b}{a+b} + 3\frac{b^2}{(a+b)^2} + 4\frac{b^3}{(a+b)^3}\right\}$$

In the problem, $a = 2, b = 3$.

$$\therefore P = \frac{4}{25} \times \left\{1 + \frac{6}{5} + \frac{27}{25} + \frac{108}{125}\right\}$$

$$= \frac{4}{25} \times \frac{518}{125} = \frac{2072}{3125}$$

$$\therefore P' = 1 - \frac{2072}{3125} = \frac{1053}{3125}.$$

the chances required.

803. That two of them specified will die, and the other two be alive is $\frac{1}{10 \times 10} \times \frac{9}{10} \times \frac{9}{10} = \frac{81}{10000}$ and the combinations in 4 things taken, two and two is $\frac{4.3}{2} = 6$.

$$\therefore P = \frac{6 \times 81}{10000}$$

is the probability that some two and no more of them will die in the time.

Again, that some three and no more will die is

$$\frac{4 \times 9}{10000}$$

and that all four will die is

$$\frac{1}{10000}$$

$$\therefore P' = \frac{1 + 36 + 486}{10000} = \frac{523}{10000}$$

is the probability that some two at least will die.

$$\therefore P : P' :: \frac{6 \times 81}{10000} : \frac{523}{10000} :: 486 : 523.$$

804. The probability of throwing an ace twice only in three throws in a specified order is

$$\frac{1}{36} \times \frac{5}{6} = \frac{5}{36 \times 6}$$

and there are $\frac{3.2}{2} = 3$ different ways of being successful, $\therefore$

$$\frac{5 \times 3}{6 \times 36} = \frac{5}{72}$$

the probability required.

805. The chance of his throwing it the first time is

$$\frac{2}{36} = \frac{1}{18}$$

$\therefore \dfrac{17}{18} =$ chance of not throwing it

and $\dfrac{17^4}{18^4} =$ ditto of not throwing it once in four trials.

$\therefore \dfrac{18^4-17^4}{18^4} = \dfrac{21455}{104976}$ is the probability required.

806. Let $x + y$ be the value of A's chance at first, x being that after having lost the first game, and first suppose $p=2$, $q=1$.

When A has lost the first game

$a.\dfrac{(x+y)}{a+b}$ is his prospect of winning

and $\therefore a.\dfrac{x+y}{a+b} = x$, $\therefore y = \dfrac{bx}{a}$.

Again, before they begin, A has a chances for 3 counters, and b chances for x counters.

$$\therefore \frac{3a+bx}{a+b} = x + y = x + \frac{bx}{a}$$

$$\therefore x = \frac{3a^2}{a^2+ab+b^2}$$

and $x + y = 3.\dfrac{a^2+ab}{a^2+ab+b^2}$

and $\therefore$ B's expectation is

$$3 - \overline{x+y} = 3.\frac{b^2}{a^2+ab+b^2}$$

Again, let $p = 3$, $q = 1$.

Then putting $x + y + z =$ A's expectation at first, $x + y$ after having lost the first game, and x after having lost the two first; by like reasoning we get

$$a.\frac{x+y}{a+b} = x \text{ and } y = \frac{bx}{a}$$

$$\frac{ax+ay+bx}{a+b} = x + y \text{ and } z = \frac{by}{a} = \frac{b^2x}{a^2}$$

$$\therefore x + y + z = x + \frac{bx}{a} + \frac{b^2x}{a^2}.$$

Now it is evident that A has at first a chances for 4 counters, and b chances for $x + y$ counters,

$$\therefore \frac{4a+b.\overline{x+y}}{a+b} = x + y + z$$

which gives

$$x = \frac{4a^3}{a^3+ba^2+b^2a+b^3}$$

$$\therefore x + y + z = 4 \cdot \frac{a^3 + a^2b + b^2a}{a^3+ba^2+b^2a+b^3}$$

and B's expectation is

$$4. \frac{b^3}{a^3+ba^2+b^2a+b^3}$$

Hence if $p = p$ and $q = 1$, we get

$$P = \frac{a^n+a^{n-1}b+\dots ab^{n-1}}{a^n+a^{n-1}+\dots b^n} \times (n+1) = \overline{n+1} \cdot \frac{a^{n+1}-b\,a}{a^{n+1}-b^{n+1}}$$

$$Q = \frac{b^na-b^{n+1}}{a^{n+1}-b^{n+1}} \times (n+1).$$

Again, let $p = p$, $q = q$.

Then x being the value of A's expectation when he has lost all but one counter, we have

$$x + \frac{bx}{a} + \frac{b^2x}{a^2} + \dots\dots \frac{b^{p-1}x}{a^{p-1}} = \frac{x\left(\frac{b}{a}\right)^p - x}{\frac{b}{a} - 1}$$

the value of A's expectation at first, and

$$x + \frac{bx}{a} + \frac{b^2x}{a^2} + \dots\dots \frac{b^{p+q}x}{a^{p+q}} = \frac{x\left(\frac{b}{a}\right)^{p+q+1} - x}{\frac{b}{a} - 1}$$

for A's expectation when B has but one left.

Hence A has a chances for the whole $p + q$, and b chances for the above expectation, $\therefore$

$$\frac{(p+q)a^{p+q} \times (b-a)}{b^{p+q} - a^{p+q}}$$

and A's original expectation becomes

$$P = \frac{a^p \times (b^q - a^q)}{b^{p+q} - a^{p+q}} \times (p + q)$$

the probability required.

and $\therefore$ B's is

$$Q = \frac{b^{p+q} - a^p b^q}{b^{p+q} - a^{p+q}} \times (p + q)$$

$$\therefore P : Q :: a^p(b^q - a^q) : (b^p - a^p)b^q$$

$$:: \frac{b^q - a^q}{b^p - a^p} : \frac{b^q}{a^p}$$

Make $b = a + x$

$$\text{Then } P : Q :: \frac{qa^{q-1}x + \dots}{pb^{p-1} + \&c.} : \frac{b^q}{a^p}$$

$$:: \frac{qa^{q-1} + xR}{pa^{p-1} + xR'} : \frac{b^q}{a^p}$$

Let $x = 0$, or $a = b$.

and we get

$$P : Q :: p : q$$

a very remarkable result, and which shews most clearly *that a person always playing at games of chance for the same stake, or at games of skill with antagonists of equal skill with himself, must in the end, be utterly ruined.*

At the rouge et noir tables, for instance, even were the play fair and strictly honourable, and the banker to have no advantage (which he always has), continual play for the *same stakes*, must leave the wealthiest person in the world without a *sou*. It may also be remarked, that the safest play, is constantly to stake the same sum and adhere to the same colour.

When b is very small in respect to a

$$P : Q :: 1 : \left(\frac{b}{a}\right)^m \text{ nearly.}$$

807. Since p of them are taken at once, and there are but p of the balls specified, there is evidently but one way of being successful, and there are in all

$$\frac{2.n-1}{2} \cdot \frac{n-2}{3} \cdot \ldots\ldots \frac{n-p+1}{p}$$

ways. Consequently the probability required is

$$P = \frac{1}{n.\frac{u-1}{2} \ldots \frac{n-p+1}{p}}$$

$$= \frac{1.2.\ldots.p}{n.(n-1).(n-2).\ldots.n-p+1}$$

The question may be generalized by stating it; "*required the probability that in taking* p *at a time from a bag containing* n *balls,* n′ *of which are marked or specified,* p′ *of the* p *shall be of the sort specified.*"

Combining the n' marked balls p' and p', we have

$$n'Cp'$$

and also combining the $n - n'$ plain balls $p - p'$ and $p - p'$ together we get

$$(p - n')\, C\, (p - p')$$

for the number of ways in which they can form variations, and each of these lots being added to each of the marked allotments make up all the ways in which p' of the marked balls can appear; which are the number of cases favourable to the event; and the whole number of combinations is

$$nCp$$

$\therefore$ the probability required is

$$\frac{\{n'Cp'\} \times \{\overline{n-n'}.C.\overline{p-p'}\}}{nCp}$$

or $P = \frac{n'.(n'-1)\ldots.n'-p'+1}{1\ldots.p'} \times \frac{(n-n').(n-n'-1)..n-n'-p+p'+1}{1.2.\ldots.p-p'}$

$\times \frac{1.2\ldots.p}{n.(n-1)\ldots.n-p+1} = n'.(n'-1)\ldots\ldots(n'-p'+1) \times$

$\frac{(p'+1)(p'+2)\ldots.p}{1.2\ldots.p-p'} \times \frac{(n-n').(n-n'-1)\ldots\ldots n+p'+1-n'-p}{n.n-1\ldots\ldots n-p+1}.$

This formula is very extensively useful. It will determine the probability of your getting, for instance, p' of the n' prizes in the State Lottery consisting of n tickets, if you have purchased p tickets therein. It will, moreover, give the chances of having so many certain cards trumps, &c. &c. at whist or other games.

808. By taking one figure from each of the lines, and adding them

1.2.3.4.5.6

1.2.3.4.5.6

the number of ways in which we can make 7 is 6, and the whole number of combinations is 6^2

$$\therefore P = \frac{6}{6^2} = \frac{1}{6}$$

the probability of throwing 7 in any one trial. Hence the chance of throwing 7 twice precisely out of three times together in a given order is $\frac{1}{6^2} \times \frac{5}{6}$; and there are $\frac{3 \cdot 2}{2} = 3$ different orders

$$\therefore P' = \frac{5}{3} \cdot \frac{1}{6^2}$$

is the chance of throwing 7 twice exactly in three trials

$$\therefore P : P' :: \frac{1}{6} : \frac{5}{3} \cdot \frac{1}{6^2} :: 18 : 5.$$

809. That A, B happen it is

$$\frac{p}{p+q}, \frac{r}{r+s};$$

that they fail will be

$$\frac{q}{p+q}, \frac{s}{r+s}$$

respectively.

Hence that A happens and B fails

is $\frac{p}{p+q} \times \frac{s}{r+s}$

and then that B happens and A fails is

$$\frac{r}{r+s} \times \frac{p}{p+q}$$

and then that A happens and B fails is

$$\frac{p}{p+q} \times \frac{s}{r+s}$$

&c. &c. &c.

$\therefore$ that in $2n$ trials A and B happen alternately is

$$\frac{p^n q^n}{(p+q)^{2n}} \times \frac{r^n s^n}{(r+s)^{2n}}$$

the probability required.

810. The chance of throwing an ace any three specified throws and missing it the other two is

$$\frac{1}{6^3} \times \frac{5^2}{6^2}$$

and there are $\frac{5.4.3}{2.3}$ ways of doing this.

$$\therefore \frac{10 \times 5^2}{6^5} = \frac{2500}{7776} = \frac{1250}{3888} = \frac{625}{1944}$$

the probability required.

811. The probabilities of the events, happening and failing are respectively

$$\frac{a}{a+b}, \; \frac{b}{b+b}$$

$\therefore \frac{a^p}{(a+b)^p} \times \frac{b^q}{(a+b)^q} =$ the probability of its happening p specified trials and failing in the other q trials; but this can take place

$$(p+q)\,Cp$$

different ways

$$\therefore P = \frac{(p+q).(p+q-1)....q+1}{1.2....p} \times \frac{a^p b^q}{(a+b)^{p+q}}.$$

Again to find the number of trials necessary to make it even whether the event happens or fails,

let x be the number required.

$$\text{then } \frac{a^x}{(a+b)^x} = \frac{1}{2}$$

$$\text{and } x = -\frac{l2}{la - l\overline{a+b}} = \frac{l2}{l\overline{a+b} - la}$$

812. The number of ways of drawing 1, 3, 5, 7, &c. are $N = n + \frac{n.n-1}{2}.\frac{n-3}{3} + \&c.\ 1$ (to $\frac{n}{2}$ or $\frac{n+1}{2}$ terms) those of drawing 2, 4, 6, 8 &c. are

$$N' = \frac{n.n-1}{2} + \frac{n.n-2.n-3.n-4}{2.3.4} + \&c.1 \text{ (to } \frac{n}{2} \text{ or } \frac{n-1}{2} \text{ terms).}$$

$$\text{Now } N = \frac{(1+1)^n + (1-1)^n}{2} = 2^{n-1}$$

$$\text{and } N' = \frac{(1+1)^n - (1-1)^n}{2} 1 = 2^{n-1} - 1$$

$$\therefore N + N' = 2^n - 1$$

$$\therefore \frac{N}{N+N'} = \frac{2^{n-1}}{2^n - 1},\quad \frac{N'}{N+N} = \frac{2^{n-1} - 1}{2^n - 1}$$

$$\text{and } N : N' :: 2^{n-1} : 2^{n-1} - 1.$$

Hence it appears more probable, that an odd number of balls should be taken from the bag than an even number, by the quantity $\frac{1}{2^n - 1}$.

813. The probabilities of A, B, C, D getting a knave the first card are

$$\frac{4}{52}, \frac{4}{51}, \frac{4}{50}, \frac{4}{49}.$$

supposing B, C, D to have the opportunity. $A_1 = \frac{1}{13}$ is the first part of A's chance (1). But if this does not succeed, A's second chance will depend upon A, B, C, D each failing, the probability of which is

$$\left(1 - \frac{4}{52}\right).\left(1 - \frac{4}{51}\right).\left(1 - \frac{4}{50}\right).\left(1 - \frac{4}{49}\right) =$$

$$\frac{48.47.46.45}{52.51.50.49} = \frac{4.9.23.47}{5.13.17.49} \ldots\ldots B_1$$

whence A's second chance is

$$A_2 = \frac{4}{48}.B_1 = \frac{1}{12}B_1 = \frac{3.23.47}{5.13.17.49}$$

Again the probability that two rounds will be dealt without turning a knave is

$$B_1 \times \frac{44.43.42.41}{48.47.46.45} = \frac{11.43.41}{5.5.13.17.7} \ldots\ldots B_2$$

$$\therefore A_3 = \frac{4}{44}.B_2 = \frac{43.41.}{7.5.5.13.17}$$

Again, that three rounds fail is

$$B_2 . \frac{40.39.38.37}{44.43.42.41} = \frac{38.37}{5.7.7.17} \ldots\ldots B_3$$

$$\therefore A_4 = \frac{4}{40}.B_3 = \frac{1}{10}.B_3 = \frac{19.37}{5.5.7.7.17}.$$

Proceeding in like manner with the other rounds, we get

$$B_4 = \frac{9.11}{5.7.13}, A_5 = B_4.\frac{4}{36} = \frac{11}{5.7.13}$$

$$B_5 = \frac{8.29.31}{7.17.13.35}, A_6 = B_5.\frac{4}{32} = \frac{29.31}{7.13.17.35}$$

$$B_6 = \frac{9}{7.17}, A_7 = B_6.\frac{4}{28} = \frac{9}{7.7.17}$$

$$B_7 = \frac{6.11.23}{7.13.17.25}, A_8 = B_7.\frac{4}{24} = \frac{11.23.}{7.13.17.25}$$

$$B_8 = \frac{3.19}{5.7.13.7}, A_9 = B_8.\frac{4}{20} = \frac{3.19}{5.5.7.7.13}$$

$$B_9 = \frac{4}{5.17.7}, A_{10} = B_9.\frac{4}{16} = \frac{1}{5.7.17}$$

$$B_{10} = \frac{9.11}{5.7.7.13.17}, A_{11} = B_{10}.\frac{4}{12} = \frac{3.11}{5.7.7.13.17}$$

$$B_{11} = \frac{2}{5.7.17.13}, A_{12} = B_{11}.\frac{4}{8} = \frac{1}{5.7.13.17}$$

$$B_{12} = \frac{1}{5.5.13.17.49}, A_{13} = B_{12}. = \frac{1}{5.5.13.17.49}$$

Hence

$$A_1 + A_2 + \ldots . A_{13} = \frac{12757}{54145}$$

and A's expectation is worth

$$\frac{12757}{54145}\text{£. or about } 4s.\ 8\tfrac{1}{2}d.$$

814. The cards being arranged numerically thus

$$\left.\begin{array}{l}
2,\ 3,\ 4,\ 5,\ 6,\ 7,\ 8,\ 9,\ 10,\ 10,\ 10,\ 10,\ 11 \\
2,\ 3,\ 4,\ 5,\ 6,\ 7,\ 8,\ 9,\ 10,\ 10,\ 10,\ 10,\ 11 \\
2,\ 3,\ 4,\ 5,\ 6,\ 7,\ 8,\ 9,\ 10,\ 10,\ 10,\ 10,\ 11 \\
2,\ 3,\ 4,\ 5,\ 6,\ 7,\ 8,\ 9,\ 10,\ 10,\ 10,\ 10,\ 11
\end{array}\right\}$$

it is easily found that

$$29 = (11 + 10 + 8) \text{ in } 4 \times (4 \times 4) \times 4 \text{ different ways.}$$

$$= (11 + 9 + 9) \text{ in } 4 \times \frac{4.3}{2} \text{ different ways.}$$

$$= (10 + 10 + 9) \text{ in } 4 \times \frac{16.15}{2} \ldots\ldots\ldots$$

∴ the different ways of making up 29 are in number

$$256 + 24 + 480 = 760.$$

Again

$$\begin{aligned}
19 &= 11 + 6 + 2 \text{ in } 4^3 \text{ different ways} \\
&= 11 + 5 + 3 \text{ in } 4^3 \\
&= 11 + 4 + 4 \ldots 4^3 \\
&= 10 + 7 + 2 \ldots 4^4 \\
&= 10 + 6 + 3 \ldots 4^4 \\
&= 10 + 5 + 4 \ldots 4^4 \\
&= 9 + 8 + 2 \ldots 4^3 \\
&= 9 + 7 + 3 \ldots 4^3 \\
&= 9 + 6 + 4 \ldots 4^3 \\
&= 9 + 5 + 5 \ldots 4^3 \\
&= 8 + 8 + 3 \ldots 4.\frac{4.3}{2} \\
&= 8 + 7 + 4 \ldots 4^3 \\
&= 8 + 6 + 5 \ldots 4^3 \\
&= 7 + 7 + 5 \ldots 4.\frac{4.3}{2} \\
&= 7 + 6 + 6 \ldots 4.\frac{4.3}{2}
\end{aligned}$$

$\therefore$ the whole number of ways of producing 19 is

$$4^3.9 + 3.4^4 + 3.4.6 = 1160.$$

Again

$$9 = 5 + 2 + 2 \text{ in } 4.\frac{4.3}{2} \text{ ways}$$

$$= 4 + 3 + 2 \text{ in } 4^3 \text{ ways}$$

$\therefore$ the whole number of ways in which 9 can be produced is

$$24 + 64 = 88.$$

And the whole number of combinations in 52 things taken three and three, is

$$\frac{52.51.50}{1.2.3} = 26.17.50 = 22200.$$

Hence the probabilities of drawing 29, 19, 9 respectively, are

$$P = \frac{760}{22200} = \frac{38}{1110} = \frac{19}{555}$$

$$P' = \frac{116}{2220} = \frac{58}{1110} = \frac{29}{555}$$

$$P'' = \frac{88}{22200} = \frac{22}{5550} = \frac{11}{2775}$$

$$\text{And } P + P' + P'' = \frac{251}{2775}$$

is the probability of his getting 29, 19, or 9.

$$\therefore 1 - \frac{251}{2775} = \frac{2524}{2775}$$

is the contrary, and $\therefore$ we have

$$2524 : 251 :: 10 : 1 \text{ nearly against him.}$$

815. Since one bag contains more balls by $m-n$ than the other, supposing we first draw from the former, the probability of hitting upon a ball contained in the latter, is

$$\frac{n}{m}$$

and this happening, the probability required becomes that of drawing from the bag with n balls any specified one, or $\frac{1}{n}$.

$\therefore \frac{n}{m} \times \frac{1}{n} = \frac{1}{m}$ is the probability required.

Otherwise.

There are evidently but n ways in which a, a; b, b'; &c., can come together, and the ways in which two sets of quantities containing m and n can be combined by taking one from each is

$$mn$$

$$\therefore \frac{n}{mn} = \frac{1}{m} \text{ the same as before.}$$

816. Here $P = \frac{(m-p')\,C\,(p-p') \times m'Cp'}{mCp}$ (see 806)

and $m = 11, m' = 5, p = 7, p' = 3$

$$\therefore P = \frac{6.C.4 \times 5.C.3}{11.C.7} = \frac{6.5}{2} \times \frac{5.4}{1.2} \times \frac{1.2.3.4}{11.10.9.8}$$

$= \frac{5}{11}$ the probability required.

817. The probability of throwing the six faces in any given order, is $\frac{1}{6^6}$

and there are $6.5.4.3.2.1$ ways in which the order can be varied; $\therefore$ the probability is

$$\frac{6.5.4.3.2.1}{6^6} = \frac{20}{6^4} = \frac{5}{2^2.3^4} = \frac{5}{4.81} = \frac{5}{324}.$$

818. Since A's skill is double of B's, his chance of winning is

$$\frac{2}{3}$$

and the value of his expectation, supposing the stake a guinea, is

$$\frac{2}{3} \times 2 \text{ guineas.}$$

Now since C and A play with equal skill, if x denote C's stake and A's as before, the probability of A's winning $1 + x$ guineas is

$$\frac{1}{2}$$

and the value of his expectation is ∴

$$\frac{1}{2}(1 + x)$$

and by the question we get

$$\frac{4}{3} = \frac{1}{2}(1 + x)$$

$$\therefore x = \frac{8}{3} - 1 = \frac{5}{3} \text{ of a guinea.}$$

819. Let $\frac{x}{a+x}$ denote E's chance of winning the first game, then the probability of D's winning the first n games is

$$\left(\frac{a}{a+x}\right)^n = \frac{1}{2}$$

by the question.

$$\therefore a2^{\frac{1}{n}} = a + x$$

$$\therefore x = a(2^{\frac{n}{1}} - 1)$$

and $$\frac{x}{a+x} = \frac{a(2^{\frac{n}{1}} - 1)}{a + a(2^{\frac{n}{1}} - 1)} = \frac{2^{\frac{1}{n}} - 1}{2^{\frac{1}{n}}}$$

the probability required.

820. Let generally $p\,C\,q$ denote the number of combinations of p things taken q and q together; then the number of ways in which A may be thrown with m dice, is

$$\begin{aligned} N = {} & (A - 1).\,C.\,(m - 1) \\ & - m.\,C.\,1 \times (A - 7).\,C.\,(m - 1) \\ & + m.\,C.\,2 \times (A - 13).\,C.\,(m - 1) \\ & - m.\,C,\,3 \times (A - 19).\,C.\,(m - 1) \\ & + \&c. \end{aligned}$$

see *Parisot's Calcul Conjectural*, p. 85; and therefore the number of ways in which B may be thrown is

$$\begin{aligned} N' = &(B-1).C.(m-1) \\ &- m.C.1 \times (B-7).C.(m-1) \\ &+ m.C.2 \times (B-13).C.(m-1) \\ &- m.C.3 \times (B-19).C.(m-1) \\ &+ \&c. \end{aligned}$$

Also it may easily be shewn that 6^m is the total number of different throws of the m dice. Consequently the probability of throwing A the first time is

$$\frac{N}{6^m}$$

and that of not throwing it once in n throws is

$$\left(1 - \frac{N}{6^m}\right)^n$$

and $\therefore$ the probability of throwing A once at least in n throws is

$$P = 1 - \left(1 - \frac{N}{6^m}\right)^n \ldots\ldots\ldots (a)$$

In the same way it is shewn that the chance of throwing B at least once in n throws is

$$P' = 1 - \left(1 - \frac{N'}{6^m}\right)^n \ldots\ldots\ldots (b)$$

Hence the probability of throwing both A and B with m dice in n throws, at least once, is

$$P + P' = 2 - \left(1 - \frac{N}{6^m}\right)^n - \left(1 - \frac{N'}{6^m}\right)^n.$$

By way of example let it be *required to find the chance of throwing* 20 *and* 16 *at least once with* 4 *dice in* 6 *throws.*

Here $\begin{aligned}[t] N = &19.C.3 \\ &- 4.C.1 \times 13.C.3 \\ &+ 4.C.2 \times 7.C.3 \end{aligned}$

$$= \frac{19.18.17}{2.3} - 4 \times \frac{13.12.11}{2.3} + \frac{4.3}{2} \times \frac{7.6.5}{2.3} = 35.$$

$$N' = \frac{15.14.13}{2.3} - 4 \times \frac{9.8.7}{2.3} + \frac{4.3}{2} \times \frac{3.2.1}{2.3} = 125.$$

$$\therefore P = 1 - \left(1 - \frac{35}{6^4}\right)^6 = \frac{6^{24} - (6^4 - 35)^6}{6^{24}}$$

and $P' = 1 - \left(1 - \frac{125}{6^4}\right)^6 = \frac{6^{24} - (6^4 - 125)^6}{6^{24}}$

and so on.

821. Let the stake, each game, be S; then the probability of B's winning any specified game being

$$\frac{m}{m+1}$$

his expectation is worth

$$\left(\frac{2m}{m+1} - 1\right) S = \frac{m-1}{m+1} S;$$

and the probability of his having to play the n^{th} game being

$$\left(\frac{m}{m+1}\right)^{n-1}$$

his gain on that game is worth

$$\left(\frac{m}{m+1}\right)^{n-1} \times \frac{m-1}{m+1} \times S.$$

Hence the whole value of his expectation is

$$\frac{m-1}{m+1}. S. \left\{1 + \frac{m}{m+1} + \left(\frac{m}{m+1}\right)^2 + \&c. \infty\right\}$$

which $= \frac{m-1}{m+1}. S \times (m+1) = (m-1). S.$

822. The whole number of combinations is

$$\frac{8 \times 7 \times 6}{2.3} = 56,$$

out of which we have

1 consisting of 20£ and two 5£.

$\frac{5 \times 4}{2} = 10$ of a 20£ and two 1£.

5 of two fives and one 1£.

$2 \times \frac{5 \times 4}{2} = 20$, of one 5£ and two 1£.

$\frac{5 \times 4 \times 3}{2.3} = 10$ of three 1£.

and 10 of a 20£, a 5£, and a 1£.

Hence his expectations of winning 30 £, 11 £, 3 £, 26 £, are

$$\frac{1}{56}, \frac{10}{56}, \frac{5}{56}, \frac{20}{56}, \frac{10}{56}, \frac{10}{56}$$

respectively; and the value of his expectation is therefore

$$\frac{30 + 220 + 55 + 140 + 30 + 260}{56} £.$$

$$= \frac{735}{56} £ = \frac{105}{8} £ = £13\ 2s.\ 6d.$$

MISCELLANIES.

823. THE line which passes through O (Fig. to enunciation) resting against the wall and ground at equal distances from the bottom of the wall B, is the length of the largest ladder required; for if the longest were longer than this ladder, it would fall beyond O when in that position. Hence having the co-ordinates a, b, of the point O referred to A, we get for the parts of the ladder on each side of O the lengths

$$a\sqrt{2},\ b\sqrt{2}$$

and the length required is therefore

$$(a+b)\sqrt{2}.$$

824. Let r be the common ratio of the geometrical series; then the successive values are

$$x,\ xr,\ xr^2,\ xr^3,\ \&c.,\ xr^m,\ xr^{m+1},\ \&c.$$

and the general increment of the ratio is

$$\frac{xr^{m+1} - xr^m}{xr^m} = r - 1$$

a constant quantity.

825. The coefficient of $(n+1)^{\text{th}}$ term of $(a+x)^{2n}$ is

$$\frac{2n\,.\,(2n-1)\,\ldots\ldots\,n+1}{2\,.\,3\,\ldots\ldots\,n}.$$

But

$$(a+x)^n = a^n + na^{n-1}x + n\,.\,\frac{n-1}{2}\,a^{n-2}\,x^2 + \&c.$$

$$(a+x)^n = a^n + na^{n-1}x + n\,.\,\frac{n-1}{2}\,a^{n-2}\,x^2 + \&c.$$

and multiplying them together the coefficient of $a^n\,x^n$ will be found to be

$$1 + n^2 + \left(n \cdot \frac{n-1}{2}\right)^2 + \&c. \text{ to } \overline{n+1} \text{ terms,}$$

and the middle term of $(a + x)^{2n}$ being the $(n + 1)^{th}$, its coefficient is followed by $a^n x^n$.

$$\therefore 1 + n^2 + \left(n . \frac{n-1}{2}\right)^2 + \&c. = \text{that coefficient}$$

$$= \frac{2n \cdot \overline{2n-1} \ldots\ldots n+1}{1 \cdot 2 \cdot 3 \ldots\ldots n}$$

$$= \frac{2n \cdot \overline{2n-1} \ldots\ldots n+1}{1 \cdot 2 \cdot 3 \ldots\ldots n} \times \frac{n \cdot (n-1) \ldots\ldots 2 \cdot 1}{n \cdot (n-1) \ldots\ldots 2 \cdot 1}$$

$$= \frac{2n \cdot (2n-2) \ldots\ldots 4 \cdot 2 \times 1 \cdot 3 \cdot 5 \ldots\ldots 2n-1}{(1 \cdot 2 \cdot 3 \ldots\ldots n)^2}$$

$$= 2^n \times \frac{1 \cdot 3 \cdot 5 \ldots\ldots 2n-1}{1 \cdot 2 \cdot 3 \ldots\ldots n}.$$

Q. E. D.

826. Let $a^{mx} b^{nz} = c$. Then

$mxla + nzlb = lc$

and $mla \times dx + nlb \times dz = 0$.

Also by the question

$(mx + n) \times (nz + m) = \max.$

$\therefore mdx \cdot (nz + m) + ndz \cdot (mx + n) = 0$

$$\therefore \frac{m}{n} \frac{dx}{dz} = \frac{lb}{la} = \frac{mx+n}{nz+m}$$

$\therefore a^{mx+n} = b^{nz+m}$. Q. E. D.

Otherwise,

since $mx = \dfrac{lc - lb \times nz}{la}$, we have

$$\left(\frac{lc - lb \times nz}{la} + n\right)(nz + m) = \max.$$

$$\therefore \frac{lb}{la} \times n \times (nz + m) = \left(\frac{lc - lb \times nz}{la} + n\right) n$$

$$\therefore n^2 \frac{lb}{la} z + nm \frac{lb}{la} = \frac{nlc}{la} - \frac{n^2 lb}{la} z + n^2.$$

Hence

$$n + mx = (m + nz)\,\frac{lb}{la}$$

and $\therefore\ a^{n+mx} = b^{m+nz}$.

827. Let $2 \cos. z = x + \dfrac{1}{x}$. Then

$$1 + n \cos. z = 1 + \frac{n}{2}\cdot\left(x + \frac{1}{x}\right)$$

$$= \left(\frac{n}{2}x^2 + x + \frac{n}{2}\right) \times \frac{1}{x}$$

$$= \frac{n}{2x} \times \left(x^2 + \frac{2}{n}x + 1\right)$$

$$= \frac{n}{2x} \times \left(x + \frac{1 + \sqrt{1 - n^2}}{n}\right)\cdot\left(x + \frac{1 - \sqrt{1 - n^2}}{n}\right)$$

$$= \frac{n}{2x}\cdot\left(x + \frac{n}{1 - \sqrt{1 - n^2}}\right)(x + \mathrm{B})$$

$$= \frac{n\mathrm{B}}{2}(1 + \mathrm{B}x)\left(1 + \frac{\mathrm{B}}{x}\right)$$

$$\therefore\ l.(1 + n \cos. z) = l.\frac{n\mathrm{B}}{2} + l.(1+\mathrm{B}x) + l.\left(1 + \frac{\mathrm{B}}{x}\right)$$

$$= l.\frac{n\mathrm{B}}{2} + \mathrm{B}x - \frac{\mathrm{B}^2x^2}{2} + \frac{\mathrm{B}^3x^3}{3} - \&c.$$

$$+ \frac{\mathrm{B}}{x} - \frac{\mathrm{B}^2}{2x^2} + \frac{\mathrm{B}^3}{3x^3} - \&c.$$

$$= l.\frac{n\mathrm{B}}{2} + \mathrm{B}.\left(x + \frac{1}{x}\right) - \frac{\mathrm{B}^2}{2}\left(x^2 + \frac{1}{x^2}\right) - \&c.$$

$$= l.\frac{n\mathrm{B}}{2} + 2\mathrm{B}\cos. z - \frac{2\mathrm{B}^2}{2}\cos. 2z + \frac{2\mathrm{B}^3}{3}\cos. 3z - \&c.$$

828. Let $u = \dfrac{1}{3} + \dfrac{1}{5} + \dfrac{1}{7} + \dfrac{1}{11} + \ldots \dfrac{1}{w}$ be the series of the reciprocals of prime numbers; then by *Legendre*, p. 397, we have

$$u = \log.(\log. w - 0.08366) - 0.2215.$$

Again, we have

$$\therefore x + \frac{x^2}{2} + \frac{x^3}{3} + \&c. = -\log.(1-x)$$

and when $x = 1$,

$$u' = 1 + \frac{1}{2} + \frac{1}{3} + \&c. = -\log. 0$$

$$= \log.\frac{1}{0} = \log. \infty.$$

Also when w is indefinitely great or $= \infty$, we have

$$u = \log.(\log. \infty) = \log. u'$$

$$\therefore u' = e^u = (2.718, \&c.)^{\infty}$$

$$\text{and } \frac{u'}{u} = \frac{(2.719, \&c.)^{\infty}}{\infty}$$

$$= \frac{(1+A)^{\infty}}{\infty} = \frac{1 + \infty \times A + \frac{\infty^2}{2} A^2 + \&c.}{\infty}$$

$$= A + \frac{\infty}{2} A^2 + \frac{\infty^2}{2.3} A^3 + \&c.$$

$$= \infty \times \left(\frac{A^2}{2} + \frac{A^3}{2.3}\infty + \&c.\right)$$

so that u' is indefinitely greater than u. Q. E. D.

829. Let P (draw the figure) be the point within the circumference, aPb, bPc, the equal angles subtended by the arcs ab, bc, ab, being nearer to the diameter passing through P than bc; then bc is $>$ ab.

For let cP, bP, be produced to meet the circumference in c', b', and join ab', bc'; then since Pb' is $>$ Pc' and the angle bPc' $=$ the angle b'Pa, $\therefore$ the $\angle\, b'$ is $<$ the angle c', or since equal $\angle$s are subtended by equal arcs,

ab is $<$ bc.

830. Since $x = \dfrac{e^{az} - e^{bz}}{2a}$ and $v = \pm\sqrt{-1}$

$$\therefore\ x = \frac{e^{z\sqrt{-1}} - e^{-z\sqrt{-1}}}{2\sqrt{-1}} = \sin. z$$

$$\therefore\ dz = \frac{dx}{\sqrt{(1-x^2)}}.$$

831. Make

$$P = \sqrt{(2a^3x - x^4)} - \sqrt{(ax^3)}$$

$$\text{and } Q = a - \sqrt{ax}$$

Then

$$\frac{dP}{dx} = \frac{a^3 - 2x^3}{\sqrt{(2a^3x - x^4)}} - \frac{3}{2}\sqrt{ax}$$

$$\text{and } \frac{dQ}{dx} = -\frac{1}{2}\sqrt{\frac{a}{x}}$$

and these, when $x = a$, become

$$-3a^3 - \frac{3a}{2}, \text{ and } -\frac{1}{2}$$

$$\therefore\ \frac{dP}{dQ} = 3a\,(2a^2 + 1) \text{ when } x = a$$

which is the value required.

Again,

$$d\,.\,l\,\frac{\sqrt{a+x} + \sqrt{a-x}}{\sqrt{a+x} - \sqrt{a-x}} = d\,.\,l\,.\,\frac{(\sqrt{a+x} + \sqrt{a+x})^2}{2x}$$

$$= 2d\,.\,l\,(\sqrt{a+x} + \sqrt{a-x}) - \frac{2dx}{x}$$

$$= dx\left\{\frac{\frac{1}{\sqrt{a+x}} - \frac{1}{\sqrt{a-x}}}{\sqrt{a+x} + \sqrt{a-x}}\right\} - \frac{2dx}{x}$$

$$= dx\left\{\frac{\sqrt{a+x} - \sqrt{a+x}}{\sqrt{a^2-x^2} \times \sqrt{a+x} + \sqrt{a-x})} - \frac{2}{x}\right\}$$

$$= dx\left\{\frac{(\sqrt{a-x} - \sqrt{a+x})^2}{-2x\sqrt{a^2-x^2}} - \frac{2}{x}\right\}$$

which may be farther reduced.

832. By the nature of the series we have (calling the coefficients a_1, a_2 ...... a_n, &c., and the scale of notation being $f_1 + f_2$ ..... f_n)

$$
\begin{aligned}
a_{n+1} &= f_1\, a_n + f_2\, a_{n-1} + \ldots\ldots f_n\, a_1 \\
a_{n+2} &= f_1\, a_{n+1} + f_2\, a_n + \ldots\ldots f_n\, a_2 \\
a_{n+3} &= f_1\, a_{n+2} + f_2\, a_{n+1} + \ldots\ldots f_n\, a_3 \\
&\&c. \qquad \&c. \\
a_{2n} &= f_1\, a_{2n-1} + f_2\, a_{2n-2} + \ldots\ldots f_n\, a_n
\end{aligned}
$$

$$
\begin{aligned}
\therefore\ \Delta\, a_{n+1} &= f_1\, \Delta\, a_n + f_2\, \Delta\, a_{n-1} + \ldots\ldots f_n\, \Delta\, a_1 \\
\Delta^2 a_{n+1} &= f_1\, \Delta^2 a_n + f_2\, \Delta^2 a_{n-1} + \ldots\ldots f_n\, \Delta^2 a_1 \\
\&c. &= \&c.
\end{aligned}
$$

and $\Delta^m a_{n+1} = f_1\, \Delta^m a_n + f_2\, \Delta^m a_{n+1} + \ldots\ldots f_n\, \Delta^m a_1$.

Let $\Delta^m a_{n+1} = \Delta^m a_n = \&c. = 0$; then

$f_1 + f_2 + \ldots\ldots f_n = 1$. Q. E. D.

833. $$\int \frac{x^{\frac{1}{2}}dx}{\sqrt{(a^7 - x^7)}} = \int \frac{x^{\frac{1}{2}}}{a^{\frac{7}{2}}} \times \left(1 - \frac{x^7}{a^7}\right)^{-\frac{1}{2}} dx$$

$$= \frac{1}{a^{\frac{7}{2}}} \times \int \left\{ x^{\frac{1}{2}}dx + \frac{1}{2a^7}\, x^{\frac{15}{2}}\, dx + \frac{3}{4a^{14}}\, x^{\frac{29}{2}}\, dx + \&c. \right\}$$

$$= \frac{1}{a^{\frac{7}{2}}} \times \left\{ \frac{2}{3}\, x^{\frac{3}{2}} + \frac{1}{17a^7}\, x^{\frac{17}{2}} + \frac{3}{2.31a^{14}}\, x^{\frac{31}{2}} + \&c. \right\} + C.$$

which converges since a is $> x$.

834. Let ABC (draw the fig.) be the $\triangle$; with C as a centre and radius $=$ CB, (CB being less than CA) describe a circle cutting AC in E, produce AC to D, join BD, BE, and draw EF parallel to BD; then since

$$\angle \text{EBA} = \angle \text{CBA} - \angle \text{CEB} = \text{A} - \frac{\text{A} + \text{B}}{2}$$

$$= \frac{\text{B} - \text{A}}{2}$$

A and B denoting the angles at those points, we have

$$\text{AD}(=a+b) : \text{AE}(=a-b) :: \text{DB} : \text{EF}$$
$$:: \text{EB} . \tan. \frac{\text{A} + \text{B}}{2} : \text{EB} . \tan. \frac{\text{A} - \text{B}}{2}$$
$$:: \tan. \frac{\text{A} + \text{B}}{2} : \tan. \frac{\text{A} - \text{B}}{2}.$$

835. Since $\sqrt{\frac{(x+a)^3}{x-a}} = \textit{minimum}$

$$\therefore u = \frac{(x+a)^3}{x-a} = \text{min.}$$
$$\therefore \frac{du}{dx} = \frac{3(x+a)^2}{x-a} - \frac{(x+a)^3}{(x-a)^2} = 0$$
$$\therefore 3x - 3a = x + a$$
and $x = 2a$.

Hence the minimum value is
$$\sqrt{\frac{27a^3}{a}} = 3a\sqrt{3}.$$

836. $d(\sin.^{-1} 2y\sqrt{1-y^2}) = \dfrac{d(2y \cdot \sqrt{1-y^2})}{\sqrt{1-4y^2.(1-y^2)}}$

$$= \frac{2dy\left(\sqrt{1-y^2} - \frac{y^2}{\sqrt{1-y^2}}\right)}{\sqrt{1-4y^2+4y^4}} = 2dy . \frac{1-2y^2}{\sqrt{(1-2y^2)^2}}$$
$$= 2dy.$$

837. Since $13x + 14y = 200$, therefore
$$x + y + \frac{y}{13} = 15 + \frac{5}{13}.$$

Let $\frac{y-5}{13} = w$, w being any integer. Then
$$y = 13w + 5$$
$$\therefore x = 10 - 14w.$$

Hence if w be assumed $= 0, 1, 2, 3$, &c., the values of y will be

$5, 18, 31$, &c.

and the corresponding ones of x will be

$10, -4, -18$, &c.

838. Let $x^3 - x^2 - 8x + 12 = u$. Then by the question

$$\frac{du}{dx} = 3x^2 - 2x - 8 = 0$$

$$\text{and } x^2 - \frac{2}{3}x = \frac{8}{3}$$

$$\therefore x = \frac{1}{3} \pm \sqrt{\frac{1}{9} + \frac{3 \times 8}{9}}$$

$$= \frac{1 \pm 5}{3} = 2 \text{ or } -\frac{4}{3}.$$

839. $$d.\frac{y\sqrt{a^2-y^2}}{a^3-y^3} = dy\left\{\frac{\sqrt{a^2-y^2}}{a^3-y^3} - \frac{y^2}{(a^3-y^3)\sqrt{a^2-y^2}} + \frac{3y^3\sqrt{a^2-y^2}}{(a^3-y^3)^2}\right\} = dy\left\{\frac{a^2-2y^2}{(a^3-y^3)\sqrt{a^2-y^2}} + \frac{3y^3\sqrt{a^2-y^2}}{(a^3-y^3)^2}\right\} = \frac{(a^2-2y^2)(a^3-y^3)+3a^2y^3-3y^5}{(a^3-y^3)^2\sqrt{a^2-y^2}}$$

$$= \frac{a^5+2a^2y^3-2a^3y^2+y^5}{(a^3-y^3)^2\sqrt{a^2-y^2}}.$$

840. The several interests being $\frac{r}{100}n, \frac{r}{100}(n-1)2^3, \frac{r}{100} \times (n-2)3^3$ &c. $\frac{r}{100}n^3$ and the principals

$1, 2^3, 3^3$, &c., n^3

the whole amount at the end of n years is

$$S = 1 + \frac{r}{100}.n + 2^3\left(1 + \frac{r.\overline{n-1}}{100}\right) + 3^3\left(1 + \frac{r.\overline{n-2}}{100}\right) +$$

$$\text{\&c.} + n^3\left(1 + \frac{r}{100}\right) = 1 + 2^3 + 3^3 + \ldots\ldots n^3$$

$$+ \frac{r}{100} \times (n + 2^3.\overline{n-1} + 3^3.\overline{n-2} + \text{\&c.} + n^3)$$

$$= (1 + 2^3 + \ldots. n^3)\left(1 + \frac{nr}{100}\right) - (2^3 + 2.3^3 + 3.4^3 + \text{\&c.}$$

$$\overline{n-1}.n^3)$$

$$= \left(1 + \frac{nr}{100}\right)\Sigma n^3 - \Sigma\overline{n-1}.n^3$$

$$= \left(2 + \frac{nr}{100}\right)\Sigma n^3 - \Sigma n^4$$

$$= \left(2 + \frac{nr}{100}\right)\frac{n^2.(n-1)^2}{4} - \left(\frac{n^5}{5} - \frac{n^4}{4} + \frac{n^3}{3} - \frac{n}{30}\right) + C.$$

Let $n = 1$; then $S = 1 + \dfrac{r}{100}$

$$\therefore C = 1 + \frac{r}{100} + \frac{1}{4}.$$

The farther reduction is left to the student.

841. Since $u = \dfrac{x^2(1-x^2)}{1-a^2x^2} = \text{max.}$

$$\therefore \frac{du}{dx} = \frac{2x(1-x^2)}{1-a^2x^2} - \frac{2x^3}{1-a^2x^2} + \frac{2a^2x^3(1-x^2)}{(1-a^2x^2)^2} = 0$$

$$\therefore (1-x^2)(1-a^2x^2) - x^2(1-a^2x^2) + a^2x^2(1-x^2) = 0$$

$$\therefore (1-2x^2)(1-a^2x^2) + a^2x^2(1-x^2) = 0.$$

Hence $a^2x^4 - 2x^2 = -1$

and $x^4 - \dfrac{2}{a^2}x^2 = -\dfrac{1}{a^2}$

$$\therefore x^2 = \frac{1 \pm \sqrt{(1-a^2)}}{a^2}$$

$$\text{and } x = \pm \frac{\sqrt{1 \pm \sqrt{1-a^2}}}{a} = \pm \frac{\sqrt{1+a} \pm \sqrt{1-a}}{a\sqrt{2}},$$

which gives the values required.

842. The 1000£ stock at 110 = 1100£ sterling $= \frac{100}{84} \times 1100 = \frac{25}{21} \times 1100$ in the Threes at 84. The interest of this at the end of six months is

$$\frac{3}{200} \times \frac{25 \times 1100}{21} = \frac{33 \times 25}{42}.$$

Now had the stock remained in the Fives, its worth (at 112) would have been

$$1120 + \frac{1}{40} \times 1000 = 1120 + 25$$
$$= 1145.$$

Hence x being the rate required, we have

$$\frac{x}{100} \times 1100 + \frac{33 \times 25}{42} = 1145$$

$$\text{and } \therefore x = -\frac{25}{14} + \frac{1145}{11}$$
$$= \frac{16030 - 275}{11 \times 14} = \frac{15755}{11 \times 14}$$
$$= 102\frac{47}{54}$$

843. $$d \,.\, l \,.\, \frac{x}{\sqrt{1+x^2}} = d \,.\, lx - \frac{1}{2} d \,.\, l\,(1+x^2)$$
$$= \frac{dx}{x} - \frac{xdx}{1+x^2} = \frac{dx}{x\,(1+x^2)}.$$

Also making

$$\sin.\,\theta = 2x\sqrt{1-x^2}$$

$$\therefore\ d\theta \cos.\theta = 2dx\sqrt{1-x^2} - \frac{2x^2dx}{\sqrt{1-x^2}} = \frac{2dx(1-2x^2)}{\sqrt{1-x^2}}.$$

$$\therefore\ d\theta = \frac{2dx(1-2x^2)}{\sqrt{(1-4x^2+4x^4)}} = 2dx.$$

844. Let $u, u_1, u_2 \ldots\ldots u_n$ be the equidistant values of any function, then we know that (see *Translation of Lacroix*)

$$\Delta^n u = u_n - \frac{n}{1}u_{n-1} + n\frac{n-1}{2}u_{n-2} - \&c. \pm nu_1, \mp u = 0$$

Hence, having all but one of these equidistant values, that one may be determined.

In the problem

$$u = l.510 = 2.70757018$$
$$u_1 = l.511 = 2.70842090$$
$$u_3 = l.513 = 2.71011737$$
$$u_4 = l.514 = 2.71096312$$
$$\text{and } u_2 = \frac{4(u_1+u_3)-(u+u_4)}{6}$$
$$= 2.70926996.$$

845. Suppose two planes passing through the lines $\perp$ to any third plane, and therefore parallel to one another. Hence a plane cuts two parallel planes at right $\angle$ and the intersections (which are also the projections of the straight line in the question) are parallel.

846. Since $a^x b^y c^z = \text{min.}$

$$\therefore\ xla + ylb + zlc = \text{min.}$$
$$\text{and } (x+1).(y+1).(z+1) = Q \ldots\ldots (1)$$

Hence (see *Vince*, p. 20)

$$\frac{dx}{dy} = -\frac{lb}{la} = -\frac{x+1}{y+1}$$

$$\frac{dx}{dz} = -\frac{lc}{la} = -\frac{x+1}{z+1}$$

$$\frac{dy}{dz} = -\frac{lc}{lb} = -\frac{y+1}{z+1}.$$

Hence

$$\left.\begin{array}{l} x - \dfrac{lb}{la}\, y = \dfrac{lb}{la} - 1 \\ x - \dfrac{lc}{la}\, z = \dfrac{lc}{la} - 1 \\ y - \dfrac{lc}{lb}\, z = \dfrac{lc}{lb} - 1 \end{array}\right\}$$

Hence, and from equation (1) the relation required may be found.

847. Let $y = A + Bx + Cx^2 + Dx^3 + \&c.$ be the equation to the curve. Then if k be the increment of the ordinate, and h that of the abscissa by Taylor's Theorem, we have

$$y + k = y + \frac{dy}{dx} \cdot h + \frac{d^2y}{dx^2}\frac{h^2}{2} + \&c.$$

$$\therefore\ k = \frac{dy}{dx}\, h + \frac{d^2y}{dx^2} \cdot \frac{h^2}{2} + \&c.$$

848. Let x be one of the parts. Then

$x^2 \cdot (30 - x) = \max.$

$\therefore\ 2x\,(30 - x) = x^2$

and $x = 10$.

849. Let x be one part. Then

$x^2\sqrt{100 - x} = \max.$

$\therefore\ x^4\,(100 - x) = \max,$

and $4x^3\,(100 - x) = x^4$

Hence $x = 80$.

850. Let θ be the arc required, of the circle whose radius is 1. Then the chord is $2 \sin. \frac{\theta}{2}$, and we have, by the question,

$$\sin. \frac{\theta}{2} . \cos. \theta = \max.$$

$$\therefore \frac{d\theta}{2} \cos. \theta \times \cos. \frac{\theta}{2} = d\theta \sin. \theta . \sin. \frac{\theta}{2}$$

$$\therefore \tan. \theta . \tan. \frac{\theta}{2} = \frac{1}{2}$$

and

$$\frac{2 \tan.^2 \frac{\theta}{2}}{1 - \tan.^2 \frac{\theta}{2}} = \frac{1}{2}.$$

Hence

$$\tan. \frac{\theta}{2} = \frac{1}{\sqrt{5}}$$

$$\therefore \tan. \theta = \frac{\frac{2}{\sqrt{5}}}{1 - \frac{1}{5}} = \frac{\sqrt{5}}{2}$$

which, by means of the tables, will give the arc required.

851. Since $\sqrt{\frac{a^2+x^2}{a+x}} = \min.$

$$\therefore \frac{a^2 + x^2}{a+x} = \min. = u$$

$$\therefore \frac{du}{dx} = \frac{2x}{a+x} - \frac{a^2 + x^2}{(a+x)^2} = 0$$

$$\therefore 2ax + x^2 = a^2$$

and $x = - a \pm \sqrt{2a^2}$
$= - a (1 \mp \sqrt{2})$

852. Since $x + y + z = a \ldots\ldots\ldots\ldots (1)$
and $x^m . y^n z^r = \max.$

or $mlx + nly + rlz = \max.$

$\therefore$ by *Vince*, p. 20, we have

$$\frac{dx}{dy} = -z = -\frac{n}{m} \cdot \frac{x}{}$$

$$\frac{dy}{dz} = -x = -\frac{r}{n} \cdot \frac{y}{z},$$

whence and from equat. (1) the values of x, y, z, which give the maximum will be found.

853. Let x be the number. Then

$$x^{\frac{1}{m}} - x^{\frac{1}{n}} = \max.$$

$$\therefore \frac{1}{m} x^{\frac{1}{m}-1} = \frac{1}{n} x^{\frac{1}{n}-1}$$

$$\therefore x = \left(\frac{m}{n}\right)^{\frac{mn}{n-m}}$$

854. Let $P = a^x - b^x$. Then

$$\frac{dP}{dx} = a^x la - b^x lb$$

$$= la - lb = l\frac{a}{b} \text{ when } x = 0,$$

which is the value required.

Again, let $P = 1 - x$, $Q = \cot. \frac{\pi x}{2}$,

Then $\frac{dp}{dx} = -1, \frac{dQ}{dx} = \frac{\pi}{2 \sin.^2 \frac{\pi}{2} x}$

$$\therefore \frac{dP}{dQ} = \frac{2 \sin.^2 \frac{\pi}{2} x}{\pi} = \frac{2}{\pi}$$

when $x = 1$.

855. Let the coefficients he denoted by P_2, P_4, P_6, &c., P_{2r}, &c. Then

$$-P_2 = na,\ P_4 = n \cdot \frac{n-3}{2} a^2,\ -P_6 = \frac{n.\overline{n-4}.\overline{n-5}}{2.3} a^3$$

&c. &c., and $\pm P_{2r} = \frac{n(n-\overline{r+1})(n-\overline{r+2})\ldots(n-\overline{2r-1})}{2.3\ldots r} a^r$

which law of the coefficients, however, is not indicated in the enunciation of the problem.

Now we know that

$$S_2 = P_1 S_1 - 2P_2 = -2P_2 = 2na$$

$$S_4 = -P_2 S_2 - 4P_4 = 2n^2a^2 - 2n(n-3)^2$$

$$= 6na^2 = \frac{4 \cdot 3}{2} \times na^2$$

$$S_6 = -P_2 S_4 - P_4 S_2 - 6P_6 = 6n^2a^3 - n \cdot \frac{n-3}{2} a^2 \times 2na$$

$$= 20na^3 = \frac{6 \times 5 \times 4}{2 \cdot 3} \times na^3$$

&c. = &c.

856. $$du = \frac{d \cdot \frac{x-y}{x+y}}{1 + \left(\frac{x-y}{x+y}\right)^2}$$

$$= \frac{(dx-dy)(x+y) - (dx+dy)(x-y)}{(x+y)^2 + (x-y)^2}$$

$$= \frac{2ydx - 2xdy}{2x^2 + 2y^2}$$

$$= \frac{ydx - xdy}{x^2+y^2}.$$

857. First we have

$$\frac{x - x^2}{1 - x^2} = \frac{x}{1+x} \times \frac{1-x}{1-x} = \frac{x}{1+x}$$

which, when $x = 1$, becomes

$$\frac{1}{2}.$$

858. Since $\frac{dy}{dx} = \frac{a^2+x^2-y^2}{a^2}$, we have

$$a^2(dy - dx) + (y^2 - x^2)dx = 0,$$

which admits being simplified by making

$$y - x = v;$$

for thence we get

$$a^2 \frac{du}{dx} + 2ux + u^2 = 0.$$

Again, by making

$$-\frac{1}{u} = v$$

differentiating and substituting, we have

$$\frac{a^2 dv}{dx} - 2xv + 1 = 0$$

a linear equation. Let therefore

$$v = wz$$

then

$$\frac{a^2 zdw}{dx} + \frac{a^2 wdz}{dx} - 2xwz + 1 = 0$$

and since we are at liberty to make a second assumption, let

$$\frac{a^2 dw}{dx} - 2xw = 0$$

thence

$$\left.\begin{array}{l} \frac{a^2}{2} \cdot \frac{dw}{w} - xdx = 0 \\ \text{and } a^2 wdz + dx = 0 \end{array}\right\}$$

the former of which gives

$$x^2 = a^2\, lw + \text{const.} = a^2 lcw$$

$$\therefore w = \frac{1}{c}\, e^{\frac{x^2}{a^2}}$$

Hence

$$dz = -\frac{dx}{a^2 w} = -\frac{c}{a^2} \cdot e^{-\frac{x^2}{a^2}} dx$$

whose integral has never yet been found except between certain limits. See *Laplace Mec. Cel.* liv. X. art 5., and *Whewell's Dynamics* pp. 15 and 16.

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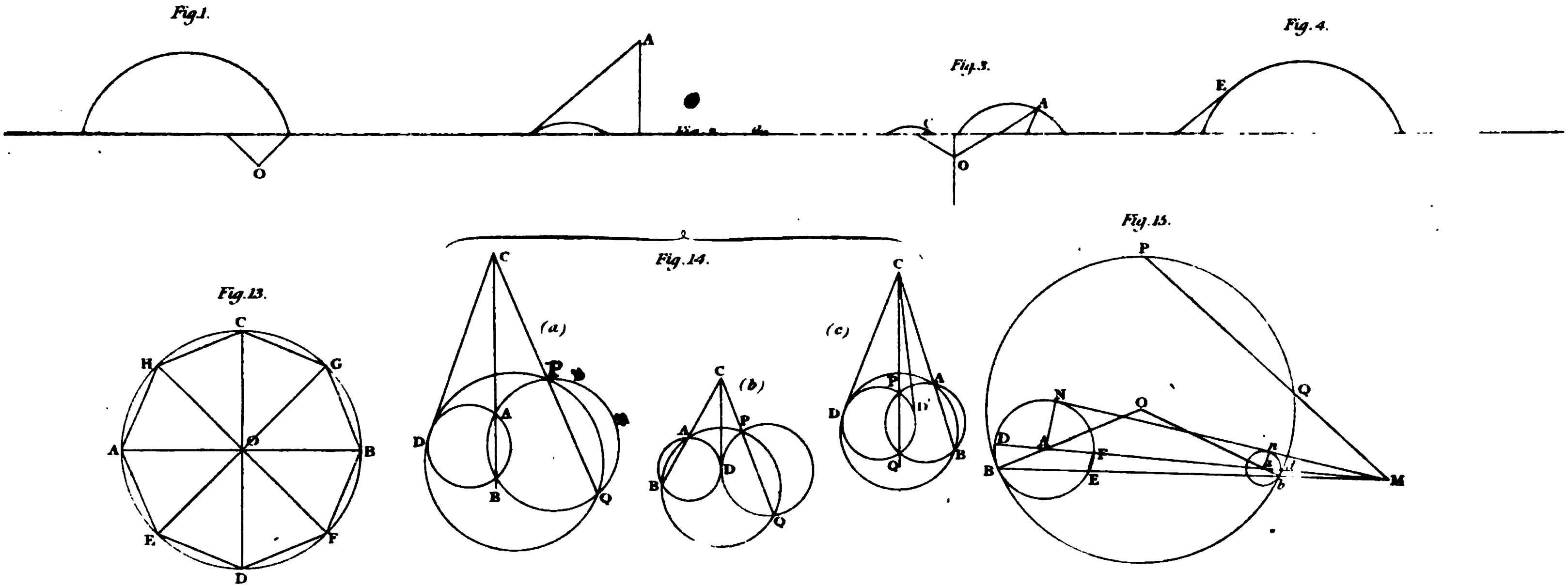

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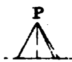
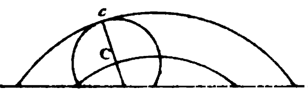


Fig. 16.



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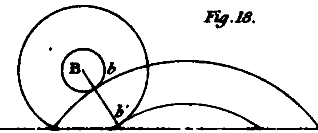
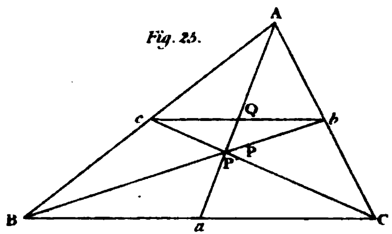


Fig. 18.



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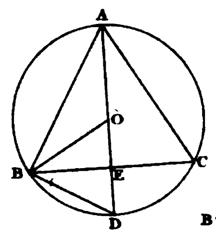


Fig. 26.

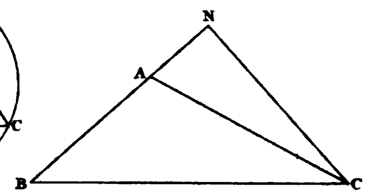
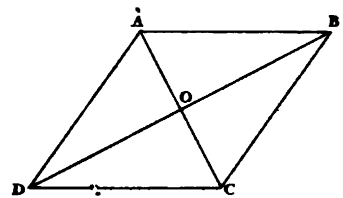


Fig. 27.



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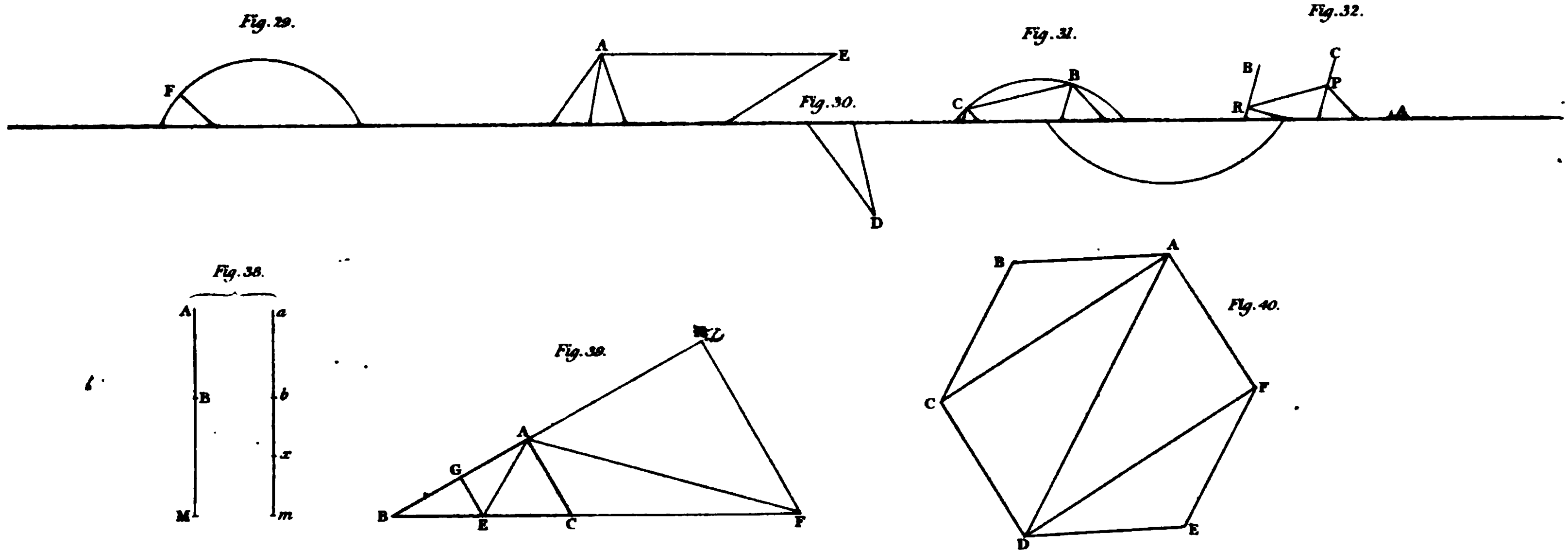

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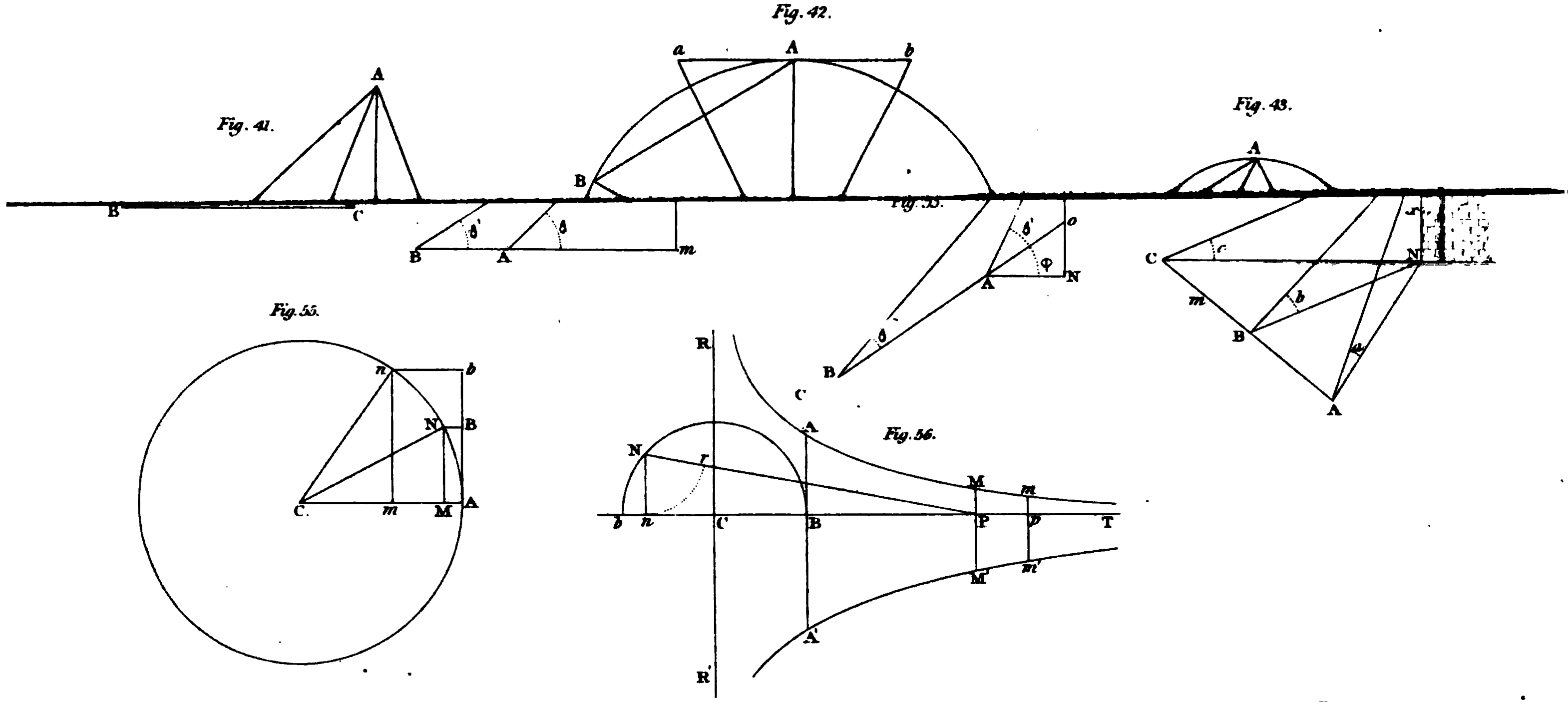

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